SOME RESULTS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS ${}_3F_{2,K}$

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TO

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Abstract

In this research work, Our aim is to give some results and applications associated with hypergeometric k-functions i.e., ${}_3F_{2,k}$ with the help of integral representation of hypergeometric k-functions. We derive summation k-theorm and give its applications. We infer some k-differential equations and then discuss solutions of these k-differential equations. The principal aim of this research is to prove Kummer k-theorem, Saalschutz k-theorem, Whipple k-theorem, Dixon k-theorem and Ramanujan's k-theorem involving hypergeometric k-functions i.e., ${}_3F_{2,k}$.

Introduction

The series

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ...$$

is called the Gauss series or the ordinary hypergeometric series. It is also represented by way of the image

$${}_2F_1(a, b; c; z).$$

The variable is z, and a,b and c are called the parameters of the function. If either of the parameters a or b is a negative integer i.e., $-n$, the series has handiest a finite wide variety of terms and turns into in fact a polynomial

$${}_2F_1(-n, b; c; z).$$

As an example,suppose that $a = -2$, then the series will become

$${}_2F_1(-2, b; c; z) = 1 + \frac{(-2)b}{c}\frac{z}{1!} + \frac{(-2)(-1)b(b+1)}{c(c+1)}\frac{z^2}{2!}.$$

In his work Arithmetica Infinitorum (1655), the Oxford teacher John Wallis (1616-1703) initially utilized the expression "hypergeometric" to indicate any series which

changed into past the ordinary geometric collection.

$$1 + x + x^2 + x^3 + ... \quad .$$

Specifically, he studied the series

$$1 + a + a(a+1) + a(a+1)(a+2) + ... \quad .$$

At some stage in the following a hundred and fifty years many other mathematicians studied similar collection, notably the Swiss Euler (1707-1783) who gave amongst many other consequences, the famous relation

$${}_2F_1(-n, b; c; z) = (1-z)^{c+n-b}{}_2F_1(c+n, c-b; c; z),$$

In 1770, the Frenchman Vandermonde (1735-1796) expressed his theorem, an extension of the binomial theorem, in the structure

$${}_2F_1(-n, b; c; 1) = \frac{(c-b)(c-b+1)(c-b+2)...(c-b+n-1)}{c(c+1)(c+2)(c+3)...(c+n-1)}$$

but for the duration of the subsequent 40 years the Göttingen faculty under Hindenberg (1741- 1808) wasted a good deal attempt on diverse complex extensions of the binomial and multinomial theorems. All this was changed dramatically, when on 20th january, 1812, Gauss (1777-1855) added his famous thesis "Disquisitiones generales circa seriem infinitam" earlier than the Royal Society in Göttingen. In it, this incredible mathematician defined the above infinite series and delivered the notation F(a,b;c;z) for it. He also proved his famous summation theorem

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and he gave many relations between two or more of these series. He showed honestly that he was already concerning

$${}_2F_1(a, b; c; z)$$

as a function in four variables, rather than a series in z, and in a note introduced 10 February, 1812, he gave a remarkably complete discussion of the convergence of such series. Gauss defined the hypergeometric function as

$$ {}_2F_1(a;b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}(z<1; Re(c)>Re(b)>0; c\neq 0,-1,-2,...). $$

The most of the theory of hypergeometric function was taken after from the work done by Gauss, who displayed some of his initial results in his publication "Disquisition generales circa seriem infinitam" [1]. For more detail about hypergeometric series [2, 3, 4, 20, 21, 10]. Secondly, solutions of system of differential equations which is, in a proper sense, holonomic and mild singularities. The differential equation

$$ z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 $$

was first established by Euler in 1769 [11], which was extensively studied by Gauss and Kummer and given a abstract treament by Bernhard Riemann $(1826-1866)$; Thirdly, as functions defined by integrals such as Mellin-Barnes integral. In 1748, Euler an integral representation for the Gauss hypergeometric function [9]

$$ F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt. $$

Hypergeometric functions have enormous applications in pure mathematics, as well as in applied areas such as acoustics, electrical current, fluid dynamics, heat conduction, solutions of wave equations, moments of inertia and quantum mechanics [5, 6, 7, 8, 9]. Hypergeometric functions have explicit series and integral representations, and thus provide ideal tools for establishing valuable summation and transformation formula. In expansion, applied problems frequently require solutions of a function in terms of parameters, rather than merely in terms of a variable and any such solution is

perfectly provided for by the parametric nature of the hypergeometric function. As a end result, the hypergeometric function can be utilized to solve physical problems in various areas of applied mathematics.

There exist large numbers of identities through which we are able to specific one Gauss hypergeometric function in terms of one or more others and considerable [10, 12, 19]. The next primary develop became made in 1836 through Kummer (1810-93), who first used the term hypergeometric for the above series. He showed that the differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c-(1+a+b)z]\frac{dy}{dz} - aby = 0$$

is satisfied by the function

$${}_2F_1(a,b;c;z)$$

and has all twenty-four answers in phrases of similar Gauss function. The first quadratic transformation were given by Kummer in Crelle's Journal (Journal fur die reine und angewandte Mathematik), Bd 15 (1836) and comprehensive lists can be found in [9] and [14]. There are many techniques for deriving these transformation identities.

In 1857, Riemann (1826-sixty six) extended this theory by means of the advent of his P function, which in a manner, are generalizations of the Gaussian

$${}_2F_1(a,b;c;z)$$

Riemann also brought the general idea of the transformation of the variable in a differential equation and this idea become carried out to Kummers work through Thomae who, in 1879, worked out in detail the relationships between Kummers twenty-four solutions.

Diaz [20, 21, 22] have introduced Pochammer k-symbol. They've brought k-gamma and k-beta functions and proved a number of their properties. They have additionally studied k-zeta functions and k-hypergeometric functions based totally on Pochhammer k-symbols for factorial function. In 2010, Kokologiannaki and Krasniqi [23, 24] observed the work of Diaz and shaped a few essential results of k-beta, k-gamma, k-zeta, and k-hypergeometric functions. Additionally, Krasniqi [25] gave a limit for the k-gamma and k-beta functions. In 2012, Mubeen and Habibullah [26, 27] introduced a variation of fractional integrals to be called k-fractional integral which turned into based on k-gamma function and gave an its utility. In addition they added an integral representation of some generalized hypergeometric-k function by the usage of the properties of Pochhammer k-function, k-gamma and k-beta functions. In 2013 Mubeen et al. [28] has recently delivered the second order linear differential equations of k-hypergeometric and confluent k-hypergeometric functions. The principle goal of this paper is to develop those k-hypergeometric differential equations by means of adopting the same technique, used by way of Rainville in [29].

Our zero chapter is on presentation and in first we characterized some definition those utilized as a part of our exploration work and in second part we demonstrate a few outcomes that we change over in third chapter in k structure.

Chapter 1

PRELIMINARIES

In this chapter, we define factorial function, Pochhammer's symbol, some important results of Pochhammer's symbol, gamma function, beta function, Hypergeometric function, differential equation of Hypergeometric function, Generalized Hypergeometric function, and give the proof of Legendre's Duplication Formula. We also define the Pochhammer k-symbol, gamma k-function, Hypergeometric k-function, beta k-function, differential equation of Hypergeometric k-function, Generalized Hypergeometric k-function.

1.1 The Factorial Function

The factorial function is formally defined by the product

$$n! = \prod_{j=1}^{n} j$$

or by recurrence relation

$$n! = \begin{cases} 1 \quad if \quad n = 0 \\ n(n-1)(n-2)...3.2.1 \quad if \quad n \in \mathbb{N}, \end{cases}$$

where $\mathbb{N}$ is a set of natural numbers.

1.2 Gamma Function

Let $z \in \mathbb{C}$ ($\mathbb{C}$ is a set of complex numbers), the gamma function is defined by Euler in integral form as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad Re(z) > 0.$$

Weierstrass defined the function $\Gamma(z)$ by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[(1+\frac{z}{n})exp(\frac{-z}{n})\right],$$

where γ is Euler constant. The gamma function in factorial form can be written as

$$\Gamma(\alpha) = (\alpha-1)!, \quad \alpha \in \mathbb{N}$$

1.3 Pochhammer's Symbol

The Pochhammer's symbol or Pochhammer's function is denoted and defined by

$$(\alpha)_n = \begin{cases} 1 \quad if \quad n = 0 \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1) \quad if \quad n \in \mathbb{N} \;\; and \;\; \alpha \neq 0. \end{cases}$$

Due to the fact that $(1)_n = n!$, it's far looked upon as a generalization of the simple factorial; therefore, the symbol $(\alpha)_n$ is also cited shifted factorial. In manipulations with $(\alpha)_n$ it's miles important to take into account that $(\alpha)_n$ is a product of n factors, beginning with α and with every factor large by unity than the previous factor.

1.4 Beta Function

The beta function, or Eulerian integral of first kind,is defined by the Euler integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad (Re(x) > 0, Re(y) > 0)$$

and is related to the gamma function through the following relation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

1.5 The Hypergeometric Function

The hypergeometric function is defined for $|z| < 1$ by the power series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

provided that c does not equal to 0, -1, -2, ... , where $(a)_n$ is the Pochhammer's symbol. Notice that the series terminates if either a or b is a negative integer.

1.6 The Hypergeometric Differential Equation

The hypergeometric differential equation is an ordinary linear second-order differential equation which can be defined as

$$z(1-z)w'' + [c-(a+b+1)]w' - abw \quad = \quad 0,$$

with three regular singular points 0, 1 and ∞. Every ordinary linear second-order differential equation with at most three regular singular points can be transformed into the hypergeometric differential equation.

1.7 Generalized Hypergeometric Function

The generalized hypergeometric functions with $p+q$ parameters, p parameters in numerator and q parameters in denominator, is defined by

$${}_pF_q(\alpha_1,\alpha_2,\cdots,\alpha_p;\beta_1,\beta_2,\cdots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n} \frac{z^n}{n!},$$

for all $\alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \cdots, \quad |z| < 1$ where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$.

It is known that

i) if $p \leq q$, the series converges for all finite z;

ii) if $p = q + 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$;

iii) if $p > q + 1$, the series diverges for $z \neq 0$ unless the series terminates.

1.8 Some important results of Pochhbemer's symbol

In this section, we shall discribe some important results.

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}. \tag{1.8.1}$$

$$(\alpha)_m(\alpha+m)_n = (\alpha)_{m+n}. \tag{1.8.2}$$

$$\frac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m} = (\alpha)_{n-m}. \tag{1.8.3}$$

In particular, if $\alpha = 1$ in above equation, we obtain

$$(n-m)! = \frac{(-1)^m n!}{(-n)_m}. \tag{1.8.4}$$

1.9 Another proof of Legendre's Duplication Formula

Since

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 u^{m-1}(1-u)^{n-1}du.$$

Let $m = n = z$ then

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \int_0^1 u^{z-1}(1-u)^{z-1}du.$$

Put $u = \frac{1+x}{2}$ $\quad \Rightarrow du = \frac{dx}{2}$, Therefore

$$\begin{aligned}
\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} &= \int_{-1}^1 \left(\frac{1+x}{2}\right)^{z-1}\left(1-\frac{1+x}{2}\right)^{z-1}\frac{dx}{2} \\
&= \frac{1}{2^{1+2(z-1)}}\int_{-1}^1 (1-x^2)^{z-1}dx \\
&= 2^{1-2z}\left[\int_{-1}^1 (1-x^2)^{z-1}dx\right] \\
&= 2^{1-2z}\left[2\int_0^1 (1-x^2)^{z-1}dx\right]
\end{aligned}$$

since

$$2\int_0^1 (1-x^2)^{z-1}dx = \int_0^1 t^{\frac{-1}{2}}(1-t)^{z-1}dt$$

because in $\int_0^1 t^{-1/2}(1-t)^{z-1}dt$

put $t = x^2$ $\quad dt = 2xdx$ when $t = 0$ $\quad \Rightarrow x = 0$ $t = 1$ $\quad \Rightarrow x = 1$

$$\int_0^1 t^{\frac{-1}{2}}(1-t)^{z-1}dt = \int_0^1 (x^2)^{\frac{-1}{2}}(1-x^2)^{z-1}2xdx = 2\int_0^1 (1-x^2)^{z-1}dx$$

so,

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z}B(1/2,z) = \frac{2^{1-2z}\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(z+\frac{1}{2})}$$

$$\Rightarrow \quad \Gamma(2z) = (2\pi)^{\frac{-1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}) \quad \left(\because \Gamma(\frac{1}{2}) = \sqrt{\pi}\right)$$

$$\Rightarrow \quad \sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

1.10 Pochhammer k-Symbol

Let $k \in \mathbb{R}^+$ where $\mathbb{R}^+$ is a set of positive real number, then the Pochhammer's k-symbol is defined by $(a)_{n,k} = a(a+k)(a+2k)\cdots(a+(n-1)k)$ for $n \geq 1, a \neq 0$ and $(a)_{0,k} = 0$.

1.11 Gamma k-Function

For $k \in \mathbb{R}^+$ and $z \in \mathbb{C}$, the gamma k-function (Γ_k) is defined in integral form as,

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}} dt.$$

The relation between Pochhammer k-symbol and gamma k-function is given as

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}.$$

1.12 Beta k-Function

The beta function, or Eulerian integral of first kind, is defined by the Euler integral in k-symbol,

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt \qquad (Re(x) > 0, Re(y) > 0).$$

The subsequent identity, connection of Γ_k and β_k function, is well known that

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}.$$

1.13 Hypergeometric k-Function

The hypergeometric k-function with three parameters a, b and c, two parameters a, b in numerator and one parameter c in denominator, is defined by

$${}_2F_{1,k}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!},$$

for all $a, b, c \in \mathbb{C}, c \neq 0, -1, -2, -3, \cdots, \quad |z| < 1.$

1.14 The Hypergeometric k-Differential Equation

The Hypergeometric k-differential equation is an ordinary linear second order differential equation defined as,

$$kz(1-kz)w'' + [c-(a+b+k)kz]w' - abw \quad = \quad 0.$$

1.15 Generalized Hypergeometric k-Function

The generalized hypergeometric k-function with $p+q$ parameters, p parameters in numerator and q parameters in denominator, is defined by

$${}_pF_{q,k}(\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}\cdots(\alpha_p)_{n,k}}{(\beta_1)_{n,k}(\beta_2)_{n,k}\cdots(\beta_q)_{n,k}} \frac{z^n}{n!},$$

for all $\alpha_i, \beta_j \in \mathbb{C}$, $\beta_j \neq 0, -1, -2, \cdots$, $|z| < 1$, $k \in \mathbb{R}^+$, $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$.

It is known that

i) if $p \leq q$, the series converges for all finite z;

ii) if $p = q+1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$;

iii) if $p > q+1$, the series diverges for $z \neq 0$ unless the series terminates.

Chapter 2

SOME RESULTS INVOLVING HYPERGEOMETRIC FUNCTIONS

In this chapter, we prove some results involving gamma and beta function and the properties of Gauss hypergeometric function. We solve some differential equations and then find the solutions of these differential equations. We use these differential equations and solutions for evaluating many results of hypergeometric functions. We prove Kummer's theorem, Saalschutz' theorem, Whipple's theorem, Dixon's theorem and Ramanujan's theorem.

Lemma 2.0.1. *If n is a positive integer, then*

$$\beta(p, n+1) = \frac{n!}{(p)_{n+1}}.$$

Proof:-*Since*

$$\begin{aligned}
\beta(p, n+1) &= \frac{\Gamma(p)\Gamma(n+1)}{\Gamma(p+n+1)} \\
&= \frac{\Gamma(p)\Gamma(n+1)}{(p+1)_n\Gamma(p+1)} \quad \left(\because (p+1)_n = \frac{\Gamma(p+1+n)}{\Gamma(p+1)}\right) \\
&= \frac{\Gamma(p)n!}{(p+1)_n p\Gamma(p)} \quad \left(\because \Gamma(n) = (n-1)!, n \in \mathbb{N}\right) \\
&= \frac{n!}{(p)_{n+1}}.
\end{aligned}$$

Problem:-Show that,

$$\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}dx = 2^{p+q-1}\beta(p,q).$$

Solution:-Put $y = \frac{1+x}{2}$ in

$$\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}dx \Rightarrow 2^{p+q-1}\beta(p,q).$$

Problem:-Use the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z}.$$

and the elementary result

$$sinxsiny = \frac{1}{2}[cos(x-y) - cos(x+y)].$$

to prove that

$$1 - \frac{\Gamma(c)\Gamma(1-c)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(c-a)\Gamma(a+1-c)\Gamma(c-b)\Gamma(b+1-c)}$$
$$= \frac{\Gamma(2-c)\Gamma(c-1)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}.$$

Solution:-Note that

$$1-(c-a-b) = a+b+1-c$$

$$1-(c-a) = a+1-c$$

$$1-(c-b) = b+1-c$$

$$1 - \frac{\Gamma(c)\Gamma(1-c)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(c-a)\Gamma(a+1-c)\Gamma(c-b)\Gamma(b+1-c)} = 1 - \frac{\pi^2 sin\pi(c-a)sin\pi(c-b)}{\pi^2 sin(\pi c)sin\pi(c-a-b)}$$
$$= \frac{sin(\pi c)sin\pi(c-a-b) - sin\pi(c-a)sin\pi(c-b)}{sin(\pi c)sin\pi(c-a-b)}$$
$$= \frac{\frac{1}{2}[cos\pi(a+b) - cos\pi(2c-a-b)] - \frac{1}{2}[cos\pi(b-a) - cos\pi(2c-a-b)]}{sin(\pi c)sin\pi(c-a-b)}$$

$$
\begin{aligned}
&= \frac{-sin(\pi a)sin(\pi b)}{sin(\pi c)sin\pi(c-a-b)} \\
&= \frac{sin(\pi a)sin(\pi b)}{sin(\pi c)sin\pi(a+b-c)} \\
&= \frac{sin(\pi a)sin(\pi b)}{sin(\pi c-1)sin\pi(a+b-(c-1))} \\
&= \frac{\Gamma(2-c)\Gamma(c-1)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}.
\end{aligned}
$$

Theorem 2.0.2. *If $|z| < 1$ and if $Re(c) > Re(b) > 0$,*

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt.$$

Lemma 2.0.3. *Prove that if $Re(c-a-b) > 0$, c is neither zero nor a negative integer, then*

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The value of $F(a,b;c;1)$ will play a vital role in many of the results of properties of hypergeometric such as given in the following example.

Example 1. *Show that if $Re(b) > 0$ and if n is a non negative integer,*

$$F[\frac{-1}{2}n, \frac{-1}{2}n + \frac{1}{2}; b + \frac{1}{2}; 1] = \frac{2^n(b)_n}{(2b)_n}.$$

Solution:-

$$L.H.S = F[\frac{-1}{2}n, \frac{-1}{2}n + \frac{1}{2}; b + \frac{1}{2}; 1]$$

by equation (2.3.1)

$$
\begin{aligned}
F[\frac{-1}{2}n, \frac{-1}{2}n + \frac{1}{2}; b + \frac{1}{2}; 1] &= \frac{\Gamma(b+\frac{1}{2})\Gamma(b+\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}n-\frac{1}{2})}{\Gamma(b+\frac{1}{2}+\frac{1}{2}n-\frac{1}{2})\Gamma(b+\frac{1}{2}+\frac{1}{2}n)} \\
&= \frac{\Gamma(b+\frac{1}{2})\Gamma(b)\Gamma(b+n)}{\Gamma(b)\Gamma(b+\frac{1}{2}n)\Gamma(b+\frac{1}{2}n+\frac{1}{2})} \\
&= \frac{(b)_n(2)^{1-2b}\sqrt{\pi}\Gamma(2b)}{(2)^{1-2b-n}\sqrt{\pi}\Gamma(2b+n)}
\end{aligned}
$$

because the Legendre's duplication for gamma function give us,

$$\Gamma(b)\Gamma(b+\frac{1}{2}) = 2^{1-2b}\sqrt{\pi}\Gamma(2b)$$

and

$$\Gamma(b+\frac{1}{2}n)\Gamma(b+\frac{1}{2}+\frac{1}{2}n) = 2^{1-2b-n}\sqrt{\pi}\Gamma(2b)$$

therefor,

$$F[\frac{-1}{2}n, \frac{-1}{2}n+\frac{1}{2}; b+\frac{1}{2}; 1] = \frac{2^n(b)_n}{(2b)_n}.$$

Lemma 2.0.4.

$$F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}.$$

Proof:-

$$\begin{aligned} F(-n, b; c; 1) &= \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c-b)\Gamma(c+n)} \quad \left(\because F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right) \\ &= \frac{(c-b)_n}{(c)_n}. \end{aligned}$$

Theorem 2.0.5. *If* $|z| < 1$ *and* $\frac{|z|}{|(1-z)|} < 1$, *then*

$$F(a, b; c; z) = (1-z)^{-a}F(a, c-b; c; \frac{-z}{1-z}).$$

Proof:-

$$
\begin{aligned}
R.H.S. &= (1-z)^{-a}F(a,c-b,c;\frac{-z}{1-z}) \\
&= \sum_{m=0}^{\infty}\frac{(a)_m(c-b)_m}{(c)_m}\quad\frac{(\frac{-z}{(1-z)})^m}{m!}(1-z)^{-a} \\
&= \sum_{m=0}^{\infty}\frac{(a)_m(c-b)_m}{(c)_m}\quad\frac{(-1)^m(z)^m(1-z)^{-(a+m)}}{(m!)} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_m(c-b)_m}{(c)_m}\quad\frac{(-1)^m(z)^m}{(m!)}\frac{(z)^n(a+m)_n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{m+n}(c-b)_m}{(c)_m}\quad\frac{(-1)^m(z)^{m+n}}{m!n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(a)_n(c-b)_m}{(c)_m}\frac{(-1)^m(z)^n}{m!(n-m)!}\quad\left(\because\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m)\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(a)_n(c-b)_m(-1)^m(z)^n(-n)_m}{(m!)(-1)^m(n!)(c)_m}\quad\left(\because(n-m)!=\frac{(-1)^m(n!)}{(-n)_m}\right) \\
&= \sum_{n=0}^{\infty}F(-n,c-b;c;1)(a)_n\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(c-(c-b))_n(a)_nz^n}{(c)_nn!}\quad\left(\because F(-n,a;b;1)=\frac{(b-a)_n}{(b)_n}\right) \\
&= F(a,b;c;z).
\end{aligned}
$$

Theorem 2.0.6. *If $|z|<1$, then*

$$F\left(a,b;c;z\right)=(1-z)^{c-a-b}F\left(c-a,c-b;c;z\right).$$

Proof:-*Consider*

$$F(a,b;c;z)=(1-z)^{-a}F(a,c-b;c;\frac{-z}{1-z}).$$

By putting $y=\frac{-z}{1-z}$ in the second factor of R.H.S of above equation and use it again, we obtain

$$
\begin{aligned}
F(a,c-b;c;\frac{-z}{1-z}) &= F(c-b,a;c;y) \\
&= (1-y)^{-(c-b)}F(c-b,c-a;c;\frac{-y}{1-y}) \\
&= (1-y)^{-c+b}F(c-a,c-b;c;\frac{-y}{1-y}).
\end{aligned}
$$

But $y = \frac{-z}{1-z}$ *and* $1 - y = (1-z)^{-1}$,
Therefore,

$$F(a, c-b; c; \frac{-z}{1-z}) = (1-z)^{-(-c+b)} F(c-a, c-b; c; z)$$

so,

$$F(a, b; c; z) = (1-z)^{c-b-a} F(c-a, c-b; c; z).$$

2.1 Derivation of differential equations

The Hypergeometric differential equation is

$$z(1-z)w'' + [2b - (a+b+1)z]w' - abw = 0.$$

Let $z = \frac{4x}{(1+x)^2}$

Then,

$$\frac{dz}{dx} = \frac{4(1-x)}{(1+x)^3}$$

and

$$1 - z = \frac{(1-x)^2}{(1+x)^2}.$$

Now

$$w' = \frac{dw}{dz} = \frac{dw}{dx}\frac{dx}{dz} = \frac{dw}{dx}\frac{(1+x)^3}{4(1-x)}$$

and

$$
\begin{aligned}
w'' &= \frac{dw'}{dx}\frac{dx}{dz} \\
&= \frac{d}{dx}\left[\frac{(1+x)^3}{4(1-x)}\frac{dw}{dx}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+x)^3}{4(1-x)} + \frac{dw}{dx}\frac{d}{dx}\frac{(1+x)^3}{4(1-x)}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+x)^3}{4(1-x)} + \frac{dw}{dx}\frac{1}{4}\left[(1+x)^3(-1)(1-x)^{-2}(-1) + (1-x)^{-1}3(1+x)^2\right]\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+x)^3}{4(1-x)} + \frac{dw}{dx}\frac{1}{2}\frac{(1+x)^2(2-x)}{(1-x)^2}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+x)^3}{4(1-x)} + \frac{dw}{dx}\frac{1}{2}\frac{(1+x)^2(2-x)}{(1-x)^2}\right]\frac{(1+x)^3}{4(1-x)} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+x)}{2} + \frac{dw}{dx}\frac{(2-x)}{1-x}\right]\frac{(1+x)^5}{8(1-x)^2}
\end{aligned}
$$

since

$$z(1-z)w'' + [2b-(a+b+1)z]w' - abw = 0$$

putting the values of w' and w'' in above equation, we obtain

$$
\begin{aligned}
&\frac{x(1+x)^2}{4}\frac{d^2w}{dx^2} + \frac{x(1+x)}{2}\frac{dw}{dx}\frac{(2-x)}{1-x} + \frac{b(1+x)^3}{2(1-x)}\frac{dw}{dx} \\
&\qquad -\frac{(a+b+1)x(1+x)}{1-x}\frac{dw}{dx} - abw = 0 \\
\Rightarrow\ &\frac{4(1-x)x(1+x)^2}{4}\frac{d^2w}{dx^2} + \frac{x4(1-x)(1+x)}{2}\frac{dw}{dx}\frac{(2-x)}{1-x} + \frac{4(1-x)b(1+x)^3}{2(1-x)}\frac{dw}{dx} \\
&\qquad -\frac{4(1-x)(a+b+1)x(1+x)}{1-x}\frac{dw}{dx} - 4(1-x)abw = 0
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow (1-x)x(1+x)^2\frac{d^2w}{dx^2} + 2x(1+x)(2-x)\frac{dw}{dx} + 2b(1+x)^3\frac{dw}{dx} \\
&\qquad -4x(a+b+1)(1+x)\frac{dw}{dx} - 4(1-x)abw = 0 \\
\Rightarrow\ &x(1-x)(1+x)^2\frac{d^2w}{dx^2} + 2(1+x)(b-2ax+bx^2-x^2)\frac{dw}{dx} - 4ab(1-x)w = 0
\end{aligned}
$$

of which one solution is $w = F(a, b; 2b; \frac{4x}{(1+x)^2})$.

put $w = (1+x)^{2a}y$ in above equation, we get

$$w' = \frac{dw}{dx} = (1+x)^{2a}\frac{dy}{dx} + 2a(1+x)^{2a-1}y$$

and

$$\begin{aligned} w'' &= \frac{dw'}{dx} \\ &= \frac{d}{dx}\left(\frac{dw}{dx}\right) \\ &= \frac{d}{dx}\left[(1+x)^{2a}\frac{dy}{dx} + 2a(1+x)^{2a-1}y\right] \\ &= \frac{d^2y}{dx^2}(1+x)^{2a} + 4a(1+x)^{2a-1}\frac{dy}{dx} + 2a(2a-1)(1+x)^{2a-2}y \end{aligned}$$

so, the equation becomes

$$\begin{aligned} x(1-x)(1+x)^2\left[(1+x)^{2a}\frac{d^2y}{dx^2} + \frac{4a(1+x)^{2a}}{1+x}\frac{dy}{dx} + \frac{2a(2a-1)(1+x)^{2a}y}{(1+x)^2}\right] \\ +2(1+x)(b-2ax+bx^2-x^2)\left[(1+x)^{2a}\frac{dy}{dx} + \frac{2a(1+x)^{2a}y}{(1+x)}\right] - 4ab(1-x)(1+x)^{2a}y = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow x(1-x)(1+x)(1+x)^{2a+1}\frac{d^2y}{dx^2} + x(1-x)4a(1+x)^{2a+1}\frac{dy}{dx} + x(1-x)2a(2a-1)(1+x)^{2a}y + \\ 2(1+x)^{2a+1}(b-2ax+bx^2-x^2)\frac{dy}{dx} \\ +2(b-2ax+bx^2-x^2)2a(1+x)^{2a}y - 4ab(1-x)(1+x)^{2a}y = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow x(1-x)(1+x)(1+x)^{2a+1}\frac{d^2y}{dx^2} + 2(1+x)^{2a+1}\left[b-(2a-b+1)x^2\right]\frac{dy}{dx} \\ -2ax(1+x)^{2a+1}y(2a+1-2b)y = 0 \end{aligned}$$

since $(1+x)^{2a+1} \neq 0$ so, the equation becomes

$$x(1-x^2)y'' + 2\left[b-(2a-b+1)x^2\right]y' - 2ax(2a+1-2b)y = 0$$

of which one solution is $w = (1+x)^{-2a}F(a,b;2b;\frac{4x}{(1+x)^2})$. The above differential equation is invariant under a change from x to $-x$. Put $v = x^2 \quad \Rightarrow \sqrt{v} = x$ and $\frac{dv}{dx} = 2x = 2\sqrt{v}$

$$y' = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = 2\sqrt{v}\frac{dy}{dv}$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(2\sqrt{v}\frac{dy}{dv}\right) = \frac{d}{dv}\left(2\sqrt{v}\frac{dy}{dv}\right)\frac{dv}{dx} = \left(2\sqrt{v}\frac{d^2y}{dv^2} + \frac{1}{\sqrt{v}}\frac{dy}{dv}\right)2\sqrt{v} = 4v\frac{d^2y}{dv^2} + 2\frac{dy}{dv}$$

putting the values of y, y' in above equation, we obtain

$$\sqrt{v}(1-v)\left(4v\frac{d^2y}{dv^2} + 2\frac{dy}{dv}\right) + 2\left[b - (2a-b+1)v\right]2\sqrt{v}\frac{dy}{dv} - 2a\sqrt{v}(1+2a-2b)y = 0$$

$$\Rightarrow 4v^{\frac{3}{2}}(1-v)\frac{d^2y}{dv^2} + 2\sqrt{v}(1-v)\frac{dy}{dv} + 4b\sqrt{v}\frac{dy}{dv} - 4v^{\frac{3}{2}}(2a-b+1)\frac{dy}{dv} - 2a\sqrt{v}(2a+1-2b)y = 0$$

$$\Rightarrow 2\sqrt{v}\left[2v(1-v)\frac{d^2y}{dv^2} + (1-v)\frac{dy}{dv} + 2b\frac{dy}{dv} - 2v(2a-b+1)\frac{dy}{dv} - a(2a-2b+1)y\right] = 0$$

$$\Rightarrow 2v(1-v)\frac{d^2y}{dv^2} + 2\left[\frac{1}{2} - \frac{3v}{2} + b - 2av + bv\right]\frac{dy}{dv} - 2a(\frac{1}{2} + a - b)y = 0$$

$$\Rightarrow v(1-v)\frac{d^2y}{dv^2} + \left[b + \frac{1}{2} - (2a-b+\frac{3}{2})v\right]\frac{dy}{dv} - a(a-b+\frac{1}{2})y = 0.$$

The function $F(a,b;a+b+\frac{1}{2};z)$ is a solution of the differential equation

$$z(1-z)y'' + [a+b+\frac{1}{2} - (a+b+1)z]y' - aby = 0$$

Let $z = 4x(1-x) = 4x - 4x^2 \quad \Rightarrow \frac{dz}{dx} = 4(1-2x)$

$$y' = \frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2x)}$$

$$y'' = \frac{d^2y}{dz^2} = \frac{dy'}{dz} = \frac{d}{dz}\left[\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2x)}\right] = \frac{d}{dx}\left[\frac{1}{4}\frac{dy}{dx}(1-2x)^{-1}\right]\frac{dx}{dz}$$

$$= \left[\frac{1}{4}\frac{dy}{dx}(-1)(1-2x)^{-2}(-2) + \frac{(1-2x)^{-1}}{4}\frac{d^2y}{dx^2}\right]\frac{1}{4(1-2x)}$$

$$= \left[\frac{1}{(1-2x)}\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\right]\frac{1}{8(1-2x)^2}$$

put the values of y' and y'' in above equation, we get

$$\frac{x(1-x)}{2(1-2x)}\frac{dy}{dx} + \frac{x(1-x)}{4}\frac{d^2y}{dx^2} + \left[a+b+\frac{1}{2}-(a+b+1)(4x-4x^2)\right]\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2x)} - aby = 0$$

$$\Rightarrow \frac{4x(1-x)}{2(1-2x)}\frac{dy}{dx} + x(1-x)\frac{d^2y}{dx^2} + \left[a+b+\frac{1}{2}-(a+b+1)(4x-4x^2)\right]\frac{dy}{dx}\frac{1}{(1-2x)} - 4aby = 0$$

$$\Rightarrow \frac{x(1-x)}{4}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left[\frac{2x(1-x)}{(1-2x)} + \left\{a+b+\frac{1}{2}-(a+b+1)(4x-4x^2)\right\}\frac{1}{(1-2x)}\right] - 4aby = 0$$

$$\Rightarrow \frac{x(1-x)}{4}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{1-2x}\left[a+b+\frac{1}{2}-4ax-4bx-2x+4ax^2+4bx^2+2x^2\right] - 4aby = 0$$

$$\Rightarrow \frac{x(1-x)}{4}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{1-2x}\left[1(a+b+\frac{1}{2})-4ax-4bx-2x+4x^2(a+b+\frac{1}{2})\right] - 4aby = 0$$

$$\Rightarrow \frac{x(1-x)}{4}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{1-2x}\left[(a+b+\frac{1}{2})(1+4x^2)-4x(a+b+\frac{1}{2})\right] - 4aby = 0$$

so, the above equation becomes

$$x(1-x)\frac{d^2y}{dx^2} + [a+b+\frac{1}{2}-(2a+2b+1)x]\frac{dy}{dx} - 4aby = 0.$$

of which one solution is $y = F(a, b; a+b+\frac{1}{2}; 4x(1-x))$.

2.2 Derivation of solutions of differential equations

Since

$$v(1-v)\frac{d^2y}{dv^2} + \left[b+\frac{1}{2}-(2a-b+\frac{3}{2})v\right]\frac{dy}{dv} - a(a-b+\frac{1}{2})y = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n v^{c+n}$ then $y' = \sum_{n=0}^{\infty} a_n(c+n)v^{c+n-1}$ $y'' = \sum_{n=0}^{\infty} a_n(c+n)(c+n-1)v^{c+n-2}$
putting the values of y, y' and y'' in above equation

$$\Rightarrow \sum_{n=0}^{\infty} a_n(c+n)(c+n-1)v^{c+n-1} - \sum_{n=0}^{\infty} a_n(c+n)(c+n-1)v^{c+n} + (b+\frac{1}{2})\sum_{n=0}^{\infty} a_n(c+n)v^{c+n-1}$$
$$-(2a-b+\frac{3}{2})\sum_{n=0}^{\infty} a_n(c+n)v^{c+n} - a(a-b+\frac{1}{2})\sum_{n=0}^{\infty} a_n v^{c+n} = 0$$

$$\sum_{n=0}^{\infty} a_n \left[(c+n)(c+n-1) + (b+\frac{1}{2})(c+n)\right] v^{c+n-1}$$
$$-\sum_{n=0}^{\infty} a_n \left[(c+n)(c+n-1) + (2a-b+\frac{3}{2})(c+n) + a(a-b+\frac{1}{2})\right] v^{c+n} = 0$$

change the summation index

$$v^c\left[a_o c(c-1)v^{-1} + a_o c(b+\frac{1}{2})v^{-1} + \sum_{n=0}^{\infty} a_{n+1}\left\{(c+n+1)(c+n) + (b+\frac{1}{2})(c+n+1)\right\}v^n\right.$$
$$\left. -\sum_{n=0}^{\infty} a_n\left\{(c+n)(c+n-1) + (2a-b+\frac{3}{2})(c+n) + a(a-b+\frac{1}{2})\right\}v^n\right] = 0$$

which implies that

$$a_o c(c-1) + a_o c(b+\frac{1}{2}) = 0$$

nothing is gained by taking $a_o = 0$

so,

$$c(c-1) + (b+\frac{1}{2})c = 0 \quad \Rightarrow c(c+b-\frac{1}{2}) = 0 \quad \Rightarrow c = 0 \;\; or \;\; c = \frac{1}{2} - b$$

$$a_{n+1}\left[(c+n+1)(c+n)+(b+\frac{1}{2})(c+n+1)\right] = a_n\left[(c+n)(c+n-1)+(2a-b+\frac{3}{2})(c+n)+a(a-b+\frac{1}{2})\right]$$

$$a_{n+1} = \frac{a_n\left[(c+n)(c+n-1) + (2a-b+\frac{3}{2})(c+n) + a(a-b+\frac{1}{2})\right]}{(c+n+1)(c+n) + (b+\frac{1}{2})(c+n+1)}$$

$n = 0, 1, 2, ...$

(1)If $c = 0$

$$a_{n+1} = \frac{a_n \left[n(n-1) + (2a - b + \frac{3}{2})n + a(a - b + \frac{1}{2}) \right]}{(n+1)n + (b + \frac{1}{2})(n+1)}$$

taking $n = 0$

$$a_1 = \frac{a_o a(a - b + \frac{1}{2})}{(b + \frac{1}{2})}$$

$n = 1$

$$a_2 = \frac{a_o a(a - b + \frac{1}{2})(a+1)(a - b + \frac{3}{2})}{(b + \frac{1}{2})2(b + \frac{3}{2})}$$

up to so on

$$a_n = \frac{a_o(a - b + \frac{1}{2})_n (a)_n}{n!(b + \frac{1}{2})_n}$$

so

$$y_1 = \sum_{n=0}^{\infty} \frac{a_o(a - b + \frac{1}{2})_n (a)_n v^n}{n!(b + \frac{1}{2})_n}$$
$$=_2 F_1(a, a - b + \frac{1}{2}; b + \frac{1}{2}; v)$$

(2)If $c = \frac{1}{2} - b$

$$a_{n+1} = \frac{a_n \left[(\frac{1}{2} - b + n)(\frac{1}{2} - b + n - 1) + (2a - b + \frac{3}{2})(\frac{1}{2} - b + n) + a(a - b + \frac{1}{2}) \right]}{(\frac{1}{2} - b + n + 1)(\frac{1}{2} - b + n) + (b + \frac{1}{2})(\frac{1}{2} - b + n + 1)}$$

$n = 0, 1, 2, ...$

taking $n = 0$

$$a_1 = \frac{a_o \left[(\frac{1}{2} - b)(\frac{1}{2} - b - 1) + (2a - b + \frac{3}{2})(\frac{1}{2} - b) + a(a - b + \frac{1}{2}) \right]}{(\frac{1}{2} - b + 1)(\frac{1}{2} - b) + (b + \frac{1}{2})(\frac{1}{2} - b + 1)}$$
$$= \frac{a_o(a - b + \frac{1}{2})(a + 1 - 2b)}{(\frac{3}{2} - b)}$$

$n = 1$

$$a_2 = \frac{a_o(a - b + \frac{1}{2})_n (a + 1 - 2b)_n}{(n! \frac{3}{2} - b)_n}$$

up to so on

$$a_n = \frac{a_o(a-b+\frac{1}{2})_n(a+1-2b)_n}{(n!\frac{3}{2}-b)_n}$$

so,

$$y_2 = v^{\frac{1}{2}-b}\sum_{n=0}^{\infty}\frac{a_o(a-b+\frac{1}{2})_n(a+1-2b)_n v^n}{(n!\frac{3}{2}-b)_n}$$
$$= v^{\frac{1}{2}-b}{}_2F_1(a-b+\frac{1}{2}, a-2b+1; \frac{3}{2}-b; v).$$

Since

$$x(1-x)\frac{d^2y}{dx^2} + [a+b+\frac{1}{2}-(2a+2b+1)x]\frac{dy}{dx} - 4aby = 0$$

Let $y = \sum\limits_{n=0}^{\infty} a_n x^{c+n}$ then $y' = \sum\limits_{n=0}^{\infty} a_n(c+n)x^{c+n-1}$ $y'' = \sum\limits_{n=0}^{\infty} a_n(c+n)(c+n-1)x^{c+n-2}$ now put the values of y, y' and y'' in above equation, we get

$$\sum_{n=0}^{\infty} a_n\left[(c+n)(c+n-1)+(a+b+\frac{1}{2})(c+n)\right]x^{c+n-1} - \sum_{n=0}^{\infty} a_n$$
$$\left[(c+n)(c+n-1)+(2a+2b+1)(c+n)+4ab\right]x^{c+n} = 0$$

change the summation index

$$x^c\left[a_o c(c-1)x^{-1} + a_o c(a+b+\frac{1}{2})x^{-1} + \sum_{n=0}^{\infty} a_{n+1}\left\{(c+n+1)(c+n)+(a+b+\frac{1}{2})(c+n+1)\right\}x^n\right.$$
$$\left. - \sum_{n=0}^{\infty} a_n\left\{(c+n)(c+n-1)+(2a+2b+1)(c+n)+4ab\right\}x^n\right] = 0$$

which implies that

$$a_o c(c-1) + a_o c(a+b+\frac{1}{2}) = 0$$

nothing is gained by taking $a_o = 0$

so,

$$c(c-1) + (a+b+\frac{1}{2})c = 0$$

$$\Rightarrow c(c+a+b-\frac{1}{2}) = 0 \quad \Rightarrow c = 0 \quad or \quad c = \frac{1}{2} - a - b$$

$$a_{n+1}\left[(c+n+1)(c+n)+(a+b+\frac{1}{2})(c+n+1)\right] = a_n\left[(c+n)(c+n-1)+(2a+2b+1)(c+n)+4ab\right]$$

$$a_{n+1} = \frac{a_n\left[(c+n)(c+n-1)+(2a+2b+1)(c+n)+4ab\right]}{(c+n+1)(c+n)+(a+b+\frac{1}{2})(c+n+1)}$$

$n = 0, 1, 2, ...$

(1)If $c = 0$

$$a_{n+1} = \frac{a_n\left[n(n-1)+(2a+2b+1)n+4ab\right]}{(n+1)n+(a+b+\frac{1}{2})(n+1)}$$

taking $n = 0$

$$a_1 = \frac{a_o 4ab}{(a+b+\frac{1}{2})}$$

$n = 1$

$$a_2 = \frac{a_o(2a)(2a+1)(2b)(2b+1)}{(a+b+\frac{1}{2})2(a+b+\frac{1}{2}+1)}$$

up to so on

$$a_n = \frac{a_o(2a)_n(2b)_n}{n!(a+b+\frac{1}{2})_n}$$

so,

$$y_1 = \sum_{n=0}^{\infty} \frac{a_o(2a)_n(2b)_n x^n}{n!(a+b+\frac{1}{2})_n}$$

$$=_2 F_1(2a, 2b; a+b+\frac{1}{2}; x)$$

(2)If $c = \frac{1}{2} - a - b$

$$a_{n+1} = \frac{a_n\left[(\frac{1}{2}-a-b+n)(\frac{1}{2}-a-b+n-1)+(2a+2b+1)(\frac{1}{2}-a-b+n)+4ab\right]}{(\frac{1}{2}-a-b+n+1)(\frac{1}{2}-a-b+n)+(a+b+\frac{1}{2})(\frac{1}{2}-a-b+n+1)}$$

$n = 0, 1, 2, ...$

taking $n = 0$

$$a_1 = \frac{a_o\left[(\frac{1}{2} - a - b)(\frac{-1}{2} - a - b) + (2a + 2b + 1)(\frac{1}{2} - a - b) + 4ab\right]}{(\frac{1}{2} - a - b + 1)(\frac{1}{2} - a - b) + (a + b + \frac{1}{2})(\frac{1}{2} - a - b + 1)}$$

$$= \frac{a_o(a - b + \frac{1}{2})(b - a + \frac{1}{2})}{(\frac{3}{2} - a - b)}$$

$n = 1$

$$a_2 = \frac{a_o(a - b + \frac{1}{2})(b - a + \frac{1}{2} + 1)}{2!(\frac{3}{2} - a - b)(\frac{3}{2} - a - b + 1)}$$

up to so on

$$a_n = \frac{a_o(a - b + \frac{1}{2})_n(b - a + \frac{1}{2})_n}{n!(\frac{3}{2} - a - b)_n}$$

so,

$$y_2 = x^{\frac{1}{2}-a-b}\sum_{n=0}^{\infty}\frac{(a - b + \frac{1}{2})_n(b - a + \frac{1}{2})_n x^n}{n!(\frac{3}{2} - a - b)_n}$$

$$= x^{\frac{1}{2}-a-b}{}_2F_1(a - b + \frac{1}{2}, b - a + \frac{1}{2}; \frac{3}{2} - a - b; x).$$

Theorem 2.2.1. *If $2b$ is neither zero nor a negative integer, and if both $|x| < 1$ and $|4x(1 + x)^{-2}| < 1$, then*

$$(1 + x)^{-2a}F\left(a, b; 2b; \frac{4x}{(1 + x)^2}\right) = F\left(a, a - b + \frac{1}{2}; b + \frac{1}{2}; x^2\right).$$

Proof:- *The equation*

$$v(1 - v)\frac{d^2y}{dv^2} + \left[b + \frac{1}{2} - (2a - b + \frac{3}{2})v\right]\frac{dy}{dv} - a(a - b + \frac{1}{2})y = 0.$$

has, in $|v| < 1$, the general solution

$$y = AF\left(a, a - b + \frac{1}{2}; b + \frac{1}{2}; v\right)$$

$$+Bv^{\frac{1}{2}-b}F\left(a - b + \frac{1}{2}, a - 2b + 1; \frac{3}{2} - b; v\right)$$

The differential equation

$$x(1-x^2)y'' + 2\left[b-(2a-b+1)x^2\right]y' - 2ax(1+2a-2b)y = 0$$

has a solution $y = (1+x)^{-2a}F(a,b;2b;\frac{4x}{(1+x)^2})$ *valid in* $|\frac{4x}{(1+x)^2}| < 1$ *as long as* $2b$ *is neither zero nor a negative integer.At the same time equation*

$$x(1-x^2)y'' + 2\left[b-(2a-b+1)x^2\right]y' - 2ax(1+2a-2b)y = 0.$$

has the general solution (2.3.11) with $v = x^2$*, this solution is valid in* $|x| < 1$*. Therefore, if both* $|x| < 1$ *and* $|\frac{4x}{(1+x)^2}| < 1$ *and if* $2b$ *is neither zero nor a negative integer, there exist constant* A *and* B *such that*

$$\begin{aligned}(1+x)^{-2a}F\left(a,b;2b;\frac{4x}{(1+x)^2}\right) &= AF\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right)\\ &+Bx^{1-2b}F\left(a-b+\frac{1}{2},a-2b+1;\frac{3}{2}-b;x^2\right)\end{aligned}$$

In equation (2.3.12) the left member and the first term on the right are analytic at $x = 0$*, but the last term is not analytic at* $x = 0$ *because of the factor* x^{1-2b}*. Hence* B *and* A *is easily determined by using* $x = 0$ *in the resultant identity*

$$(1+x)^{-2a}F\left(a,b;2b;\frac{4x}{(1+x)^2}\right) = AF\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right)$$

Thus $A = 1$

$$(1+x)^{-2a}F\left(a,b;2b;\frac{4x}{(1+x)^2}\right) = F\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right).$$

Theorem 2.2.2. *If* $2b$ *is neither zero nor a negative integer and if* $|y| < \frac{1}{2}$ *and* $|\frac{y}{(1-y)}| < 1$*, then*

$$(1-y)^{-a}F\left(\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};\frac{1}{2}+b;\frac{y^2}{(1-y)^2}\right) = F(a,b;2b;2y).$$

Proof:-

$$\begin{aligned}
L.H.S. &= (1-y)^{-a}F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}+b; \frac{y^2}{(1-y)^2}\right) \\
&= (1-y)^{-a}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_m(\frac{1}{2}a+\frac{1}{2})_m}{(b+\frac{1}{2})_m m!}\quad\left(\frac{y^2}{(1-y)^2}\right)^m \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m}y^{2m}}{(b+\frac{1}{2})_m m!2^{2m}}\ \frac{(a+2m)_n y^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+n}y^{2m+n}}{(b+\frac{1}{2})_m m!n!2^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(a)_n y^n}{(b+\frac{1}{2})_m m!(n-2m)!2^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(a)_n y^n(-n)_{2m}}{(b+\frac{1}{2})_m m!n!2^{2m}(-1)^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(a)_n y^n 2^{2m}(\frac{-n}{2})_m(\frac{-n+1}{2})_m}{(b+\frac{1}{2})_m m!n!2^{2m}} \\
&= \sum_{n=0}^{\infty}\quad F(\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}; b+\frac{1}{2}; 1)\frac{(a)_n y^n}{n!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{(b)_n}{(2b)_n}\frac{(a)_n 2y^n}{n!}\quad\left(\because F(\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}; b+\frac{1}{2}; 1) = \frac{2^n(b)_n}{(2b)_n}\right) \\
&= \sum_{n=0}^{\infty}\frac{2^n(b)_n}{(2b)_n}\frac{(a)_n y^n}{n!} \\
&= F\left(a, b; 2b; 2y\right).
\end{aligned}$$

Theorem 2.2.3. *If $1+a-b$ is neither zero nor a negative integer, and if $|z|<1$ and $|\frac{-4z}{(1-z)^2}|<1$, then*

$$(1-z)^{-a}F\left(\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; 1+a-b; \frac{-4z}{(1-z)^2}\right) = F\left(a, b; 1+a-b; z\right).$$

Proof:-*In theorem (2.3.7) put*

$$y = \frac{2x}{(1+x)^2}\qquad \Rightarrow 1-y = \frac{1+x^2}{(1+x)^2}$$

and

$$\frac{y}{1-y} = \frac{2x}{(1+x)^2}.$$

so theorem (2.3.7) becomes

$$\frac{(1+x^2)^{-a}}{(1+x)^{-2a}} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}+b; \frac{4x^2}{(1+x^2)^2}\right) = F\left(a, b; 2b; \frac{4x}{(1+x)^2}\right)$$

$$\Rightarrow (1+x^2)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}+b; \frac{4x^2}{(1+x^2)^2}\right) = (1+x)^{-2a} F\left(a, b; 2b; \frac{4x}{(1+x)^2}\right)$$

using theorem (2.3.6) on R.H.S

$$(1+x^2)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}+b; \frac{4x^2}{(1+x^2)^2}\right) = F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; x^2\right)$$

now put $x^2 = z$ *and replace* b *by* $\frac{1}{2}+a-b$ *in above equation*

$$(1+z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}+a-b+\frac{1}{2}; \frac{4z}{(1+z)^2}\right) = F\left(a, a-\frac{1}{2}-a+b+\frac{1}{2}; \frac{1}{2}+a-b+\frac{1}{2}; z\right)$$

$$(1+z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; 1+a-b; \frac{4z}{(1+z)^2}\right) = F\left(a, b; 1+a-b; z\right)$$

using theorem

$$F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; \frac{-z}{1-z}).$$

on R.H.S of above equation

$$F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; 1+a-b; \frac{4z}{(1+z)^2}\right) = \left(1-\frac{4z}{(1+z)^2}\right)^{\frac{-a}{2}}$$

$$F\left(\frac{1}{2}a, 1+a-b-\frac{1}{2}a-\frac{1}{2}; 1+a-b; \frac{\frac{-4z}{(1+z)^2}}{1-\frac{4z}{(1+z)^2}}\right)$$

$$= \left[\frac{(1-z)^2}{(1+z)^2}\right]^{\frac{-a}{2}}$$

$$F\left(\frac{1}{2}a, 1+a-b-\frac{1}{2}a-\frac{1}{2}; 1+a-b; \frac{-4z}{(1+z)^2}\frac{(1+z)^2}{(1-z)^2}\right)$$

so,

$$(1-z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; 1+a-b; \frac{-4z}{(1-z)^2}\right) = F\left(a, b; 1+a-b; z\right).$$

Theorem 2.2.4. *If $a+b+\frac{1}{2}$ is neither zero nor a negative integer, and if $|x|<1$ and $|4x(1-x)|<1$, then*

$$F(a,b;a+b+\frac{1}{2};4x(1-x)) = F(2a,2b;a+b+\frac{1}{2};x).$$

Proof:-*The equation*

$$x(1-x)\frac{d^2y}{dx^2}+[a+b+\frac{1}{2}-(2a+2b+1)x]\frac{dy}{dx}-4aby=0.$$

is hypergeometric in character and has the general solution if $|x|<1$ and $|4x(1-x)|<1$, $a+b+\frac{1}{2}$ is neither zero nor a negative integer, there exist constants A and B such that

$$y = AF\left(2a,2b;a+b+\frac{1}{2};x\right)$$
$$+Bx^{\frac{1}{2}-a-b}F\left(\frac{1}{2}+a-b,\frac{1}{2}+b-a;\frac{3}{2}-a-b;x\right)$$

as well as have the solution $y=F(a,b;a+b+\frac{1}{2};4x(1-x))$

$$F\left(a,b;a+b+\frac{1}{2};4x(1-x)\right) = AF\left(2a,2b;a+b+\frac{1}{2};x\right)$$
$$+Bx^{\frac{1}{2}-a-b}F\left(\frac{1}{2}+a-b,\frac{1}{2}+b-a;\frac{3}{2}-a-b;x\right)$$

In equation (2.3.15) the left member and the first term on the right are analytic at x=0, but the last term is not analytic at x=0 because of the factor $x^{\frac{1}{2}-a-b}$. Hence B=0 and A is easily determined by using x=0 in the resultant identity

$$F\left(a,b;a+b+\frac{1}{2};4x(1-x)\right) = AF\left(2a,2b;a+b+\frac{1}{2};x\right)$$

Thus A=1

$$F\left(a,b;a+b+\frac{1}{2};4x(1-x)\right) = F\left(2a,2b;a+b+\frac{1}{2};x\right).$$

Theorem 2.2.5. *If c is neither zero nor a negative integer, and if both $|x|<1$ and $|4x(1-x)|<1$, then*

$$F(a,1-a;c;x) = (1-x)^{c-1}F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};c;4x(1-x)).$$

Proof:-*In the identity of theorem (2.3.11) replace a by* $\frac{1}{2}c-\frac{1}{2}a$ *and b by* $\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}$

$$F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}+\frac{1}{2};4x(1-x))$$
$$=F(2(\frac{1}{2}c-\frac{1}{2}a),2(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2});\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}+\frac{1}{2};x)$$

$$F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};c;4x(1-x))=F(c-a,c+a-1;c;x)$$

using theorem

$$F\left(a,b;c;z\right)=(1-z)^{c-a-b}F\left(c-a,c-b;c;z\right)$$

on R.H.S of the above equation, we obtain

$$\begin{aligned}F(c-a,c+a-1;c;x) &= (1-x)^{c-c+a-c-a+1}F(c-c+a,c-c-a+1;c;x)\\ &= (1-x)^{1-c}F(a,1-a;c;x)\end{aligned}$$

so,

$$F(a,1-a;c;x)=(1-x)^{c-1}F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};c;4x(1-x))$$

Theorem 2.2.6.

$$F\left[-n,1-b-n;a;1\right]=\frac{(a+b-1)_{2n}}{(a)_n(a+b-1)_n}.$$

Proof:-

$$\begin{aligned}L.H.S. &= F\left[-n,1-b-n;a;1\right]\\ &= \frac{\Gamma(a)}{\Gamma(1-b-n)\Gamma(a+b+n-1)}\int_0^1 t^{1-b-n-1}(1-t)^{a+b+2n-1-1}dt\\ &= \frac{\Gamma(a)}{\Gamma(1-b-n)\Gamma(a+b+n-1)}\quad\frac{\Gamma(1-b-n)\Gamma(a+b+2n-1)}{\Gamma(1-b-n+a+b+2n-1)}\\ &= \frac{\Gamma(a)\Gamma(a+b+2n-1)}{\Gamma(a+n)\Gamma(a+b+n-1)}\\ &= \frac{\Gamma(a)}{\Gamma(a+n)}\quad\frac{\Gamma(a+b+2n-1)}{\Gamma(a+b+n-1)}\quad\frac{\Gamma(a+b-1)}{\Gamma(a+b-1)}\\ &= \frac{1}{(a)_n}\quad\frac{1}{(a+b-1)_n}\quad(a+b-1)_{2n}\\ &= \frac{(a+b-1)_{2n}}{(a)_n(a+b-1)_n}.\end{aligned}$$

Theorem 2.2.7.

$$F(a,b;\frac{1}{2}(a+b+1);\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(1+a+b)]}{\Gamma[\frac{1}{2}(1+a)]\Gamma[\frac{1}{2}(1+b)]}.$$

Proof:-*Since*

$$F(a,b;a+b+\frac{1}{2};4x(1-x)) = F(2a,2b;a+b+\frac{1}{2};x).$$

Replace a *by* $\frac{a}{2}$ *and* b *by* $\frac{b}{2}$

$$F(\frac{a}{2},\frac{b}{2};\frac{a}{2}+\frac{b}{2}+\frac{1}{2};4x(1-x)) = F(a,b;\frac{a}{2}+\frac{b}{2}+\frac{1}{2};x)$$

put $x = \frac{1}{2}$

$$\begin{aligned} F(a,b;\frac{a}{2}+\frac{b}{2}+\frac{1}{2};\frac{1}{2}) &= F(\frac{a}{2},\frac{b}{2};\frac{a}{2}+\frac{b}{2}+\frac{1}{2};1) \\ &= \frac{\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}-\frac{a}{2}-\frac{b}{2})}{\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}-\frac{a}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}-\frac{b}{2})} = \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(1+a+b)]}{\Gamma[\frac{1}{2}(1+a)]\Gamma[\frac{1}{2}(1+b)]}. \end{aligned}$$

Theorem 2.2.8.

$$F(a,1-a;c;\frac{1}{2}) = \frac{\Gamma(\frac{1}{2}c)\Gamma[\frac{1}{2}(c+1)]}{\Gamma[\frac{1}{2}(a+c)]\Gamma[\frac{1}{2}(1+c-a)]}.$$

Proof:-*Since*

$$F(a,1-a;c;x) = (1-x)^{c-1}F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};c;4x(1-x)).$$

Put $\quad x = \frac{1}{2}$

$$\begin{aligned} F(a,1-a;c;\frac{1}{2}) &= (1-\frac{1}{2})^{c-1}F(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2};c;1) \\ &= \frac{(\frac{1}{2})^{c-1}\Gamma(c)\Gamma(c-\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}{\Gamma(c-\frac{1}{2}c+\frac{1}{2}a)\Gamma(c-\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} \\ &= \frac{2^{1-c}\Gamma(c)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} \end{aligned}$$

since

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z).$$

Put $z = \frac{c}{2}$ *in above equation, we get*

$$\frac{2^{c-1}\Gamma(\frac{c}{2})\Gamma[\frac{1}{2}(c+1)]}{\sqrt{\pi}} = \Gamma(c)$$

so,

$$\begin{aligned} F(a, 1-a; c; \frac{1}{2}) &= \frac{2^{1-c}2^{c-1}\Gamma(\frac{c}{2})\Gamma[\frac{1}{2}(c+1)]\sqrt{\pi}}{\sqrt{\pi}\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})} \quad \left(\because \Gamma(\frac{1}{2}) = \sqrt{\pi}\right) \\ &= \frac{\Gamma(\frac{1}{2}c)\Gamma[\frac{1}{2}(c+1)]}{\Gamma[\frac{1}{2}(a+c)]\Gamma[\frac{1}{2}(1+c-a)]}. \end{aligned}$$

Theorem 2.2.9.

$$\frac{d^n}{dx^n}\left[x^{a-1+n}F(a,b;c;x)\right] = (a)_n x^{a-1}F(a+n,b;c;x).$$

Proof:-

$$\begin{aligned} L.H.S &= \frac{d^n}{dx^n}\left[x^{a-1+n}F(a,b;c;x)\right] \\ &= \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}\frac{d^n}{dx^n}x^{m+n+a-1} \\ &= \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}(n+m+a-1)(n+m+a-2)...(n+m+a-n)x^{n+m+a-n-1} \\ &= \sum_{m=0}^{\infty}\frac{(a)_m(a)_{n+m}(b)_m x^{m+a-1}}{(a)_m(c)_m m!} \\ &= \sum_{m=0}^{\infty}\frac{(a+n)_m(a)_n(b)_m x^{m+a-1}}{(c)_m m!} \\ &= (a)_n x^{a-1}F(a+n,b;c;x). \end{aligned}$$

Kummer's Theorem

Theorem 2.2.10. *If (1+a-b) is neither zero nor a negative integer, and* $Re(b) < 1$ *for convergence,*

$$F(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+a)}.$$

Proof:-*Since*

$$(1-z)^{-a}F\left(\frac{1}{2}a,\frac{1}{2}+\frac{1}{2}a-b;1+a-b;\frac{-4z}{(1-z)^2}\right)=F(a,b;1+a-b;z).$$

put $z=-1$ *in above equation, then we have*

$$(2)^{-a}F\left(\frac{1}{2}a,\frac{1}{2}+\frac{1}{2}a-b;1+a-b;1\right)=F(a,b;1+a-b;-1)$$

$$\frac{\Gamma(1+a-b)\Gamma(1+a-b-\frac{1}{2}a-\frac{1}{2}a-\frac{1}{2}+b)}{2^a\Gamma(1+a-b-\frac{1}{2}a)\Gamma(1+a-b-\frac{1}{2}-\frac{1}{2}a)+b}=F(a,b;1+a-b;-1)$$

$$\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a)}=F(a,b;1+a-b;-1)$$

By Legendre's Duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2})=\Gamma(\frac{1}{2})2^{1-2z}\Gamma(2z)$$

put $z=\frac{a+1}{2}$ *in above equation, implies*

$$\frac{2^a\Gamma(\frac{1}{2}+\frac{1}{2}a)+\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)}=\Gamma(\frac{1}{2})$$

so,

$$F(a,b;1+a-b;-1)=\frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+a)}.$$

Theorem 2.2.11.

$$(1-x)^{1-c}F(a,1-a;c;x)=(1-2x)^{a-c}F\left(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2};c;\frac{4x(x-1)}{(1-2x)^2}\right).$$

Proof:-*Since*

$$(1-x)^{1-c}F(a,1-a;c;x)=F\left(\frac{c-a}{2},\frac{c+a-1}{2};c;4x(1-x)\right)$$

and

$$F(a,b;c;z)=(1-z)^{-a}F(a,c-b;c;\frac{-z}{1-z}).$$

therefore,

$$\begin{aligned}(1-x)^{1-c}F(a,1-a;c;x) &= (1-4x(1-x))^{-\frac{c-a}{2}}F\left(\frac{c-a}{2},c-\frac{c+a-1}{2};c;\frac{-4x(1-x)}{1-4x(1-x)}\right)\\ &= (1-2x)^{2\frac{a-c}{2}}F\left(\frac{c-a}{2},\frac{c-a+1}{2};c;\frac{4x(x-1)}{(1-2x)^2}\right)\\ &= (1-2x)^{a-c}F\left(\frac{c-a}{2},\frac{c-a+1}{2};c;\frac{4x(x-1)}{(1-2x)^2}\right).\end{aligned}$$

Theorem 2.2.12.

$$F(2a,2b;a+b+\frac{1}{2};\frac{1}{2})=\frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}.$$

Proof:-*Since*

$$F(2a,2b;a+b+\frac{1}{2};x)=F(a,b;a+b+\frac{1}{2};4x(1-x)).$$

Put $x=\frac{1}{2}$ *so,*

$$\begin{aligned}F(2a,2b;a+b+\frac{1}{2};\frac{1}{2}) &= F(a,b;a+b+\frac{1}{2};1)\\ &= \frac{\Gamma(a+b+\frac{1}{2})\Gamma(a+b+\frac{1}{2}-a-b)}{\Gamma(a+b+\frac{1}{2}-a)\Gamma(a+b+\frac{1}{2}-b)}=\frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}.\end{aligned}$$

Theorem 2.2.13.

$$F(-n,a+n;c;1)=\frac{(-1)^n(1+a-c)_n}{(c)_n}.$$

Proof:-

$$\begin{aligned}F(-n,a+n;c;1) &= \frac{\Gamma(c)\Gamma(c-a)}{\Gamma(c+n)\Gamma(c-a-n)} \quad \left(\because F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)\\ &= \frac{\Gamma(c-a)}{(c)_n\Gamma(c-a-n)}.\end{aligned}$$

since

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)}=\frac{(-1)^n}{(\alpha)_n}.$$

Put $\alpha = 1 - c + a$ *in the above equation, we get*

$$\frac{\Gamma(c-a-n)}{\Gamma(c-a)} = \frac{(-1)^n}{(1-c+a)_n}$$

therefore,

$$F(-n, a+n; c; 1) = \frac{(1-c+a)_n}{(c)_n(-1)^n} = \frac{(-1)^n(1-c+a)_n}{(c)_n}.$$

Theorem 2.2.14. *If* $p \leq q+1$*, if* $Re(b_1) > Re(a_1) > 0$*, if no one of* $b_1, b_2 ... b_q$ *is zero or a negative integer, and if* $|z| < 1$*,*

$${}_pF_q\left(a_1, a_2, ...a_p; b_1, b_2, ...b_q; z\right) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)}$$
$$\int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}{}_{p-1}F_{q-1}\left(a_2, ...a_p; b_2, ...b_q; zt\right)dt.$$

Saalschutz'theorem

Theorem 2.2.15. *If* n *is a non negative integer and if* a, b *and* c *are independent of* n*, then*

$${}_3F_2(-n, a, b; c, 1-c+a+b-n; 1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Proof:-*Since*

$$F\left(a, b; c; z\right) = (1-z)^{c-a-b}F\left(c-a, c-b; c; z\right).$$

Replace a *by* $c-a$ *and* b *by* $c-b$*, we obtain*

$$F\left(c-a, c-b; c; z\right) = (1-z)^{a+b-c}F\left(a, b; c; z\right)$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{(c-a)_n(c-b)_n}{(c)_n}\frac{z^n}{n!} = (1-z)^{-(c-a-b)}F\left(a, b; c; z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(c-a-b)_n z^n}{n!} \sum_{m=0}^{\infty} \frac{(a)_m(b)_m z^m}{(c)_m m!} \quad \left(\because (1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(c-a-b)_n(a)_m(b)_m z^{n+m}}{(c)_m n! m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(c-a-b)_{n-m}(a)_m(b)_m z^n}{(c)_m(n-m)!m!} \quad \left(\because \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-n)_m(c-a-b)_{n-m}(a)_m(b)_m z^n}{(c)_m(-1)^m n!m!} \quad \left(\because (n-m)!=\frac{(-1)^m(n!)}{(-n)_m}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-n)_m(c-a-b)_n(-1)^m(a)_m(b)_m z^n}{(c)_m(-1)^m(1-c+a+b-n)_m n!m!} \quad \left(\because (a)_{n-m}=\frac{(-1)^m(a)_n}{(1-a-n)_m}\right)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{(-n)_m(a)_m(b)_m}{(c)_m(1-c+a+b-n)_m m!}\right]\frac{z^n(c-a-b)_n}{n!}$$

comparing the coefficients of like terms

$$\frac{(c-a)_n(c-b)_n}{(c)_n} =_3 F_2(-n,a,b;c,1-c+a+b-n;1)(c-a-b)_n$$

$$\Rightarrow \qquad {}_3F_2(-n,a,b;c,1-c+a+b-n;1)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Theorem 2.2.16. *If n is non negative integer and if a and b are independent of n, then*

$${}_3F_2(-n,a+n,\frac{1}{2}+\frac{1}{2}a-b;1+a-b,\frac{1}{2}a+\frac{1}{2};1)=\frac{(b)_n}{(1+a-b)_n}.$$

Proof:-*Since*

$$F\left(a,b;1+a-b;z\right)=(1-z)^{-a}F\left(\frac{1}{2}a,\frac{1}{2}+\frac{1}{2}a-b;1+a-b;\frac{-4z}{(1-z)^2}\right).$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(1+a-b)_n n!} = \sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}a-b)_m}{(1+a-b)_m m!}\frac{(-4z)^m}{(1-z)^{2m+a}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m 2^{2m} z^{m+n}(a+2m)_n(\frac{a}{2})_m(\frac{a+1}{2}-b)_m}{(1+a-b)_m m!n!} \quad \left(\because (1-y)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n y^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m 2^{2m} z^{m+n}(a)_{2m+n}(\frac{a}{2})_m(\frac{a+1}{2}-b)_m}{(1+a-b)_m(a)_{2m}m!n!} \quad \left(\because (a)_m(a+m)_n=(a)_{m+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m 2^{2m} z^{n}(a)_{m+n}(\frac{a}{2})_m(\frac{a+1}{2}-b)_m}{(1+a-b)_m(a)_{2m}m!(n-m)!} \quad \left(\because \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m 2^{2m} z^n (a)_{m+n} (\frac{a}{2})_m (\frac{a+1}{2}-b)_m}{(1+a-b)_m m! 2^{2m} (\frac{a}{2})_m (\frac{a+1}{2})_m (n-m)!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{z^n (a)_n (a+n)_m (\frac{a+1}{2}-b)_m (-n)_m}{(1+a-b)_m m! (\frac{a+1}{2})_m n!} \quad \left(\because (n-m)! = \frac{(-1)^m (n!)}{(-n)_m} \right)$$

$$= \sum_{n=0}^{\infty} {}_3F_2(-n, a+n, \frac{1}{2}+\frac{1}{2}a-b; 1+a-b, \frac{1}{2}a+\frac{1}{2}; 1) \frac{z^n (a)_n}{n!}$$

comparing the coefficients of like terms

$$\frac{(b)_n}{(1+a-b)_n} = {}_3F_2(-n, a+n, \frac{1}{2}+\frac{1}{2}a-b; 1+a-b, \frac{1}{2}a+\frac{1}{2}; 1).$$

Whipple's theorem

Theorem 2.2.17. *If neither $(a-b), (a-c)$ nor a is a negative integer, then*

$$F(a,b,c;1+a-b,1+a-c;x) = (1-x)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, 1+a-b-c; 1+a-b, 1+a-c; \frac{-4x}{(1-x)^2}\right).$$

Proof:-

$$\begin{aligned} L.H.S &= F(a,b,c;1+a-b,1+a-c;x) \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(1+a-b)_m (1+a-c)_m} \frac{x^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m \Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b+m)\Gamma(1+a-c+m)} \frac{x^m}{m!} \end{aligned}$$

multiplying and dividing by $\Gamma(1+a)\Gamma(1+a-b-c)$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m \Gamma(1+a+2m)\Gamma(1+a-b-c)}{\Gamma(1+a-b+m)\Gamma(1+a-c+m)(1+a)_{2m}} \frac{x^m}{m!}$$

since

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

therefore,

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{m=0}^{\infty} F(c+m, b+m; 1+a+2m; 1) \frac{(a)_m (b)_m (c)_m}{(1+a)_{2m}} \frac{x^m}{m!}$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(c+m)_n (b+m)_n}{(1+a+2m)_n n!} \right] \frac{(a)_m (b)_m (c)_m}{(1+a)_{2m}} \frac{x^m}{m!}$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_{m+n}(c)_{m+n}}{(1+a)_{2m+n}} \frac{x^m}{m!n!} \quad \left(\because (a)_m(a+m)_n = (a)_{m+n} \right)$$

$$\begin{aligned}
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_m(b)_n(c)_n}{(1+a)_{m+n}} \frac{x^m}{m!(n-m)!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_m(b)_n(c)_n(-n)_m(-1)^m}{(1+a+n)_m(1+a)_n n!} \frac{x^m}{m!} \quad \left(\because (n-m)! = \frac{(-1)^m n!}{(-n)_m} \right) \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(a)_m(-n)_m(-x)^m}{(1+a+n)_m m!} \right] \frac{(b)_n(c)_n}{(1+a)_n n!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} F(-n,a;1+a+n;-x) \frac{(b)_n(c)_n}{(1+a)_n n!}
\end{aligned}$$

since

$$F(a,b;1+a-b;z) = (1+z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; 1+a-b; \frac{4z}{(1+z)^2} \right).$$

put $z = -x, \qquad b = -n$

$$\Rightarrow F(a,-n;1+a+n;-x) = (1-x)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; 1+a+n; \frac{-4x}{(1-x)^2} \right)$$

putting this value in the second last equation gives,

$$\begin{aligned}
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} (1-x)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; 1+a+n; \frac{-4x}{(1-x)^2} \right) \frac{(b)_n(c)_n}{(1+a)_n n!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m(\frac{1}{2}a+\frac{1}{2})_m(-1)^m 2^{2m} x^m}{(1+a+n)_m(1+a)_n(1-x)^{a+2m}} \frac{(b)_n(c)_n}{n!m!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m(\frac{1}{2}a+\frac{1}{2})_m(-1)^m 2^{2m} x^m}{(1+a)_{m+n}(1-x)^{a+2m}} \frac{(b)_n(c)_n}{n!m!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{2m}(-1)^m x^m (b)_n(c)_n}{(1+a)_{m+n}(1-x)^{a+2m} m!n!} \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{n!(1+a+m)_n} \right] \frac{(a)_{2m}(-1)^m x^m}{m!(1+a)_m(1-x)^{a+2m}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)}\sum_{m=0}^{\infty}F(b,c;1+a+m;1)\frac{(a)_{2m}(-1)^m x^m}{m!(1+a)_m(1-x)^{a+2m}}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)}\sum_{m=0}^{\infty}\frac{\Gamma(1+a+m)\Gamma(1+a+m-b-c)}{\Gamma(1+a+m-b)\Gamma(1+a+m-c)}\frac{(a)_{2m}(-1)^m x^m}{m!(1+a)_m(1-x)^{a+2m}}\\
&= (1-x)^{-a}\sum_{m=0}^{\infty}\frac{(-1)^m(a)_{2m}(1+a-b-c)_m(1+a)_m x^m}{m!(1+a-b)_m(1+a-c)_m(1-x)^{2m}(1+a)_m}\\
&= (1-x)^{-a}F\left(\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},1+a-b-c;1+a-b,1+a-c;\frac{-4x}{(1-x)^2}\right).
\end{aligned}
$$

Dixon's theorem

Theorem 2.2.18.

$$F(a,b,c;1+a-b,1+a-c;1)=\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.$$

Proof:-

$$
\begin{aligned}
L.H.S &= F(a,b,c;1+a-b,1+a-c;1)\\
&= \sum_{m=0}^{\infty}\frac{(a)_m(b)_m(c)_m}{(1+a-b)_m(1+a-c)_m m!}\\
&= \sum_{m=0}^{\infty}\frac{(a)_m(b)_m(c)_m\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b+m)\Gamma\Gamma(1+a-c+m)m!}
\end{aligned}
$$

$$= \Gamma(1+a-b)\Gamma(1+a-c)\sum_{m=0}^{\infty}\frac{(a)_m(b)_m(c)_m}{\Gamma(1+a-b+m)\Gamma\Gamma(1+a-c+m)m!}$$

using

$$(1+a)_{2m}=\frac{\Gamma(1+a+2m)}{\Gamma(1+a)}$$

and multiplying and dividing by $\Gamma(1+a-b-c)(1+a)_{2m}$ *in above equation*

$$
\begin{aligned}
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{m=0}^{\infty}\frac{(a)_m(b)_m(c)_m\Gamma(1+a+2m)\Gamma(1+a-b-c)}{\Gamma(1+a-b+m)\Gamma(1+a-c+m)m!(1+a)_{2m}}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{m=0}^{\infty}\frac{(a)_m(b)_m(c)_m}{m!(1+a)_{2m}}F(c+m,b+m;1+a+2m;1)\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c+m)_n(b+m)_n(a)_m(b)_m(c)_m}{(1+a+2m)_n m!(1+a)_{2m}n!}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c)_{m+n}(b)_{m+n}(a)_m}{m!(1+a)_{2m+n}n!}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(c)_n(b)_n(a)_m}{m!(1+a)_{m+n}(n-m)!}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(c)_n(b)_n(a)_m(-1)^m(-n)_m}{m!(1+a+n)_m(1+a)_n n!}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{n=0}^{\infty}F(a,-n;1+a+n;-1)\frac{(b)_n(c)_n}{(1+a)_n n!}
\end{aligned}
$$

since

$$F(a,b;1+a+b;-1)=\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+a)}$$

put $b=-n$

$$F(a,-n;1+a-n;-1)=\frac{\Gamma(1+a+n)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}+n)\Gamma(1+a)}=\frac{(1+a)_n}{(1+\frac{a}{2})_n}$$

using this value in the second last equation, we get

$$
\begin{aligned}
F(a,b,c;1+a-b,1+a-c;1) &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\sum_{n=0}^{\infty}\frac{(1+a)_n}{(1+\frac{a}{2})_n}\frac{(b)_n(c)_n}{(1+a)_n n!}\\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}F(b,c;1+\frac{a}{2};1)\\
&= \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.
\end{aligned}
$$

Theorem 2.2.19. *If* $Re(\alpha)>0, Re(\beta)>0$, *and if* m *and* s *are positive integers, then inside the region of convergence of the resultant series*

$$
\begin{aligned}
\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_pF_q\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;cx^m(t-x)^s\right)dx = B(\alpha,\beta)t^{\alpha+\beta-1}{}_{p+m+s}F_{q+m+s}&\\
\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+1}{m},...,\frac{\alpha+m-1}{m},\frac{\beta}{s},\frac{\beta+1}{s},...,\frac{\beta+s-1}{s}\right.&\\
\left.;b_1,...,b_q,\frac{\alpha+\beta}{m+s},\frac{\alpha+\beta+1}{m+s},...\frac{\alpha+\beta+m+s-1}{m+s};\frac{m^m s^s c t^{m+s}}{m+s^{m+s}}\right).&
\end{aligned}
$$

Proof:-

$$L.H.S = \int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_pF_q\left(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; cx^m(t-x)^s\right)dx$$

Let $x = tv \Rightarrow \quad \frac{dx}{dv} = t$ *when* $x = 0 \Rightarrow \quad v = 0$ *when* $x = t \Rightarrow \quad v = 1$

$$= \int_0^1 tv^{\alpha-1}(t-tv)^{\beta-1}{}_pF_q\left(a_1, a_2, ...a_p; b_1, b_2, ..., b_q; ct^{m+s}v^m(1-v)^s\right)tdv$$

$$= \int_0^1 t^{\alpha-1}v^{\alpha-1}t^{\beta-1}(1-v)^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^nt^{n(m+s)}v^{mn}(1-v)^{sn}}{n!}tdv$$

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^nt^{n(m+s)}}{n!}\int_0^1 v^{\alpha+mn-1}(1-v)^{\beta+sn-1}dv$$

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^nt^{n(m+s)}}{n!}B(\alpha+mn, \beta+sn) \quad \left(\because B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt\right)$$

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^nt^{n(m+s)}}{n!}\frac{\Gamma(\alpha+mn)\Gamma(\beta+sn)}{\Gamma(\alpha+mn+\beta+sn)}$$

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^nt^{n(m+s)}}{n!}\frac{(\alpha)_{mn}(\beta)_{sn}\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)_{(m+s)n}\Gamma(\alpha+\beta)}$$

since

$$(\alpha)_{mn} = m^{nm}(\frac{\alpha}{m})_n(\frac{\alpha+1}{m})_n...(\frac{\alpha+m-1}{m})_n = m^{nm}\prod_{i=1}^{m}(\frac{\alpha+i-1}{m}).$$

therefore,

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n, ..., (a_p)_n}{(b_1)_n, ..., (b_q)_n}\frac{c^nt^{n(m+s)}s^{ns}m^{nm}}{n!}\frac{(\frac{\alpha}{m})_n(\frac{\alpha+1}{m})_n, ..., (\frac{\alpha+m-1}{m}{}_n(\frac{\beta}{s})_n(\frac{\beta+1}{s})_n, ..., (\frac{\beta+s-1}{s})_n}{(m+s)^{n(m+s)}(\frac{\alpha+\beta}{m+s})_n(\frac{\alpha+\beta+1}{m+s})_n, ..., (\frac{\alpha+\beta+m+s-1}{m+s})_n}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= B(\alpha,\beta)t^{\alpha+\beta-1}{}_{p+m+s}F_{q+m+s}\left(a_1, a_2, ..., a_p, \frac{\alpha}{m}, \frac{\alpha+1}{m}, ..., \frac{\alpha+m-1}{m}, \frac{\beta}{s}, \frac{\beta+1}{s}, ..., \frac{\beta+s-1}{s}\right.$$
$$\left.; b_1, ..., b_q, \frac{\alpha+\beta}{m+s}, \frac{\alpha+\beta+1}{m+s}, ..., \frac{\alpha+\beta+m+s-1}{m+s}; \frac{m^ms^sct^{m+s}}{m+s^{m+s}}\right)$$

Theorem 2.2.20. *If $Re(\alpha) > 0, Re(\beta) > 0$, and if m and s are positive integers, then inside the region of convergence of the resultant series*

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_pF_q\left(a_1, a_2, ...a_p; b_1, b_2, ..., b_q; cx^m\right)dx = B(\alpha,\beta)t^{\alpha+\beta-1}{}_{p+m}F_{q+m}$$

$$\left(a_1, a_2, ..., a_p, \frac{\alpha}{m}, \frac{\alpha+1}{m}, ..., \frac{\alpha+m-1}{m}; b_1, ..., b_q, \frac{\alpha+\beta}{m}, \frac{\alpha+\beta+1}{m}, ..., \frac{\alpha+\beta+m-1}{m}; ct^m\right).$$

Proof:-

$$L.H.S = \int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_pF_q\left(a_1, a_2, ...a_p; b_1, b_2, ...b_q; cx^m\right)dx$$

Let $x = tv \Rightarrow \quad \frac{dx}{dv} = t$ *when* $x = 0 \Rightarrow \quad v = 0$ *when* $x = t \Rightarrow \quad v = 1$

$$\begin{aligned}
&= \int_0^1 tv^{\alpha-1}(t-tv)^{\beta-1}{}_pF_q\left(a_1, a_2, ...a_p; b_1, b_2, ...b_q; ct^m v^m\right)tdv \\
&= \int_0^1 t^{\alpha-1}v^{\alpha-1}t^{\beta-1}(1-v)^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^n t^{mn}v^{mn}}{n!}tdv \\
&= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^n t^{mn}}{n!}\int_0^1 v^{\alpha+mn-1}(1-v)^{\beta-1}dv
\end{aligned}$$

$$\begin{aligned}
&= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^n t^{mn}}{n!}B(\alpha+mn,\beta) \quad \left(\because B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt\right) \\
&= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^n t^{mn}}{n!}\frac{\Gamma(\alpha+mn)\Gamma(\beta)}{\Gamma(\alpha+mn+\beta)} \\
&= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{c^n t^{mn}}{n!}\frac{(\alpha)_{mn}\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)_{mn}\Gamma(\alpha+\beta)}
\end{aligned}$$

since

$$(\alpha)_{mn} = m^{nm}(\frac{\alpha}{m})_n(\frac{\alpha+1}{m})_n...(\frac{\alpha+m-1}{m})_n = m^{nm}\prod_{i=1}^{m}(\frac{\alpha+i-1}{m})$$

therefore,

$$= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{(a_1)_n, ..., (a_p)_n}{(b_1)_n, ..., (b_q)_n}\frac{c^n t^{mn}m^{nm}}{n!}\frac{(\frac{\alpha}{m})_n(\frac{\alpha+1}{m})_n, ..., (\frac{\alpha+m-1}{m})_n}{m^{nm}(\frac{\alpha+\beta}{m})_n(\frac{\alpha+\beta+1}{m})_n, ..., (\frac{\alpha+\beta+m-1}{m})_n}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= B(\alpha,\beta)t^{\alpha+\beta-1}{}_{p+m}F_{q+m}\left(a_1,a_2,...a_p,\frac{\alpha}{m},\frac{\alpha+1}{m},...\frac{\alpha+m-1}{m}\right.$$

$$\left.;b_1,...b_q,\frac{\alpha+\beta}{m},\frac{\alpha+\beta+1}{m},...\frac{\alpha+\beta+m-1}{m};ct^m\right).$$

Problem:- show that

$$\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}\left[1-x^2(t-x)^2\right]^{\frac{-1}{2}}dx=\frac{1}{2}\pi tF\left(\frac{1}{4},\frac{3}{4};1;\frac{t^4}{16}\right).$$

Solution:-

$$L.H.S=\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}\left[1-x^2(t-x)^2\right]^{\frac{-1}{2}}dx$$

since

$$(1-y)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n y^n}{n!}.$$

$$\therefore\qquad \left[1-x^2(t-x)^2\right]^{\frac{-1}{2}}=\sum_{n=0}^{\infty}(\frac{1}{2})_n\left(\frac{x^2(t-x)^2}{n!}\right)^n={}_1F_0\left(\frac{1}{2};-;x^2(t-x)^2\right)$$

so,

$$L.H.S=\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}{}_1F_0\left(\frac{1}{2};-;x^2(t-x)^2\right)dx$$

now using the following result in above equation

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_pF_q\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;cx^m(t-x)^s\right)dx=B(\alpha,\beta)t^{\alpha+\beta-1}{}_{p+m+s}F_{q+m+s}$$

$$\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+1}{m},...,\frac{\alpha+m-1}{m},\frac{\beta}{s},\frac{\beta+1}{s},...,\frac{\beta+s-1}{s}\right.$$

$$\left.;b_1,...,b_q,\frac{\alpha+\beta}{m+s},\frac{\alpha+\beta+1}{m+s},...\frac{\alpha+\beta+m+s-1}{m+s};\frac{m^m s^s ct^{m+s}}{m+s^{m+s}}\right).$$

and comparing L.H.S of result with the above equation L.H.S $\alpha-1=\frac{1}{2}\quad\Rightarrow\quad\alpha=\frac{3}{2}$,

$\beta-1=\frac{-1}{2}\quad\Rightarrow\beta=\frac{1}{2}$ $p=1,q=0,k=2,s=2,c=1$ so,

$$\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}{}_1F_0\left(\frac{1}{2};-;x^2(t-x)^2\right)dx = B\left(\frac{3}{2},\frac{1}{2}\right)t^{\frac{3}{2}+\frac{1}{2}-1}{}_5F_4$$

$$\left(\frac{1}{2},\frac{\frac{3}{2}}{2},\frac{\frac{3}{2}+1}{2},\frac{\frac{1}{2}}{2},\frac{\frac{1}{2}+1}{2};-,\frac{\frac{3}{2}+\frac{1}{2}}{4},\frac{\frac{3}{2}+\frac{1}{2}+1}{4},\frac{\frac{3}{2}+\frac{1}{2}+2}{4},\frac{\frac{3}{2}+\frac{1}{2}+3}{4};\frac{2^2 2^2 t^4}{(2+2)^4}\right)$$

$$= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}+\frac{1}{2})}t{}_5F_4\left(\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{1}{4},\frac{3}{4};-,\frac{1}{2},\frac{3}{4},1,\frac{5}{4};\frac{t^4}{16}\right)$$

$$= \frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{2})}{\Gamma(2)}t\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{3}{4})_n(\frac{5}{4})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(\frac{1}{2})_n(\frac{3}{4})_n(1)_n(\frac{5}{4})_n n!}(\frac{t^4}{16})^n$$

$$= \frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})t\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{3}{4})_n}{(1)_n n!}(\frac{t^4}{16})^n \quad \left(\because \Gamma(2)=1, \Gamma(1+z)=z\Gamma(z)\right)$$

$$= \frac{1}{2}\sqrt{\pi}\sqrt{\pi}tF\left(\frac{1}{4},\frac{3}{4};1;\frac{t^4}{16}\right)$$

$$= \frac{1}{2}\pi tF\left(\frac{1}{4},\frac{3}{4};1;\frac{t^4}{16}\right)$$

Theorem 2.2.21.

$$\frac{d^n}{dz^n}{}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right) = \frac{\prod_{m=1}^{p}(a_m)_n}{\prod_{j=1}^{q}(b_j)_n}{}_pF_q\left(a_1+n,...,a_p+n;b_1+n,...,b_q+n;z\right).$$

Proof:-

$$L.H.S = \frac{d^n}{dz^n}{}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{d^n}{dz^n}\frac{z^n}{n!}$$

$$= \sum_{n=n}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^{n-n}n!}{n!}$$

$$= \sum_{n=n}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^{n-n}}{(n-n)!}$$

replace n by n+n

$$= \sum_{n=0}^{\infty} \frac{(a_1)_{n+n}...(a_p)_{n+n}}{(b_1)_{n+n}...(b_q)_{n+n}} \frac{z^n}{n!}$$

since

$$(a_1)_{n+n} = (a_1)_n (a_1+n)_n$$

therefore,

$$= \frac{\prod_{m=1}^{p} (a_m)_n}{\prod_{j=1}^{q} (b_j)_n} {}_pF_q\left(a_1+n, ..., a_p+n; b_1+n, ..., b_q+n; z\right).$$

Theorem 2.2.22.

$${}_0F_1\left(-; a; x\right) {}_0F_1\left(-; b; x\right) = {}_2F_3\left(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}; a, b, a+b-1; 4x\right).$$

Proof:-

$$\begin{aligned} L.H.S &= {}_0F_1\left(-; a; x\right) {}_0F_1\left(-; b; x\right) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(a)_n n!} \sum_{m=0}^{\infty} \frac{x^m}{(b)_m m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^{m+n}}{(a)_n (b)_m m! n!} \end{aligned}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^n}{(a)_{n-m} (b)_m m! (n-m)!} \quad \left(\because \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m, n-m) \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^n (1-a-n)_m (-n)_m}{(a)_n (b)_m (-1)^m m! (-1)^m n!} \quad \left(\because (a)_{n-m} = \frac{(-1)^m (a)_n}{(1-a-n)_m} \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^n (1-a-n)_m (-n)_m}{(a)_n (b)_m m! n!} \quad \left(\because (n-m)! = \frac{(-1)^m (n!)}{(-n)_m} \right) \\ &= \sum_{n=0}^{\infty} F\left(-n, 1-a-n; b; 1\right) \frac{x^n}{(a)_n n!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(b+a-1)_{2n}}{(b)_n(b+a-1)_n} \frac{x^n}{(a)_n n!} \quad \left(\because F(-n, 1-a-n; b; 1) = \frac{(b+a-1)_{2n}}{(b)_n(b+a-1)_n} \right) \\
&= \sum_{n=0}^{\infty} \frac{2^{2n}(\frac{b+a-1}{2})_n(\frac{b+a}{2})_n x^n}{(b)_n(b+a-1)_n(a)_n n!} \quad \left(\because (a)_{2m} = 2^{2m}(\frac{a}{2})_m(\frac{a+1}{2})_m \right) \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a+\frac{1}{2}b)_n(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2})_n}{(b)_n(b+a-1)_n(a)_n} \frac{(4x)^n}{n!} \\
&= {}_2F_3\left(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}; a, b, a+b-1; 4x \right)
\end{aligned}
$$

Theorem 2.2.23.

$$\frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)} = \frac{cos\frac{1}{2}\pi a\Gamma(1-a)}{\Gamma(1-\frac{1}{2}a)}.$$

and that

$$\frac{\Gamma(1+a-b)}{\Gamma(1+\frac{1}{2}a-b)} = \frac{sin\pi(b-\frac{1}{2}a)\Gamma(b-\frac{1}{2}a)}{sin\pi(b-a)\Gamma(b-a)}.$$

and

$$
\begin{aligned}
F(a, b, c; 1+a-b, 1+a-c; 1) &= \frac{cos\frac{1}{2}\pi a sin\pi(b-\frac{1}{2}a)}{sin\pi(b-a)} \\
&\times \frac{\Gamma(1-a)\Gamma(b-\frac{1}{2}a)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1-\frac{1}{2}a)\Gamma(b-a)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.
\end{aligned}
$$

Proof:-*Firstly we show that*

$$\frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)} = \frac{cos\frac{1}{2}\pi a\Gamma(1-a)}{\Gamma(1-\frac{1}{2}a)}$$

$$
\begin{aligned}
R.H.S &= \frac{cos\frac{1}{2}\pi a\Gamma(1-a)}{\Gamma(1-\frac{1}{2}a)} \\
&= \frac{cos\frac{1}{2}\pi a\Gamma(1-a)\Gamma(a)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(1-\frac{1}{2}a)\Gamma(\frac{a}{2})} \\
&= \frac{cos\frac{\pi a}{2}\frac{\pi}{sin\pi a}\Gamma(\frac{a}{2})}{\Gamma(a)\frac{\pi}{sin\frac{\pi a}{2}}} \quad \left(\because \Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z} \right) \\
&= \frac{\Gamma(\frac{a}{2})2sin\frac{\pi a}{2}cos\frac{\pi a}{2}}{\Gamma(a)2sin\pi a} \\
&= \frac{\Gamma(\frac{a}{2})sin\frac{2\pi a}{2}}{\Gamma(a)2sin\pi a} \quad \left(\because 2sinacosa = sin2a \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\Gamma(1+\frac{a}{2})}{\frac{a}{2}}\frac{a}{\Gamma(1+a)} \quad \left(\because \Gamma(1+z) = z\Gamma(z)\right) \\
&= \frac{\Gamma(1+\frac{a}{2})}{\Gamma(1+a)} \\
&= L.H.S
\end{aligned}
$$

Now we show

$$
\frac{\Gamma(1+a-b)}{\Gamma(1+\frac{1}{2}a-b)} = \frac{sin\pi(b-\frac{1}{2}a)\Gamma(b-\frac{1}{2}a)}{sin\pi(b-a)\Gamma(b-a)}
$$

$$
\begin{aligned}
L.H.S &= \frac{sin\pi(b-\frac{1}{2}a)\Gamma(b-\frac{1}{2}a)}{sin\pi(b-a)\Gamma(b-a)} \\
&= \frac{\frac{\pi}{sin\pi(b-a)}}{\frac{\pi}{sin\pi(b-\frac{a}{2})}}\frac{\Gamma(b-\frac{a}{2})}{\Gamma(b-a)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(b-a)\Gamma(1-b+a)}{\Gamma(b-\frac{a}{2})\Gamma(1-b+\frac{a}{2})}\frac{\Gamma(b-\frac{a}{2})}{\Gamma(b-a)} \quad \left(\because \Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z}\right) \\
&= \frac{\Gamma(1+a-b)}{\Gamma(1+\frac{a}{2}-b)} \\
&= R.H.S
\end{aligned}
$$

Dixon's theorem is,

$$
F(a,b,c;1+a-b,1+a-c;1) = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.
$$

put

$$
\frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)} = \frac{cos\frac{1}{2}\pi a\Gamma(1-a)}{\Gamma(1-\frac{1}{2}a)}
$$

and

$$
\frac{\Gamma(1+a-b)}{\Gamma(1+\frac{1}{2}a-b)} = \frac{sin\pi(b-\frac{1}{2}a)\Gamma(b-\frac{1}{2}a)}{sin\pi(b-a)\Gamma(b-a)}
$$

in Dixon's theorem gives,

$$
\begin{aligned}
F(a,b,c;1+a-b,1+a-c;1) &= \frac{cos\frac{1}{2}\pi a sin\pi(b-\frac{1}{2}a)}{sin\pi(b-a)} \\
&\times \frac{\Gamma(1-a)\Gamma(b-\frac{1}{2}a)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1-\frac{1}{2}a)\Gamma(b-a)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.
\end{aligned}
$$

Theorem 2.2.24.

$$F(-2n,\alpha,1-\beta-2n;1-\alpha-2n,\beta;1)=\frac{(2n)!(\alpha)_n(\beta-\alpha)_n}{n!(\alpha)_{2n}(\beta)_n}.$$

Proof:-*Since*

$$F(a,b,c;1+a-b,1+a-c;1)=\frac{cos\frac{1}{2}\pi asin\pi(b-\frac{1}{2}a)}{sin\pi(b-a)}$$
$$\times\frac{\Gamma(1-a)\Gamma(b-\frac{1}{2}a)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1-\frac{1}{2}a)\Gamma(b-a)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.$$

Put

$$a=-2n,\quad b=\alpha,\quad c=1-\beta-2n$$

in above equation gives,

$$F(-2n,\alpha,1-\beta-2n;1-\alpha-2n,\beta;1)$$
$$=\frac{cos(-\pi n)sin\pi(\alpha+n)\Gamma(1+2n)\Gamma(\alpha+n)\Gamma(\beta)\Gamma(\beta-\alpha+n)}{sin\pi(\alpha+2n)\Gamma(1+n)\Gamma(\alpha+2n)\Gamma(\beta+n)\Gamma(\beta-\alpha)}$$

$$=\frac{(-1)^n sin\pi(\alpha+n)(2n)!(\alpha)_n\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta-\alpha+n)(\beta-\alpha)_n}{sin\pi(\alpha+2n)n!(\alpha)_{2n}\Gamma(\alpha)(\beta)_n\Gamma(\beta)\Gamma(\beta-\alpha+n)}\quad\left(\because(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\right)$$
$$=\frac{(-1)^n sin\pi(\alpha+n)(2n)!(\alpha)_n(\beta-\alpha)_n}{sin\pi(\alpha+2n)n!(\alpha)_{2n}(\beta)_n}\quad\left(\because\Gamma(n)=(n-1)!,n\in\mathbb{N}\right)$$
$$=\frac{(-1)^n\frac{\pi}{sin\pi(\alpha+2n)}(2n)!(\alpha)_n(\beta-\alpha)_n}{\frac{\pi}{sin\pi(\alpha+n)}n!(\alpha)_{2n}(\beta)_n}$$

$$=\frac{(-1)^n\Gamma(\alpha+2n)\Gamma(1-\alpha-2n)(2n)!(\alpha)_n(\beta-\alpha)_n}{\Gamma(\alpha+n)\Gamma(1-\alpha-n)n!(\alpha)_{2n}(\beta)_n}\quad\left(\because\Gamma(z)\Gamma(1-z)=\frac{\pi}{sin\pi z}\right)$$
$$=\frac{(-1)^n(\alpha)_{2n}\Gamma(\alpha)\Gamma(1-\alpha-2n)(2n)!(\alpha)_n(\beta-\alpha)_n}{(\alpha)_n\Gamma(\alpha)\Gamma(1-\alpha-n)n!(\alpha)_{2n}(\beta)_n}$$

$$=\frac{(-1)^n(\alpha)_{2n}\Gamma(\alpha)(\alpha)_n(-1)^{2n}\Gamma(1-\alpha)(2n)!(\alpha)_n(\beta-\alpha)_n}{(\alpha)_n\Gamma(\alpha)\Gamma(1-\alpha)(-1)^n(\alpha)_{2n}n!(\alpha)_{2n}(\beta)_n}\quad\left(\because\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)}=\frac{(-1)^n}{(\alpha)_n}\right)$$
$$=\frac{(2n)!(\alpha)_n(\beta-\alpha)_n}{n!(\alpha)_{2n}(\beta)_n}.$$

Ramanujan's theorem

Theorem 2.2.25.

$$F(\alpha;\beta;x)F(\alpha;\beta;-x) = F(\alpha,\beta-\alpha;\beta,\frac{1}{2}\beta,\frac{1}{2}\beta+\frac{1}{2};\frac{x^2}{4}).$$

Proof:-

$$\begin{aligned} F(\alpha;\beta;x)F(\alpha;\beta;-x) &= \sum_{n=0}^{\infty}\frac{(\alpha)_n x^n}{(\beta)_n n!}\sum_{m=0}^{\infty}\frac{(\alpha)_m(-x)^m}{(\beta)_m m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_{n-m}(\alpha)_m(-1)^m x^n}{(\beta)_{n-m}(\beta)_m(n-m)!m!} \end{aligned}$$

since

$$(n-m)! = \frac{(-1)^m n!}{(-n)_m} \quad and \quad (a)_{n-m} = \frac{(-1)^m(a)_n}{(1-a-n)_m}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(\alpha)_m(\alpha)_n(-1)^m(1-\beta-n)_m(-n)_m x^n}{(1-\alpha-n)_m(\beta)_m(-1)^m(\beta)_n(-1)^m n!m!} \\ &= \sum_{n=0}^{\infty}F(-n,\alpha,1-\beta-n;\beta,1-\alpha-n;1)\frac{(\alpha)_n x^n}{(\beta)_n n!} \end{aligned}$$

since

$$F(-2n,\alpha,1-\beta-2n;1-\alpha-2n,\beta;1) = \frac{(2n)!(\alpha)_n(\beta-\alpha)_n}{n!(\alpha)_{2n}(\beta)_n}.$$

using this value in the above equation gives,

$$= \sum_{\frac{n}{2}=0}^{\infty}\frac{n!(\alpha)_{\frac{n}{2}}(\beta-\alpha)_{\frac{n}{2}}}{(\frac{n}{2})!(\alpha)_n(\beta)_{\frac{n}{2}}}\frac{(\alpha)_n x^n}{(\beta)_n n!}$$

$$= \sum_{\frac{n}{2}=0}^{\infty}\frac{(\alpha)_{\frac{n}{2}}(\beta-\alpha)_{\frac{n}{2}}}{(\frac{n}{2})!(\beta)_{\frac{n}{2}}}\frac{x^n}{(\beta)_n}$$

since

$$(\beta)_{2n} = 2^{2n}(\frac{\beta}{2})_n(\frac{\beta+1}{2})_n$$

replace n by $\frac{n}{2}$

$$(\beta)_{2(\frac{n}{2})} = 2^{2(\frac{n}{2})}(\frac{\beta}{2})_{\frac{n}{2}}(\frac{\beta+1}{2})_{\frac{n}{2}} = 4^{\frac{n}{2}}(\frac{\beta}{2})_{\frac{n}{2}}(\frac{\beta+1}{2})_{\frac{n}{2}}$$

therefore the above equation becomes,

$$= \sum_{\frac{n}{2}=0}^{\infty} \frac{(\alpha)_{\frac{n}{2}}(\beta-\alpha)_{\frac{n}{2}}(x^2)^{\frac{n}{2}}}{(\frac{n}{2})!(\beta)_{\frac{n}{2}}4^{\frac{n}{2}}(\frac{\beta}{2})_{\frac{n}{2}}(\frac{\beta+1}{2})_{\frac{n}{2}}}$$

$$= \sum_{\frac{n}{2}=0}^{\infty} \frac{(\alpha)_{\frac{n}{2}}(\beta-\alpha)_{\frac{n}{2}}(\frac{x^2}{4})^{\frac{n}{2}}}{(\frac{n}{2})!(\beta)_{\frac{n}{2}}(\frac{\beta}{2})_{\frac{n}{2}}(\frac{\beta+1}{2})_{\frac{n}{2}}}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta-\alpha)_r(\frac{x^2}{4})^r}{r!(\beta)_r(\frac{\beta}{2})_r(\frac{\beta+1}{2})_r}$$

$$= F(\alpha, \beta-\alpha; \beta, \frac{1}{2}\beta, \frac{1}{2}\beta+\frac{1}{2}; \frac{x^2}{4}).$$

Problem:-Let

$$\Upsilon_n = F(-n, 1-a-n, 1-b-n; a, b; 1)$$

show that

$$\Upsilon_{2n} = \frac{(-1)^n(2n)!(a+b-1)_{3n}}{n!(a)_n(b)_n(a+b-1)_{2n}}.$$

Solution:-

$$\Upsilon_{2n} = F(-2n, 1-a-2n, 1-b-2n; a, b; 1)$$

Since

$$F(a,b,c;1+a-b,1+a-c;1) = \frac{cos\frac{1}{2}\pi a sin\pi(b-\frac{1}{2}a)}{sin\pi(b-a)}$$
$$\times \frac{\Gamma(1-a)\Gamma(b-\frac{1}{2}a)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1-\frac{1}{2}a)\Gamma(b-a)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.$$

Put

$$a = -2n, \quad b = 1-a-2n, \quad and \quad c = 1-b-2n$$

in the above equation gives,

$$F(-2n, 1-a-2n, 1-b-2n; a, b; 1) = \frac{cos\frac{\pi}{2}(-2n)sin\pi(1-a-2n+n)}{sin\pi(1-a-2n+2n)}$$
$$\frac{\Gamma(1+2n)\Gamma(1-a-2n+n)\Gamma(1-2n-1+b+2n)\Gamma(1-n-1+a+2n-1+b+2n)}{\Gamma(1+n)\Gamma(1-a-2n+2n)\Gamma(1-n-1+b+2n)\Gamma(1-2n-1+a+2n-1+b+2n)}$$

$$\begin{aligned}
\Upsilon_{2n} &= \frac{cosn\pi sin\pi(1-a-n)(2n)!\Gamma(1-a-n)\Gamma(b)\Gamma(a+b-1+3n)\Gamma(a+b-1)}{n!sin\pi(1-a)\Gamma(1-a)\Gamma(b+n)\Gamma(a+b-1+2n)\Gamma(a+b-1)} \\
&= \frac{(-1)^n(2n)!(a+b-1)_{3n}\frac{\pi}{sin\pi(1-a)}\Gamma(1-a-n)}{n!(b)_n(a+b-1)_{2n}\frac{\pi}{sin\pi(1-a-n)}\Gamma(1-a)} \quad \left(\because \Gamma(n)=(n-1)!, n\in\mathbb{N}\right) \\
&= \frac{(-1)^n(2n)!(a+b-1)_{3n}\Gamma(1-a)\Gamma(1-1+a)\Gamma(1-a-n)}{n!(b)_n(a+b-1)_{2n}\Gamma(1-a-n)\Gamma(a+n)\Gamma(1-a)} \\
&= \frac{(-1)^n(2n)!(a+b-1)_{3n}\Gamma(a)}{n!(b)_n(a+b-1)_{2n}\Gamma(a+n)} \\
&= \frac{(-1)^n(2n)!(a+b-1)_{3n}}{n!(b)_n(a+b-1)_{2n}(a)_n}.
\end{aligned}$$

Theorem 2.2.26.

$$F(-;a,b;t)F(-;a,b;-t) =_3 F_8\left(\frac{1}{3}(a+b-1),\frac{1}{3}(a+b),\frac{1}{3}(a+b+1);a,b,\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},\right.$$
$$\left.\frac{1}{2}b,\frac{1}{2}b+\frac{1}{2},\frac{1}{2}(a+b-1),\frac{1}{2}(a+b);\frac{-27t^2}{64}\right).$$

Proof:-

$$\begin{aligned}
L.H.S &= F(-;a,b;t)F(-;a,b;-t) \\
&= \sum_{n=0}^{\infty}\frac{t^n}{(a)_n(b)_n n!}\sum_{m=0}^{\infty}\frac{(-t^m)}{(a)_m(b)_m m!}
\end{aligned}$$

since

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m)$$

$$\therefore \quad = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m t^n}{(a)_{n-m}(b)_{n-m}(a)_m(b)_m(n-m)!m!}$$

since

$$(n-m)! = \frac{(-1)^m(n!)}{(-n)_m} \quad and \quad (a)_{n-m} = \frac{(-1)^m(a)_n}{(1-a-n)_m}$$

$$\begin{aligned}
\therefore \qquad &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m t^n (1-a-n)_m (1-b-n)_m (-n)_m}{(-1)^m (a)_n (-1)^m (b)_n (a)_m (b)_m (-1)^m n! m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{t^n (1-a-n)_m (1-b-n)_m (-n)_m}{(a)_n (b)_n (a)_m (b)_m n! m!} \\
&= \sum_{n=0}^{\infty} F(-n, 1-a-n, 1-b-n; a, b; 1)\frac{t^n}{n!(a)_n (b)_n}
\end{aligned}$$

$$\because \qquad F(-2n, 1-a-2n, 1-b-2n; a, b; 1) = \frac{(-1)^n (2n)!(a+b-1)_{3n}}{n!(b)_n (a+b-1)_{2n}(a)_n}$$

$$\begin{aligned}
\therefore \qquad &= \sum_{n=0}^{\infty}\frac{(-1)^n (2n)!(a+b-1)_{3n}}{n!(b)_n (a+b-1)_{2n}(a)_n}\frac{t^{2n}}{(2n)!(a)_{2n}(b)_{2n}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 3^{3n}(\frac{a+b-1}{3})_n(\frac{a+b}{3})_n(\frac{a+b+1}{3})_n}{n!(a)_n (b)_n 2^{2n}(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n}\frac{(t^2)^n}{2^{2n}(\frac{1}{2}b)_n(\frac{1}{2}b+\frac{1}{2})_n 2^{2n}(\frac{a+b-1}{2})_n(\frac{a+b}{2})_n} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n (27)^n(\frac{a+b-1}{3})_n(\frac{a+b}{3})_n(\frac{a+b+1}{3})_n}{n!(a)_n (b)_n(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n}\frac{(t^2)^n}{(\frac{1}{2}b)_n(\frac{1}{2}b+\frac{1}{2})_n(\frac{a+b-1}{2})_n(\frac{a+b}{2})_n(64)^n} \\
&= \sum_{n=0}^{\infty}\frac{(\frac{a+b-1}{3})_n(\frac{a+b}{3})_n(\frac{a+b+1}{3})_n}{n!(a)_n (b)_n(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n}\frac{(-27t^2)^n}{(\frac{1}{2}b)_n(\frac{1}{2}b+\frac{1}{2})_n(\frac{a+b-1}{2})_n(\frac{a+b}{2})_n(64)^n}
\end{aligned}$$

$$=_3 F_8\left(\frac{1}{3}(a+b-1), \frac{1}{3}(a+b), \frac{1}{3}(a+b+1); a, b, \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}(a+b-1), \frac{1}{2}(a+b); \frac{-27t^2}{64}\right).$$

Chapter 3

SOME RESULTS INVOLVING HYPERGEOMETRIC k-FUNCTIONS

In this chapter, we prove some results involving gamma $k-$function and beta $k-$function and the properties of Gauss hypergeometric $k-$function. We solve some differential $k-$equations and then find the solutions of these $k-$differential equations. We use these differential equations and solutions for evaluating many results of hypergeometric $k-$functions. We prove Kummer's $k-$theorem, Saalschutz'$k-$theorem, Whipple's $k-$theorem, $k-$Dixon's theorem and Ramanujan's $k-$theorem.

Lemma 3.0.27. *If n is a positive integer, $k > 0$, then*

$$\beta_k(p,(n+1)k) = \frac{k^n n!}{(p)_{n+1,k}}.$$

Proof:-

$$\begin{aligned}
\beta_k(p, nk+k) &= \frac{\Gamma_k(p)\Gamma_k(nk+k)}{\Gamma_k(p+nk+k)} \\
&= \frac{\Gamma_k(p)\Gamma_k(nk+k)}{(p+k)_{n,k}\Gamma_k(p+k)} \quad \left(\because (p+k)_{n,k} = \frac{\Gamma_k(p+k+nk)}{\Gamma_k(p+k)}\right) \\
&= \frac{\Gamma_k(p)n!k^n}{(p+k)_{n,k}p\Gamma_k(p)} \quad \left(\because \Gamma_k(nk) = k^{n-1}(n-1)!, k>0, n\in\mathbb{N}\right) \\
&= \frac{n!k^n}{p(p+k)_{n,k}} = \frac{n!k^n}{(p)_{n+1,k}}.
\end{aligned}$$

Lemma 3.0.28.

$$\frac{1}{k}\int_{-1}^{1}(1+x)^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}dx = 2^{\frac{p+q}{k}-1}\beta_k(p,q).$$

Proof:-$Let\ y = \frac{1+x}{2}$

$$\Rightarrow x = 2y-1 \quad and \quad 1-x = 2-2y = 2(1-y)$$

so,

$$\begin{aligned}\frac{1}{k}\int_{-1}^{1}(1+x)^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}dx &= \frac{1}{k}\int_{0}^{1}2^{\frac{p}{k}-1}y^{\frac{p}{k}-1}2^{\frac{q}{k}-1}(1-y)^{\frac{q}{k}-1}2dy \\ &= 2^{\frac{p+q}{k}-1}\frac{1}{k}\int_{0}^{1}y^{\frac{p}{k}-1}(1-y)^{\frac{q}{k}-1}dy \\ &= 2^{\frac{p+q}{k}-1}B_k(p,q).\end{aligned}$$

3.0.1 Some important results of Pochhbemer k-symbol:-

(1)

$$\begin{aligned}\frac{\Gamma_k(k-\alpha-nk)}{\Gamma_k(k-\alpha)} &= \frac{\Gamma_k(k-\alpha-nk)}{(k-\alpha-k)(k-\alpha-2k)...(k-\alpha-nk)\Gamma_k(k-\alpha-nk)} \\ &= \frac{(-1)^n}{(\alpha+k-k)(\alpha+2k-k)...(\alpha+nk-k)} \\ &= \frac{(-1)^n}{\alpha(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)} = \frac{(-1)^n}{(\alpha)_{n,k}}.\end{aligned}$$

(2)

$$\begin{aligned}&(\alpha)_{m,k}(\alpha+mk)_{n,k} \\ &= (\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(m-1)k)(\alpha+mk)(\alpha+mk+k)... \\ &(\alpha+mk+(n-1)k) \\ &= (\alpha)_{m+n,k}.\end{aligned}$$

(3)

$$\begin{aligned}
\frac{(-1)^m(\alpha)_{n,k}}{(k-\alpha-nk)_{m,k}} &= \frac{(-1)^m(\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)}{(k-\alpha-nk)(k-\alpha-nk+k)...(k-\alpha-nk+(m-1)k)} \\
&= \frac{(-1)^m(\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)}{(k-\alpha-nk)(2k-\alpha-nk)...(mk-\alpha-nk)} \\
&= \frac{(\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)}{(\alpha+nk-k)(\alpha+nk-2k)...(\alpha+nk-mk)} \\
&= \frac{(\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k)}{(\alpha+nk-mk)(\alpha+nk-(m-1)k)...(\alpha+nk-k)} \\
&= (\alpha)(\alpha+k)(\alpha+2k)...(\alpha+(n-m-1)k) = (\alpha)_{n-m,k}.
\end{aligned}$$

(4)Since

$$(\alpha)_{n-m,k} = \frac{(-1)^m(\alpha)_{n,k}}{(k-\alpha-nk)_{m,k}}$$

taking $\alpha = k$

$$(k)_{n-m,k} = \frac{(-1)^m(k)_{n,k}}{(k-k-nk)_{m,k}}$$

so

$$k^{n-m}(1)_{n-m} = \frac{(-1)^m(k)^n(1)_n}{(-nk)_{m,k}} \qquad \left(\because (\alpha)_{n,k} = k^n(\frac{\alpha}{k})_n\right)$$

$$(n-m)!k^{n-m} = \frac{(-1)^m(k)^n n!}{(-nk)_{m,k}}$$

$$(n-m)! = \frac{(-1)^m(k)^n n!}{(-nk)_{m,k}k^{n-m}}$$

$$= \frac{(-1)^m(k)^m n!}{(-nk)_{m,k}}$$

Here,

$$(-nk)_{m,k} = (-nk)(-nk+k)...(-nk+(m-1)k)$$

$$= k^m(-n)(-n+1)...(-n+(m-1))$$

$$= k^m(-n)_m$$

so,

$$(n-m)! = \frac{(-1)^m (k)^m n!}{k^m (-n)_m} = \frac{(-1)^m n!}{(-n)_m}.$$

Lemma 3.0.29.

$$\Gamma_k(nk) = k^{n-1}(n-1)!, \qquad k > 0.$$

Proof:-*Since*

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{v^k}{k}} v^{x-1} dv.$$

Put $\quad \frac{v^k}{k} = t \Rightarrow \quad v = (tk)^{\frac{1}{k}} \quad$ *and* $\quad dv = \frac{1}{(tk)^{1-\frac{1}{k}}} dt$ *so,*

$$\begin{aligned} \Gamma_k(x) &= \int_0^\infty e^{-t} (tk)^{\frac{1}{k}(x-1)} \frac{1}{(tk)^{1-\frac{1}{k}}} dt \\ &= k^{\frac{x}{k}-1} \int_0^\infty e^{-t} t^{\frac{x}{k}-1} dt \\ &= k^{\frac{x}{k}-1} \Gamma(\frac{x}{k}) \end{aligned}$$

Let $x = nk, \quad n \in \mathbb{N}, k > 0,$ *then*

$$\begin{aligned} \Gamma_k(nk) &= k^{\frac{nk}{k}-1} \Gamma(\frac{nk}{k}) = k^{n-1}\Gamma(n) \\ &= k^{n-1}(n-1)!. \quad \left(\because \Gamma(n) = (n-1)! \right) \end{aligned}$$

Lemma 3.0.30.

$$\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{k sin \frac{\pi x}{k}}, \qquad k > 0.$$

Proof:-

$$\begin{aligned} \Gamma_k(x)\Gamma_k(k-x) &= k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) k^{\frac{k-x}{k}-1}\Gamma(\frac{k-x}{k}) \quad \left(\because \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) \right) \\ &= k^{-1}\Gamma(\frac{x}{k})\Gamma(1-\frac{x}{k}) = \frac{\pi}{k sin \frac{\pi x}{k}} \quad \left(\because \Gamma(x)\Gamma(1-x) = \frac{\pi}{sin \pi z} \right) \end{aligned}$$

Lemma 3.0.31.

$$\Gamma_k(\frac{k}{2}) = \sqrt{\frac{\pi}{k}}, \qquad k > 0.$$

Proof:-*Since*

$$\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{ksin\frac{\pi x}{k}}.$$

put $\quad x = \frac{k}{2}$

$$\Rightarrow \quad \Gamma_k(\frac{k}{2})\Gamma_k(k-\frac{k}{2}) = \frac{\pi}{ksin\frac{\pi}{2}}$$

$$\Gamma_k(\frac{k}{2})\Gamma_k(\frac{k}{2}) = \frac{\pi}{k}$$

$$\Gamma_k(\frac{k}{2}) = \sqrt{\frac{\pi}{k}}.$$

Problem:-Use the relation

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{ksin\frac{\pi z}{k}}, \quad k > 0.$$

and the elementary result

$$sinxsiny = \frac{1}{2}[cos(x-y) - cos(x+y)].$$

to prove that

$$1 - \frac{\Gamma_k(c)\Gamma_k(k-c)\Gamma_k(c-a-b)\Gamma_k(a+b+k-c)}{\Gamma - k(c-a)\Gamma_k(a+k-c)\Gamma_k(c-b)\Gamma_k(b+k-c)} = \frac{\Gamma_k(2k-c)\Gamma_k(c-k)\Gamma_k(c-a-b)\Gamma_k(a+b+k-c)}{\Gamma_k(a)\Gamma_k(k-a)\Gamma_k(b)\Gamma_k(k-b)}.$$

Solution:-Note that

$$k-(c-a-b) = a+b+k-c$$

$$k-(c-a) = a+k-c$$

$$k-(c-b) = b+k-c$$

$$1-\frac{\Gamma_k(c)\Gamma_k(k-c)\Gamma_k(c-a-b)\Gamma_k(a+b+k-c)}{\Gamma_k(c-a)\Gamma_k(a+k-c)\Gamma_k(c-b)\Gamma_k(b+k-c)}=1-\frac{\pi^2k^2sin\frac{\pi(c-a)}{k}sin\frac{\pi(c-b)}{k}}{\pi^2k^2sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}}$$

$$=\frac{sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}-sin\frac{\pi(c-a)}{k}sin\frac{\pi(c-b)}{k}}{sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}}$$

$$=\frac{\frac{1}{2}[cos\frac{\pi(a+b)}{k}-cos\frac{\pi(2c-a-b)}{k}]-\frac{1}{2}[cos\frac{\pi(b-a)}{k}-cos\frac{\pi(2c-a-b)}{k}]}{sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}}$$

$$=\frac{-\frac{1}{2}[cos\frac{\pi(b-a)}{k}-cos\frac{\pi(a+b)}{k}]}{sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}}$$

$$=\frac{-sin\frac{(\pi a)}{k}sin\frac{(\pi b)}{k}}{sin\frac{(\pi c)}{k}sin\frac{\pi(c-a-b)}{k}}$$

$$=\frac{sin\frac{(\pi a)}{k}sin\frac{(\pi b)}{k}}{[-sin\frac{(\pi c)}{k}][-sin\frac{\pi(a+b-c)}{k}]}$$

$$=\frac{sin\frac{(\pi a)}{k}sin\frac{(\pi b)}{k}}{sin\frac{\pi(c-k)}{k}sin\frac{\pi(a+b-(c-k))}{k}}$$

$$=\frac{\Gamma_k(2k-c)\Gamma_k(c-k)\Gamma_k(c-a-b)\Gamma_k(a+b+k-c)}{\Gamma_k(a)\Gamma_k(k-a)\Gamma_k(b)\Gamma_k(k-b)}.$$

3.1 Proof of Legendre's $k-$Duplication formula

Since

$$\beta_k(m,n)=\frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}=\frac{1}{k}\int_0^1 u^{\frac{m}{k}-1}(1-u)^{\frac{n}{k}-1}du$$

Let $m=n=z$

$$\Rightarrow\quad \frac{\Gamma_k(z)\Gamma_k(z)}{\Gamma_k(2z)}=\frac{1}{k}\int_0^1 u^{\frac{z}{k}-1}(1-u)^{\frac{z}{k}-1}du$$

put $u = \frac{1+x}{2} \Rightarrow du = \frac{dx}{2}$, Therefore

$$\begin{aligned}
\frac{\Gamma_k(z)\Gamma_k(z)}{\Gamma_k(2z)} &= \frac{1}{k}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{\frac{z}{k}-1}\left(1-\frac{1+x}{2}\right)^{\frac{z}{k}-1}\frac{dx}{2} \\
&= \frac{1}{k2^{1+2(\frac{z}{k}-1)}}\int_{-1}^{1}(1-x^2)^{\frac{z}{k}-1}dx \\
&= \frac{1}{k}2^{1-\frac{2z}{k}}\left[\int_{-1}^{1}(1-x^2)^{\frac{z}{k}-1}dx\right] \\
&= 2^{1-\frac{2z}{k}}\left[\frac{2}{k}\int_{0}^{1}(1-x^2)^{\frac{z}{k}-1}dx\right]
\end{aligned}$$

Since

$$\frac{2}{k}\int_0^1(1-x^2)^{\frac{z}{k}-1}dx = \frac{1}{k}\int_0^1 t^{\frac{k}{2k}-1}(1-t)^{\frac{z}{k}-1}dt$$

because in $\frac{1}{k}\int_0^1 t^{\frac{k}{2k}-1}(1-t)^{\frac{z}{k}-1}dt$ put $t = x^2 \quad dt = 2xdx$ when $t = 0 \quad \Rightarrow x = 0$

$t = 1 \quad \Rightarrow x = 1$

$$\Rightarrow \frac{1}{k}\int_0^1 t^{\frac{k}{2k}-1}(1-t)^{\frac{z}{k}-1}dt = \frac{1}{k}\int_0^1 (x^2)^{\frac{k}{2k}-1}(1-x^2)^{\frac{z}{k}-1}2xdx = \frac{1}{k}\int_0^1 (x^2)^{\frac{k}{2k}-1}(1-x^2)^{\frac{z}{k}-1}2dx$$

so,

$$\frac{\Gamma_k(z)\Gamma_k(z)}{\Gamma_k(2z)} = 2^{1-2\frac{z}{k}}B_k(k/2,z) = \frac{2^{1-2\frac{z}{k}}\Gamma_k(\frac{k}{2})\Gamma_k(z)}{\Gamma_k(z+\frac{k}{2})}$$

$$\Rightarrow \quad \Gamma_k(2z) = \frac{\Gamma_k(z)\Gamma_k(\frac{k}{2}+z)}{2^{1-\frac{2z}{k}}\sqrt{\frac{\pi}{k}}} \quad \left(\because \Gamma_k(\frac{k}{2}) = \sqrt{\frac{\pi}{k}}\right)$$

$$\Rightarrow \quad \sqrt{\frac{\pi}{k}}\Gamma_k(2z) = 2^{\frac{2z}{k}-1}\Gamma_k(z)\Gamma_k(z+\frac{k}{2}).$$

Theorem 3.1.1. *If $|z| < 1$, $k > 0$ and if $Re(c) > Re(b) > 0$,*

$$F_k(a,b;c;z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-ktz)^{\frac{-a}{k}}dt$$

Summation k-theorem

Theorem 3.1.2. *If $Re(c-b-a)>0$, $k>0$ and if c is neither zero nor a negative integer,*

$$ {}_2F_{1k}\left((a,k),(b,k);(c,k);\frac{1}{k}\right)=\frac{\Gamma_k(c)\Gamma_k(c-b-a)}{\Gamma_k(c-b)\Gamma_k(c-a)} $$

Proof:-

$$ F_k(a,b;c;z)=\frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{(\frac{b}{k}-1)}(1-t)^{(\frac{c-b}{k}-1)}(1-ktz)^{\frac{-a}{k}}dt $$

put $z=\frac{1}{k}$

$$ \begin{aligned} F_k((a,k),(b,k);(c,k);\frac{1}{k}) &= \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(c-b)}\frac{1}{k}\int_0^1 t^{(\frac{b}{k}-1)}(1-t)^{(\frac{c-b}{k}-1)}(1-t)^{\frac{-a}{k}}dt \\ &= \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(c-b)}\frac{1}{k}\int_0^1 t^{(\frac{b}{k}-1)}(1-t)^{(\frac{c-b-a}{k}-1)}dt \\ &= \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(c-b)}B_k(b,c-b-a) \\ &= \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(c-b)}\frac{\Gamma_k(b)\Gamma_k(c-b-a)}{\Gamma_k(b+c-b-a)} \\ &= \frac{\Gamma_k(c)\Gamma_k(c-b-a)}{\Gamma_k(c-b)\Gamma_k(c-a)} \end{aligned} $$

The value of ${}_2F_{1k}\left((a,k),(b,k);(c,k);\frac{1}{k}\right)$ will play a vital role in many of the results of properties of k-hypergeometric such as given in the following example.

Example 2. *Show that if $Re(b)>0$, $k>0$ and if n is a non negative integer,*

$$ {}_2F_{1k}\left[(\frac{-nk}{2},k),(\frac{-nk+k}{2},k);(b+\frac{k}{2},k);\frac{1}{k}\right]=\frac{2^n(b)_{n,k}}{(2b)_{n,k}}. $$

Solution:-

$$ L.H.S={}_2F_{1k}\left[(\frac{-nk}{2},k),(\frac{-nk+k}{2},k);(b+\frac{k}{2},k);\frac{1}{k}\right] $$

$$
\begin{aligned}
&= \frac{\Gamma_k(b+\frac{k}{2})\Gamma_k(b+\frac{k}{2}+\frac{1}{2}nk+\frac{1}{2}nk-\frac{k}{2})}{\Gamma_k(b+\frac{k}{2}+\frac{1}{2}nk-\frac{k}{2})\Gamma_k(b+\frac{k}{2}+\frac{1}{2}nk)} \\
&= \frac{\Gamma_k(b+\frac{k}{2})\Gamma_k(b)\Gamma_k(b+nk)}{\Gamma_k(b)\Gamma_k(b+\frac{1}{2}nk)\Gamma(b+\frac{1}{2}nk+\frac{k}{2})} \\
&= \frac{(b)_{n,k}\Gamma_k(b)\Gamma_k(b+\frac{k}{2})}{\Gamma_k(b+\frac{1}{2}nk)\Gamma_k(b+\frac{1}{2}nk+\frac{k}{2})}
\end{aligned}
$$

Legendre's duplication formula for gamma k-function gives,

$$\Gamma_k(b)\Gamma_k(b+\frac{k}{2}) = 2^{1-\frac{2b}{k}}\sqrt{\frac{\pi}{k}}\Gamma_k(2b)$$

and

$$\Gamma_k(b+\frac{1}{2}nk)\Gamma_k(b+\frac{k}{2}+\frac{1}{2}nk) = 2^{1-\frac{2b}{k}-n}\sqrt{\frac{\pi}{k}}\Gamma_k(2b)$$

therefor,

$$
\begin{aligned}
{}_2F_{1k}\left[(\frac{-nk}{2},k),(\frac{-nk+k}{2},k);(b+\frac{k}{2},k);\frac{1}{k}\right] &= \frac{(b)_{n,k}(2)^{1-\frac{2b}{k}}\sqrt{\frac{\pi}{k}}\Gamma_k(2b)}{(2)^{1-\frac{2b}{k}-n}\sqrt{\frac{\pi}{k}}\Gamma_k(2b+nk)} \\
&= \frac{(b)_{n,k}(2)^n\Gamma_k(2b)}{\Gamma_k(2b+nk)} \\
&= \frac{2^n(b)_{n,k}}{(2b)_{n,k}}.
\end{aligned}
$$

Lemma 3.1.3.

$$F_k(-nk,b;c;\frac{1}{k}) = \frac{(c-b)_{n,k}}{(c)_{n,k}}.$$

Proof:-

$$
\begin{aligned}
F_k(-nk,b;c;\frac{1}{k}) &= \frac{\Gamma_k(c)\Gamma_k(c-b+n)}{\Gamma_k(c-b)\Gamma_k(c+n)} \quad \left(\because F_k(a,b;c;\frac{1}{k}) = \frac{\Gamma_k(c)\Gamma_k(c-a-b)}{\Gamma_k(c-a)\Gamma_k(c-b)}\right) \\
&= \frac{(c-b)_{n,k}}{(c)_{n,k}}.
\end{aligned}
$$

Theorem 3.1.4. *If* $|z|<1$ *and* $\frac{|z|}{|(1-kz)|}<1$, $k>0$, *then*

$$F_k(a,b;c;z) = (1-kz)^{\frac{-a}{k}}F_k(a,c-b;c;\frac{-z}{1-kz}).$$

Proof:-

$$\begin{aligned} R.H.S. &= (1-kz)^{\frac{-a}{k}} F_k(a, c-b, c; \frac{-z}{1-kz}) \\ &= \sum_{m=0}^{\infty} \frac{(a)_{m,k}(c-b)_{m,k}}{(c)_{m,k}} \frac{(\frac{-z}{(1-kz)})^m}{m!} (1-kz)^{\frac{-a}{k}} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_{m,k}(c-b)_{m,k}}{(c)_{m,k}} \quad \frac{(-1)^m (z)^m (1-kz)^{-(\frac{a+mk}{k})}}{(m!)}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{m,k}(c-b)_{m,k}}{(c)_{m,k}} \frac{(-1)^m (z)^m}{(m!)} \frac{(z)^n (a+mk)_{n,k}}{n!} \quad \left(\because (1-kx)^{\frac{-a}{k}} = \sum_{n=0}^{\infty} (a)_{n,k} \frac{x^n}{n!} \right)$$

$$\begin{aligned} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{m+n,k}(c-b)_{m,k}}{(c)_{m,k}} \frac{(-1)^m (z)^{m+n}}{m!n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_{n,k}(c-b)_{m,k}}{(c)_{m,k}} \frac{(-1)^m (z)^n}{m!(n-m)!} \quad \left(\because \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_k(m, n-m) \right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_{n,k}(c-b)_{m,k}(-1)^m (z)^n (-nk)_{m,k}}{k^m (m!)(-1)^m (n!)(c)_{m,k}} \quad \left(\because (n-m)! = \frac{(-1)^m k^m (n!)}{(-nk)_{m,k}} \right) \\ &= \sum_{n=0}^{\infty} F_k\left((-nk, k), (c-b, k); (c, k); \frac{1}{k} \right) (a)_{n,k} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(c-(c-b))_{n,k}(a)_{n,k} z^n}{(c)_{n,k} n!} \quad \left(\because F_k((-nk, k), (a, k); (b, k); \frac{1}{k}) = \frac{(b-a)_{n,k}}{(b)_{n,k}} \right) \\ &= \sum_{n=0}^{\infty} \frac{(b)_{n,k}(a)_{n,k}(z)^n}{(c)_{n,k}(n!)} = F_k(a, b; c; z). \end{aligned}$$

Theorem 3.1.5. *If $|z| < 1$, $k > 0$, then*

$$F_k(a, b; c; z) = (1-kz)^{\frac{c-a-b}{k}} F_k(c-a, c-b; c; z).$$

Proof:-*Consider*

$$F_k(a, b; c; z) = (1-kz)^{\frac{-a}{k}} F_k(a, c-b; c; \frac{-z}{1-kz}).$$

By putting $y = \frac{-z}{1-kz}$ *in the second factor of R.H.S of above equation and use it again, we obtain*

$$\begin{aligned} F_k(a, c-b; c; \frac{-z}{1-kz}) &= F_k(c-b, a; c; y) \\ &= (1-ky)^{-\frac{(c-b)}{k}} F_k(c-b, c-a; c; \frac{-y}{1-ky}) \\ &= (1-ky)^{\frac{-c+b}{k}} F_k(c-a, c-b; c; \frac{-y}{1-ky}) \end{aligned}$$

But $y = \frac{z}{kz-1}$ *and* $1-ky = (1-kz)^{-1}$, *Therefore,*

$$F_k(a, c-b; c; \frac{-z}{1-kz}) = (1-kz)^{-\frac{(-c+b)}{k}} F_k(c-a, c-b; c; z)$$

so,

$$F_k(a, b; c; z) = (1-kz)^{\frac{c-b-a}{k}} F_k(c-a, c-b; c; z).$$

3.2 Derivation of k-differential equations

The Hypergeometric $k-$differential equation is

$$kz(1-kz)w'' + [2b - (a+b+k)kz]w' - abw = 0.$$

Let $z = \frac{4x}{(1+kx)^2}$ Then,

$$\frac{dz}{dx} = \frac{4(1-kx)}{(1+kx)^3}$$

and

$$1 - kz = \frac{(1-kx)^2}{(1+kx)^2}.$$

Now

$$w' = \frac{dw}{dz} = \frac{dw}{dx}\frac{dx}{dz} = \frac{dw}{dx}\frac{(1+kx)^3}{4(1-kx)}$$

and

$$
\begin{aligned}
w'' &= \frac{dw'}{dx}\frac{dx}{dz} \\
&= \frac{d}{dx}\left[\frac{(1+kx)^3}{4(1-x)}\frac{dw}{dx}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{d}{dx}\frac{(1+kx)^3}{4(1-kx)}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{1}{4}[(1+kx)^3(-1)(1-kx)^{-2}(-k) + (1-kx)^{-1}3(1+kx)^2k]\right]\frac{dx}{dz}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{1}{4}[\frac{(1+kx)}{(1-kx)} + 3]k\frac{(1+kx)^2}{(1-kx)}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{1}{4}k\frac{(1+kx)^2}{(1-kx)}[\frac{4-2kx}{(1-kx)}]\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{1}{2}k\frac{(1+kx)^2(2-kx)}{(1-kx)^2}\right]\frac{dx}{dz} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)^3}{4(1-kx)} + \frac{dw}{dx}\frac{k(1+kx)^2(2-kx)}{2(1-kx)^2}\right]\frac{(1+kx)^3}{4(1-kx)} \\
&= \left[\frac{d^2w}{dx^2}\frac{(1+kx)}{2} + \frac{dw}{dx}\frac{k(2-kx)}{1-kx}\right]\frac{(1+kx)^5}{8(1-kx)^2}
\end{aligned}
$$

Since

$$kz(1-kz)w'' + [2b-(a+b+k)kz]w' - abw = 0$$

put the values of w' and w'' in above equation, we obtain

$$\frac{4kx}{(1+kx)^2}\frac{(1-kx)^2}{(1+kx)^2}\frac{(1+kx)^5}{8(1-kx)^2}\left[\frac{d^2w}{dx^2}\frac{(1+kx)}{2}+\frac{dw}{dx}\frac{k(2-x)}{1-kx}\right]+\left[2b-(a+b+k)\frac{4kx}{(1+kx)^2}\right]\frac{dw}{dx}\frac{(1+kx)^3}{4(1-kx)}-abw=0$$

$$\Rightarrow \frac{kx(1+kx)^2}{4}\frac{d^2w}{dx^2}+\frac{k^2x(1+kx)}{2}\frac{dw}{dx}\frac{(2-kx)}{1-kx}+\frac{b(1+kx)^3}{2(1-kx)}\frac{dw}{dx}-\frac{(a+b+k)kx(1+kx)}{1-kx}\frac{dw}{dx}-abw=0$$

$$\Rightarrow \frac{4(1-kx)kx(1+kx)^2}{4}\frac{d^2w}{dx^2}+\frac{k^2x4(1-kx)(1+kx)}{2}\frac{dw}{dx}\frac{(2-kx)}{1-kx}+\frac{4(1-kx)b(1+kx)^3}{2(1-kx)}\frac{dw}{dx}-\frac{4(1-kx)(a+b+k)kx(1+kx)}{1-kx}\frac{dw}{dx}-4(1-kx)abw=0$$

$$\Rightarrow (1-kx)kx(1+kx)^2\frac{d^2w}{dx^2}+2k^2x(1+kx)(2-kx)\frac{dw}{dx}+2b(1+kx)^3\frac{dw}{dx}-4kx(a+b+k)(1+kx)\frac{dw}{dx}-4(1-kx)abw=0$$

$$\Rightarrow (1-kx)kx(1+kx)^2\frac{d^2w}{dx^2}+\frac{dw}{dx}2(1+kx)\left[k^2x(2-kx)+b(1+kx)^2-2kx(a+b+k)\right]-4(1-kx)abw=0$$

$$\Rightarrow kx(1-kx)(1+kx)^2\frac{d^2w}{dx^2}+2(1+kx)(b-2akx+bk^2x^2-k^3x^2)\frac{dw}{dx}-4ab(1-kx)w=0.$$

of which one solution is $w=F(a,b;2b;\frac{4x}{(1+kx)^2})$. put $w=(1+kx)^{\frac{2a}{k}}y$ in above equation, we get

$$w'=\frac{dw}{dx}=(1+kx)^{\frac{2a}{k}}\frac{dy}{dx}+k\frac{2a}{k}(1+kx)^{\frac{2a}{k}-1}y$$

and

$$\begin{aligned}
w'' &= \frac{dw'}{dx} \\
&= \frac{d}{dx}\left(\frac{dw}{dx}\right) \\
&= \frac{d}{dx}\left[(1+kx)^{\frac{2a}{k}}\frac{dy}{dx} + 2a(1+kx)^{\frac{2a}{k}-1}y\right] \\
&= \frac{d^2y}{dx^2}(1+kx)^{\frac{2a}{k}} + 4a(1+kx)^{\frac{2a}{k}-1}\frac{dy}{dx} + 2a(2a-k)(1+kx)^{\frac{2a}{k}-2}y
\end{aligned}$$

put the values of w' and w'' in above equation, we obtain

$$\begin{aligned}
kx(1-kx)(1+kx)^2\left[(1+kx)^{\frac{2a}{k}}\frac{d^2y}{dx^2} + \frac{4a(1+kx)^{\frac{2a}{k}}}{1+kx}\frac{dy}{dx} + \frac{2a(2a-k)(1+kx)^{\frac{2a}{k}}y}{(1+kx)^2}\right] \\
+2(1+kx)(b-2akx+bk^2x^2-k^3x^2)\left[(1+kx)^{\frac{2a}{k}}\frac{dy}{dx} + \frac{2a(1+kx)^{\frac{2a}{k}}y}{(1+kx)}\right] \\
-4ab(1-kx)(1+kx)^{\frac{2a}{k}}y = 0 \\
\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2} + kx(1-kx)4a(1+kx)^{\frac{2a}{k}+1}\frac{dy}{dx} \\
+kx(1-kx)\frac{2a(2a-k)(1+kx)^{\frac{2a}{k}}y}{+}2(1+kx)^{\frac{2a}{k}+1}(b-2akx+bk^2x^2-k^3x^2)\frac{dy}{dx} \\
+2(b-2akx+bk^2x^2-k^3x^2)2a(1+kx)^{\frac{2a}{k}}y - 4ab(1-kx)(1+kx)^{\frac{2a}{k}}y = 0
\end{aligned}$$

$$\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}2(1+kx)^{\frac{2a}{k}+1}\Big[2akx(1-kx)$$
$$+(b-2akx+bk^2x^2-k^3x^2)\Big]+2a(1+kx)^{\frac{2a}{k}}y\Big[(kx-k^2x^2)(2a-k)$$
$$+2b-4akx+2bk^2x^2-2k^3x^2-2b+2bkx\Big]=0$$
$$\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}2(1+kx)^{\frac{2a}{k}+1}\Big[b-(2a-b+k)kx^2\Big]$$
$$+2a(1+kx)^{\frac{2a}{k}}y\Big[-2akx-k^2x-2ak^2x^2-k^3x^2+2bk^2x^2+2bkx\Big]=0$$
$$\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}2(1+kx)^{\frac{2a}{k}+1}\Big[b-(2a-b+k)kx^2\Big]$$
$$-2akx(1+kx)^{\frac{2a}{k}}y\Big[2a+k+2akx+k^2x-2bkx-2b\Big]=0$$
$$\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}2(1+kx)^{\frac{2a}{k}+1}\Big[b-(2a-b+k)kx^2\Big]$$
$$-2akx(1+kx)^{\frac{2a}{k}}y\Big[(1+kx)(2a+k-2b)\Big]=0$$
$$\Rightarrow kx(1-kx)(1+kx)(1+kx)^{\frac{2a}{k}+1}\frac{d^2y}{dx^2}+\frac{dy}{dx}2(1+kx)^{\frac{2a}{k}+1}\Big[b-(2a-b+k)kx^2\Big]$$
$$-2akx(1+kx)^{\frac{2a}{k}+1}y(2a+k-2b)=0$$

since $(1+kx)^{\frac{2a}{k}+1}\neq 0$ so, the equation become

$$kx(1-(kx)^2)y''+2\left[b-(2a-b+k)(kx)^2\right]y'-2akx(2a+k-2b)y=0$$

of which one solution is $w=(1+kx)^{\frac{-2a}{k}}F_k(a,b;2b;\frac{4x}{(1+kx)^2})$. The above differential equation is invariant under a change from x to $-x$. Put $kv=(kx)^2 \quad \Rightarrow \sqrt(kv)=kx$ and $v=kx^2\Rightarrow \frac{dv}{dx}=2kx=2\sqrt(kv)$

$$y'=\frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{dx}=2\sqrt(kv)\frac{dy}{dv}$$

$$y''=\frac{dy'}{dx}=\frac{d}{dx}\left(2\sqrt(kv)\frac{dy}{dv}\right)=\frac{d}{dv}\left(2\sqrt(kv)\frac{dy}{dv}\right)\frac{dv}{dx}=\left(2\sqrt(kv)\frac{d^2y}{dv^2}+\frac{k}{\sqrt(kv)}\frac{dy}{dv}\right)2\sqrt(kv)$$
$$=4kv\frac{d^2y}{dv^2}+2k\frac{dy}{dv}$$

put the values of y' and y'' in the above equation, we obtain

$$\sqrt(kv)(1-kv)\left(4kv\frac{d^2y}{dv^2}+2k\frac{dy}{dv}\right)+2\left[b-(2a-b+k)kv\right]2\sqrt(kv)\frac{dy}{dv}$$
$$-2a\sqrt(kv)(k+2a-2b)y=0$$
$$\Rightarrow 2\sqrt(kv)\left[2kv(1-kv)\frac{d^2y}{dv^2}+k(1-kv)\frac{dy}{dv}+2b\frac{dy}{dv}-2kv(2a-b+k)\frac{dy}{dv}-a(2a-2b+k)y\right]=0$$
$$\Rightarrow 2\sqrt(kv)\left[2kv(1-kv)\frac{d^2y}{dv^2}+\frac{dy}{dv}\left[k-k^2v+2b-4akv+2bkv-2k^2v\right]-a(2a-2b+k)y\right]=0$$
$$\Rightarrow 2kv(1-kv)\frac{d^2y}{dv^2}+2\left[\frac{k}{2}-\frac{3k^2v}{2}+b-2akv+bkv\right]\frac{dy}{dv}-2a(\frac{k}{2}+a-b)y=0$$
$$\Rightarrow kv(1-kv)\frac{d^2y}{dv^2}+\left[b+\frac{k}{2}-(2a-b+\frac{3k}{2})kv\right]\frac{dy}{dv}-a(a-b+\frac{k}{2})y=0.$$

Theorem 3.2.1. *The function $F_k(a,b;a+b+\frac{k}{2};z)$ is a solution of the differential equation*

$$kz(1-kz)y''+[a+b+\frac{k}{2}-(a+b+k)kz]y'-aby=0$$

put $z=4x(1-kx)$ to obtain

$$kx(1-kx)\frac{d^2y}{dx^2}+[a+b+\frac{k}{2}-(2a+2b+k)kx]\frac{dy}{dx}-4aby=0$$

of which one solution is $y=F_k(a,b;a+b+\frac{k}{2};4x(1-kx))$
Proof:-*Since*

$$kz(1-kz)\frac{d^2y}{dz^2}+[a+b+\frac{k}{2}-(a+b+k)kz]\frac{dy}{dz}-aby=0$$

Let $z=4x(1-kx)=4x-4kx^2 \qquad \Rightarrow \frac{dz}{dx}=4(1-2kx)$

$$y'=\frac{dy}{dz}=\frac{dy}{dx}\frac{dx}{dz}=\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2kx)}$$

$$y''=\frac{d^2y}{dz^2}=\frac{dy'}{dz}=\frac{d}{dz}\left[\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2kx)}\right]$$
$$=\frac{d}{dx}\left[\frac{1}{4}\frac{dy}{dx}(1-2kx)^{-1}\right]\frac{dx}{dz}$$

$$= \left[\frac{1}{4}\frac{dy}{dx}(-1)(1-2kx)^{-2}(-2k) + \frac{(1-2kx)^{-1}}{4}\frac{d^2y}{dx^2}\right]\frac{1}{4(1-2kx)}$$

$$= \left[\frac{k}{2(1-2kx)^2}\frac{dy}{dx} + \frac{1}{4(1-2kx)}\frac{d^2y}{dx^2}\right]\frac{1}{4(1-2kx)}$$

$$= \left[\frac{k}{(1-2kx)}\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\right]\frac{1}{8(1-2kx)^2}$$

put the values of y', y'' *in above equation*

$$\frac{4kx(1-kx)(1-4kx+4k^2x^2)}{8(1-2kx)^2}\left[\frac{k}{(1-2kx)}\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\right]$$
$$+\left[a+b+\frac{k}{2}-(a+b+k)(4kx-4k^2x^2)\right]\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2kx)} - aby = 0$$

$$\frac{kx(1-kx)}{2}\left[\frac{k}{(1-2kx)}\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\right]$$
$$+\left[a+b+\frac{k}{2}-(a+b+k)(4kx-4k^2x^2)\right]\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2kx)} - aby = 0$$

$$\frac{k^2x(1-kx)}{2(1-2kx)}\frac{dy}{dx} + \frac{kx(1-kx)}{4}\frac{d^2y}{dx^2}$$
$$+\left[a+b+\frac{k}{2}-(a+b+k)(4kx-4k^2x^2)\right]\frac{1}{4}\frac{dy}{dx}\frac{1}{(1-2kx)} - aby = 0$$

multiplying by 4

$$\frac{4k^2x(1-kx)}{2(1-2kx)}\frac{dy}{dx} + kx(1-kx)\frac{d^2y}{dx^2}$$
$$+\left[a+b+\frac{k}{2}-(a+b+k)(4kx-4k^2x^2)\right]\frac{dy}{dx}\frac{1}{(1-2kx)} - 4aby = 0$$

now we solve

$$\frac{dy}{dx}\left[\frac{2k^2x(1-kx)}{(1-2kx)} + \left\{a+b+\frac{k}{2}-(a+b+k)(4kx-4k^2x^2)\right\}\frac{1}{(1-2kx)}\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[2k^2x - 2k^3x^2 + a + b + \frac{k}{2} - \left\{4akx + 4bkx + 4k^2x - 4ak^2x^2 - 4bk^2x^2 - 4k^3x^2\right\}\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[2k^2x - 2k^3x^2 + a + b + \frac{k}{2} - 4akx - 4bkx - 4k^2x + 4ak^2x^2 + 4bk^2x^2 + 4k^3x^2\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[-2k^2x + 2k^3x^2 + a + b + \frac{k}{2} - 4akx - 4bkx + 4ak^2x^2 + 4bk^2x^2\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[a + b + \frac{k}{2} - 4akx - 4bkx - 2k^2x + 4ak^2x^2 + 4bk^2x^2 + 2k^3x^2\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[1(a + b + \frac{k}{2}) - 4akx - 4bkx - 2k^2x + 4k^2x^2(a + b + \frac{k}{2})\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[(a + b + \frac{k}{2})(1 + 4k^2x^2) - 4akx - 4bkx - 2k^2x\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[(a + b + \frac{k}{2})(1 + 4k^2x^2) - 4kx(a + b + \frac{k}{2})\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[(a + b + \frac{k}{2})(1 + 4k^2x^2 - 4kx)\right]$$

$$= \frac{dy}{dx}\frac{1}{1-2kx}\left[(a + b + \frac{k}{2})(1 - 2kx)^2\right]$$

$$= \frac{dy}{dx}\left[(a + b + \frac{1}{2})(1 - 2kx)\right]$$

$$= \frac{dy}{dx}\left[a + b + \frac{k}{2} - 2akx - 2bkx - k^2x\right]$$

so above equation becomes

$$kx(1-kx)\frac{d^2y}{dx^2} + [a + b + \frac{k}{2} - (2a + 2b + k)kx]\frac{dy}{dx} - 4aby = 0$$

of which one solution is $y = F(a, b; a + b + \frac{k}{2}; 4x(1-kx))$

3.3 Derivation of solutions of k-differential equations

Since

$$kv(1-kv)\frac{d^2y}{dv^2} + \left[b + \frac{k}{2} - (2a - b + \frac{3k}{2})kv\right]\frac{dy}{dv} - a(a - b + \frac{k}{2})y = 0$$

Let $y = \sum_{n=0}^{\infty} a_n v^{c+n}$ then $y' = \sum_{n=0}^{\infty} a_n(c+n)v^{c+n-1}$ $y'' = \sum_{n=0}^{\infty} a_n(c+n)(c+n-1)v^{c+n-2}$
now put the values of y, y' and y'' in above equation

$$\Rightarrow \sum_{n=0}^{\infty} a_n k(c+n)(c+n-1)v^{c+n-1} - \sum_{n=0}^{\infty} a_n k^2(c+n)(c+n-1)v^{c+n} + (b+\frac{k}{2})\sum_{n=0}^{\infty} a_n(c+n)v^{c+n-1}$$
$$-k(2a-b+\frac{3k}{2})\sum_{n=0}^{\infty} a_n(c+n)v^{c+n} - a(a-b+\frac{k}{2})\sum_{n=0}^{\infty} a_n v^{c+n} = 0$$

$$v^c\Bigg[a_o kc(c-1)v^{-1} + a_o c(b+\frac{k}{2})v^{-1} + \sum_{n=1}^{\infty} a_n\Bigg\{k(c+n)(c+n-1) + (b+\frac{k}{2})(c+n)\Bigg\}v^{n-1}$$
$$-\sum_{n=0}^{\infty} a_n\Bigg\{k^2(c+n)(c+n-1) + k(2a-b+\frac{3k}{2})(c+n) + a(a-b+\frac{k}{2})\Bigg\}v^n\Bigg] = 0$$

change the summation index

$$v^c\Bigg[a_o kc(c-1)v^{-1} + a_o c(b+\frac{k}{2})v^{-1} + \sum_{n=0}^{\infty} a_{n+1}\Bigg\{k(c+n+1)(c+n) + (b+\frac{k}{2})(c+n+1)\Bigg\}v^n$$
$$-\sum_{n=0}^{\infty} a_n\Bigg\{k^2(c+n)(c+n-1) + k(2a-b+\frac{3k}{2})(c+n) + a(a-b+\frac{k}{2})\Bigg\}v^n\Bigg] = 0$$

which implies that

$$a_o kc(c-1) + a_o c(b+\frac{k}{2}) = 0$$

nothing is gained by taking $a_o = 0$ so,

$$kc(c-1) + (b+\frac{k}{2})c = 0 \quad \Rightarrow c(kc+b-\frac{k}{2}) = 0 \quad c = 0 \quad or \quad c = \frac{1}{2} - \frac{b}{k}$$

$$a_{n+1}\Bigg[k(c+n+1)(c+n)+(b+\frac{k}{2})(c+n+1)\Bigg] = a_n\Bigg[k^2(c+n)(c+n-1)+k(2a-b+\frac{3k}{2})(c+n)+a(a-b+\frac{k}{2})\Bigg]$$

$$a_{n+1} = \frac{a_n\Bigg[k^2(c+n)(c+n-1) + k(2a-b+\frac{3k}{2})(c+n) + a(a-b+\frac{k}{2})\Bigg]}{k(c+n+1)(c+n) + (b+\frac{k}{2})(c+n+1)}$$

$n = 0, 1, 2, ...$

(1)If $c = 0$

$$a_{n+1} = \frac{a_n \left[k^2 n(n-1) + k(2a - b + \frac{3k}{2})n + a(a - b + \frac{k}{2})\right]}{k(n+1)n + (b + \frac{k}{2})(n+1)}$$

taking $n = 0$

$$a_1 = \frac{a_o a(a - b + \frac{k}{2})}{(b + \frac{k}{2})}$$

$n = 1$

$$a_2 = \frac{a_o a(a - b + \frac{k}{2})(a + k)(a - b + \frac{3k}{2})}{(b + \frac{k}{2})2(b + \frac{3k}{2})}$$

up to so on

$$a_n = \frac{a_o(a - b + \frac{k}{2})_{n,k}(a)_{n,k}}{n!(b + \frac{k}{2})_{n,k}}$$

so

$$y_1 = \sum_{n=0}^{\infty} \frac{a_o(a - b + \frac{k}{2})_{n,k}(a)_{n,k} v^n}{n!(b + \frac{k}{2})_{n,k}}$$

$$= {}_2F_{1k}\left((a, k), (a - b + \frac{k}{2}, k); (b + \frac{k}{2}, k); v\right)$$

(2)If $c = \frac{1}{2} - \frac{b}{k}$

$$a_{n+1} = \frac{a_n \left[k^2(\frac{1}{2} - \frac{b}{k} + n)(\frac{1}{2} - \frac{b}{k} + n - 1) + k(2a - b + \frac{3k}{2})(\frac{1}{2} - \frac{b}{k} + n) + a(a - b + \frac{k}{2})\right]}{k(\frac{1}{2} - \frac{b}{k} + n + 1)(\frac{1}{2} - \frac{b}{k} + n) + (b + \frac{k}{2})(\frac{1}{2} - \frac{b}{k} + n + 1)}$$

$n = 0, 1, 2, ...$

taking $n = 0$

$$a_1 = \frac{a_o \left[k^2(\frac{1}{2} - \frac{b}{k})(\frac{1}{2} - \frac{b}{k} - 1) + k(2a - b + \frac{3k}{2})(\frac{1}{2} - \frac{b}{k}) + a(a - b + \frac{k}{2})\right]}{(k\frac{1}{2} - \frac{b}{k} + 1)(\frac{1}{2} - \frac{b}{k}) + (b + \frac{k}{2})(\frac{1}{2} - \frac{b}{k} + 1)}$$

$$= \frac{a_o(a - b + \frac{k}{2})(a + k - 2b)}{(\frac{3k}{2} - b)}$$

$n = 1$

$$a_2 = \frac{a_1\left[k^2(\frac{1}{2}-\frac{b}{k}+1)(\frac{1}{2}-\frac{b}{k})+(2a-b+\frac{3k}{2})(\frac{1}{2}-\frac{b}{k}+1)+a(a-b+\frac{k}{2})\right]}{k(\frac{1}{2}-\frac{b}{k}+2)(\frac{1}{2}-\frac{b}{k}+1)+(b+\frac{k}{2})(\frac{1}{2}-\frac{b}{k}+2)}$$

$$= \frac{a_o(a-b+\frac{k}{2})(a+k-2b)(a-b+\frac{3k}{2})(a-2b+2k)}{2!(\frac{3k}{2}-b)(\frac{5k}{2})-b}$$

up to so on

$$a_n = \frac{a_o(a-b+\frac{k}{2})_{n,k}(a+k-2b)_{n,k}}{(n!\frac{3k}{2}-b)_{n,k}}$$

so,

$$y_2 = v^{\frac{1}{2}-\frac{b}{k}}\sum_{n=0}^{\infty}\frac{a_o(a-b+\frac{k}{2})_{n,k}(a+k-2b)_{n,k}v^n}{(n!\frac{3k}{2}-b)_{n,k}}$$

$$= v^{\frac{1}{2}-\frac{b}{k}}{}_2F_{1k}\left(a-b+\frac{k}{2}, a-2b+k; \frac{3k}{2}-b; v\right).$$

Theorem 3.3.1. *Find the solutions of the differential equation*

$$kx(1-kx)\frac{d^2y}{dx^2}+[a+b+\frac{k}{2}-(2a+2b+k)kx]\frac{dy}{dx}-4aby=0$$

Proof:- *Since*

$$kx(1-kx)\frac{d^2y}{dx^2}+[a+b+\frac{k}{2}-(2a+2b+k)kx]\frac{dy}{dx}-4aby=0$$

Let $y=\sum_{n=0}^{\infty}a_nx^{c+n}$ *then* $y'=\sum_{n=0}^{\infty}a_n(c+n)x^{c+n-1}$ $y''=\sum_{n=0}^{\infty}a_n(c+n)(c+n-1)x^{c+n-2}$
now put the values of y, y', y'' *in above equation*

$$kx(1-kx)\sum_{n=0}^{\infty}a_n(c+n)(c+n-1)x^{c+n-2}+\left[a+b+\frac{k}{2}-(2a+2b+k)kx\right]\sum_{n=0}^{\infty}a_n(c+n)x^{c+n-1}$$
$$-4ab\sum_{n=0}^{\infty}a_nx^{c+n}=0$$

$$\sum_{n=0}^{\infty}ka_n(c+n)(c+n-1)x^{c+n-1}-\sum_{n=0}^{\infty}k^2a_n(c+n)(c+n-1)x^{c+n}+(a+b+\frac{k}{2})\sum_{n=0}^{\infty}a_n$$
$$(c+n)x^{c+n-1}-k(2a+2b+k)\sum_{n=0}^{\infty}a_n(c+n)x^{c+n}-4ab\sum_{n=0}^{\infty}a_nx^{c+n}=0$$

$$\sum_{n=0}^{\infty} a_n \left[k(c+n)(c+n-1) + (a+b+\frac{k}{2})(c+n) \right] x^{c+n-1} - \sum_{n=0}^{\infty} a_n$$
$$\left[k^2(c+n)(c+n-1) + k(2a+2b+k)(c+n) + 4ab \right] x^{c+n} = 0$$

$$x^c \left[a_o kc(c-1)x^{-1} + a_o c(a+b+\frac{k}{2})x^{-1} + \sum_{n=1}^{\infty} a_n \left\{ k(c+n)(c+n-1) + (a+b+\frac{k}{2})(c+n) \right\} x^{n-1} \right.$$
$$\left. - \sum_{n=0}^{\infty} a_n \left\{ k^2(c+n)(c+n-1) + k(2a+2b+k)(c+n) + 4ab \right\} x^n \right] = 0$$

change the summation index

$$x^c \left[a_o kc(c-1)x^{-1} + a_o c(a+b+\frac{k}{2})x^{-1} + \sum_{n=0}^{\infty} a_{n+1} \left\{ k(c+n+1)(c+n) + (a+b+\frac{k}{2}) \right.\right.$$
$$\left.\left. (c+n+1) \right\} x^n - \sum_{n=0}^{\infty} a_n \left\{ k^2(c+n)(c+n-1) + k(2a+2b+k)(c+n) + 4ab \right\} x^n \right] = 0$$

which implies that

$$a_o kc(c-1) + a_o c(a+b+\frac{k}{2}) = 0$$

nothing is gained by taking $a_o = 0$
so,

$$kc(c-1) + (a+b+\frac{k}{2})c = 0$$
$$kc^2 - kc + ac + bc + \frac{1}{2}kc = 0$$
$$kc^2 + ac + bc - \frac{1}{2}kc = 0$$
$$c(kc + a + b - \frac{1}{2}k) = 0$$
$$c = 0$$

or

$$c = \frac{1}{2} - \frac{a}{k} - \frac{b}{k}$$

$$a_{n+1} \left[k(c+n+1)(c+n) + (a+b+\frac{k}{2})(c+n+1) \right] = a_n \left[k^2(c+n)(c+n-1) + k(2a+2b+k)(c+n) + 4ab \right]$$

$$a_{n+1} = \frac{a_n\left[k^2(c+n)(c+n-1)+k(2a+2b+k)(c+n)+4ab\right]}{k(c+n+1)(c+n)+(a+b+\frac{k}{2})(c+n+1)}$$

$n = 0, 1, 2, ...$
(1)If $c = 0$

$$a_{n+1} = \frac{a_n\left[k^2n(n-1)+k(2a+2b+k)n+4ab\right]}{k(n+1)n+(a+b+\frac{k}{2})(n+1)}$$

taking $n = 0$

$$a_1 = \frac{a_o 4ab}{(a+b+\frac{k}{2})}$$

$n = 1$

$$a_2 = \frac{a_1\left[k(2a+2b+k)+4ab\right]}{2k+(a+b+\frac{k}{2})2}$$

$$= \frac{a_o(2a)(2a+k)(2b)(2b+k)}{(a+b+\frac{k}{2})2(a+b+\frac{k}{2}+k)}$$

up to so on

$$a_n = \frac{a_o(2a)_{n,k}(2b)_{n,k}}{n!(a+b+\frac{k}{2})_{n,k}}$$

so

$$y_1 = \sum_{n=0}^{\infty} \frac{a_o(2a)_{n,k}(2b)_{n,k}x^n}{n!(a+b+\frac{k}{2})_{n,k}}$$

$$= {}_2F_{1k}(2a, 2b; a+b+\frac{k}{2}; x)$$

(2)If $c = \frac{1}{2} - \frac{a}{k} - \frac{b}{k}$

$$a_{n+1} = \frac{a_n\left[k^2(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n-1)+k(2a+2b+k)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n)+4ab\right]}{k(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n+1)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n)+(a+b+\frac{k}{2})(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+n+1)}$$

$n = 0, 1, 2, ...$
taking $n = 0$

$$a_1 = \frac{a_o\left[k^2(\frac{1}{2}-\frac{a}{k}-\frac{b}{k})(\frac{-1}{2}-\frac{a}{k}-\frac{b}{k})+k(2a+2b+k)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k})+4ab\right]}{k(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+1)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k})+(a+b+\frac{k}{2})(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+1)}$$

$$= \frac{a_o(a-b+\frac{k}{2})(b-a+\frac{k}{2})}{(\frac{3k}{2}-a-b)}$$

$n = 1$

$$a_2 = \frac{a_1\left[k^2(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+1)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k})+k(2a+2b+k)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+1)+4ab\right]}{k(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+2)(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+1)+(a+b+\frac{k}{2})(\frac{1}{2}-\frac{a}{k}-\frac{b}{k}+2)}$$

$$= \frac{a_o(a-b+\frac{k}{2})(a-b+\frac{k}{2}+k)(b-a+\frac{k}{2})(b-a+\frac{k}{2}+k)}{2!(\frac{3k}{2}-a-b)(\frac{3k}{2}-a-b+k)}$$

up to so on

$$a_n = \frac{a_o(a-b+\frac{k}{2})_{n,k}(b-a+\frac{k}{2})_{n,k}}{n!(\frac{3k}{2}-a-b)_{n,k}}$$

so,

$$y_2 = x^{\frac{1}{2}-\frac{a}{k}-\frac{b}{k}}\sum_{n=0}^{\infty}\frac{(a-b+\frac{k}{2})_{n,k}(b-a+\frac{k}{2})_{n,k}x^n}{n!(\frac{3k}{2}-a-b)_{n,k}}$$

$$= x^{\frac{1}{2}-\frac{a}{k}-\frac{b}{k}}{}_2F_{1k}(a-b+\frac{k}{2}, b-a+\frac{k}{2}; \frac{3k}{2}-a-b; x)$$

Theorem 3.3.2. *If $2b$ is neither zero nor a negative integer, and if both $|kx| < 1$ and $|4x(1+kx)^{-2}| < 1$, $k > 0$,*

$$(1+kx)^{\frac{-2a}{k}}F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right) = F_k\left(a, a-b+\frac{k}{2}; b+\frac{k}{2}; kx^2\right).$$

Proof:-*The equation*

$$kv(1-kv)\frac{d^2y}{dv^2} + \left[b+\frac{k}{2}-(2a-b+\frac{3k}{2})kv\right]\frac{dy}{dv} - a(a-b+\frac{k}{2})y = 0$$

has, in $|v| < 1$, the general solution

$$y = AF_k\left(a, a-b+\frac{k}{2}; b+\frac{k}{2}; v\right)$$
$$+Bv^{\frac{1}{2}-\frac{b}{k}}F_k\left(a-b+\frac{k}{2}, a-2b+k; \frac{3k}{2}-b; v\right)$$

The differential equation

$$kx(1-(kx)^2)y'' + 2\left[b-(2a-b+k)(kx)^2\right]y' - 2akx(k+2a-2b)y = 0$$

has a solution $y = (1+kx)^{\frac{-2a}{k}}F_k(a, b; 2b; \frac{4x}{(1+kx)^2})$ valid in $|\frac{4x}{(1+kx)^2}| < 1$ as long as $2b$ is neither zero nor a negative integer.At the same time equation

$$kx(1-(kx)^2)y'' + 2\left[b-(2a-b+k)(kx)^2\right]y' - 2akx(k+2a-2b)y = 0$$

has the general solution (3.3.11) with $kv = (kx)^2$*, this solution is valid in* $|x| < 1$*. Therefore, if both* $|x| < 1$ *and* $|\frac{4x}{(1+kx)^2}| < 1$ *and if* $2b$ *is neither zero nor a negative integer, there exist constant* A *and* B *such that*

$$(1+kx)^{\frac{-2a}{k}} F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right) = AF_k\left(a, a-b+\frac{k}{2}; b+\frac{k}{2}; kx^2\right)$$
$$+Bx^{1-\frac{2b}{k}} k(\frac{1}{2}-\frac{b}{k}) F_k\left(a-b+\frac{k}{2}, a-2b+k; \frac{3k}{2}-b; kx^2\right)$$

In equation (3.3.12) the left member and the first term on the right are analytic at $x = 0$*, but the last term is not analytic at* $x = 0$ *because of the factor* $x^{1-\frac{2b}{k}}$*. Hence* $B = 0$ *and* A *is easily determined by using* $x = 0$ *in the resultant identity*

$$(1+kx)^{\frac{-2a}{k}} F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right) = AF_k\left(a, a-b+\frac{k}{2}; b+\frac{k}{2}; kx^2\right)$$

Thus $A = 1$

$$(1+kx)^{\frac{-2a}{k}} F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right) = F_k\left(a, a-b+\frac{k}{2}; b+\frac{k}{2}; kx^2\right).$$

Theorem 3.3.3. *If* $2b$ *is neither zero nor a negative integer and if* $|ky| < \frac{1}{2}$ *and* $|\frac{ky}{(1-ky)}| < 1$*,* $k > 0$*,*

$$(1-ky)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a+\frac{k}{2}; \frac{k}{2}+b; \frac{ky^2}{(1-ky)^2}\right) = F_k(a, b; 2b; 2y).$$

Proof:-

$$
\begin{aligned}
L.H.S. &= (1-ky)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a+\frac{k}{2}; \frac{k}{2}+b; \frac{ky^2}{(1-ky)^2}\right) \\
&= (1-ky)^{\frac{-a}{k}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_{m,k}(\frac{1}{2}a+\frac{k}{2})_{m,k}}{(b+\frac{k}{2})_{m,k}m!} \left(\frac{ky^2}{(1-ky)^2}\right)^m \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m,k}y^{2m}k^m}{(b+\frac{k}{2})_{m,k}m!2^{2m}} \frac{(a+2mk)_{n,k}y^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m+n,k}y^{2m+n}k^m}{(b+\frac{k}{2})_{m,k}m!n!2^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_{n,k}y^n k^m}{(b+\frac{k}{2})_{m,k}m!(n-2m)!2^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_{n,k}y^n k^m}{(b+\frac{k}{2})_{m,k}m!(n-2m)!2^{2m}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_{n,k}y^n k^m(-nk)_{2m,k}}{(b+\frac{k}{2})_{m,k}m!(-1)^{2m}k^{2m}n!2^{2m}} \quad \left(\because (n-2m)! = \frac{(-1)^{2m}k^{2m}n!}{(-nk)_{2m,k}}\right)
\end{aligned}
$$

$$
= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_{n,k}y^n 2^{2m}(\frac{-nk}{2})_{m,k}(\frac{-nk+k}{2})_{m,k}k^m}{(b+\frac{k}{2})_{m,k}m!n!(-1)^{2m}k^{2m}2^{2m}} \quad \left(\because (-nk)_{2m,k} = 2^{2m}\left(\frac{-nk}{2}\right)_{m,k}\left(\frac{-nk+k}{2}\right)_{m,k}\right)
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} F_k\left((\frac{-nk}{2},k),(\frac{-nk+k}{2},k);(b+\frac{k}{2},k);\frac{1}{k}\right)\frac{(a)_{n,k}y^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{2^n(b)_{n,k}}{(2b)_{n,k}}\frac{(a)_{n,k}y^n}{n!} \quad \left(\because F_k\left((\frac{-nk}{2},k),(\frac{-nk+k}{2},k);(b+\frac{k}{2},k);\frac{1}{k}\right) = \frac{2^n(b)_{n,k}}{(2b)_{n,k}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(b)_{n,k}}{(2b)_{n,k}}\frac{(a)_{n,k}(2y)^n}{n!} \\
&= F_k\left(a,b;2b;2y\right).
\end{aligned}
$$

Theorem 3.3.4. *If $k+a-b$ is neither zero nor a negative integer, and if $|kz|<1$ and $|\frac{4z}{(1-kz)^2}|<1$, $k>0$,*

$$
(1-kz)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{k}{2}+\frac{1}{2}a-b; k+a-b; \frac{-4z}{(1-kz)^2}\right) = F_k\left(a,b;k+a-b;z\right).
$$

Proof:-*In theorem (3.3.8) put*

$$y = \frac{2x}{(1+kx)^2} \qquad \Rightarrow 1 - ky = \frac{1+k^2x^2}{(1+kx)^2}$$

and

$$\frac{y}{1-ky} = \frac{2x}{(1+k^2x)^2}$$

so theorem (3.3.8) becomes

$$\frac{(1+k^2x^2)^{\frac{-a}{k}}}{(1+kx)^{\frac{-2a}{k}}} F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; \frac{k}{2} + b; \frac{4kx^2}{(1+k^2x^2)^2}\right) = F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right)$$

$$\Rightarrow (1+k^2x^2)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; \frac{k}{2} + b; \frac{4kx^2}{(1+k^2x^2)^2}\right) = (1+kx)^{\frac{-2a}{k}} F_k\left(a, b; 2b; \frac{4x}{(1+kx)^2}\right)$$

using theorem (3.3.7) on R.H.S

$$(1+^2 x^2)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; \frac{k}{2} + b; \frac{4kx^2}{(1+k^2x^2)^2}\right) = F_k\left(a, a - b + \frac{k}{2}; b + \frac{k}{2}; kx^2\right)$$

now put $(kx)^2 = kz$ *and replace* b *by* $\frac{k}{2} + a - b$ *in above equation*

$$(1+kz)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; \frac{k}{2} + a - b + \frac{k}{2}; \frac{4z}{(1+kz)^2}\right) =$$
$$F_k\left(a, a - \frac{k}{2} - a + b + \frac{k}{2}; \frac{k}{2} + a - b + \frac{k}{2}; z\right)$$

$$(1+kz)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; k + a - b; \frac{4z}{(1+kz)^2}\right) = F_k\left(a, b; k + a - b; z\right)$$

using theorem

$$F_k(a, b; c; z) = (1-kz)^{\frac{-a}{k}} F_k(a, c-b; c; \frac{-z}{1-kz}).$$

on R.H.S of above equation

$$F_k\left(\frac{1}{2}a, \frac{1}{2}a + \frac{k}{2}; k + a - b; \frac{4z}{(1+kz)^2}\right) = \left(1 - \frac{4kz}{(1+kz)^2}\right)^{\frac{-a}{2k}}$$
$$F_k\left(\frac{1}{2}a, k + a - b - \frac{1}{2}a - \frac{k}{2}; k + a - b; \frac{\frac{-4z}{(1+kz)^2}}{1 - \frac{4kz}{(1+kz)^2}}\right)$$
$$= \left[\frac{(1-kz)^2}{(1+kz)^2}\right]^{\frac{-a}{2k}}$$
$$F_k\left(\frac{1}{2}a, k + a - b - \frac{1}{2}a - \frac{k}{2}; k + a - b; \frac{-4z}{(1+kz)^2}\frac{(1+kz)^2}{(1-kz)^2}\right)$$

so,

$$(1-kz)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a, \frac{k}{2}+\frac{1}{2}a-b; k+a-b; \frac{-4z}{(1-kz)^2}\right) = F_k\left(a, b; k+a-b; z\right).$$

Theorem 3.3.5. *If $a+b+\frac{k}{2}$ is neither zero nor a negative integer, and if $|x|<1$ and $|4x(1-kx)|<1, k>0$,*

$$F_k(a,b;a+b+\frac{k}{2};4x(1-kx)) = F_k(2a,2b;a+b+\frac{k}{2};x)$$

Proof:-*The equation*

$$kx(1-kx)\frac{d^2y}{dx^2}+[a+b+\frac{k}{2}-(2a+2b+k)kx]\frac{dy}{dx}-4aby=0$$

is hypergeometric in character and has the general solution if $|x|<1$ and $|4x(1-kx)|<1$, $a+b+\frac{k}{2}$ is neither zero nor a negative integer there exist constants A and B such that

$$y = AF_k\left(2a,2b;a+b+\frac{k}{2};x\right)$$
$$+Bx^{\frac{1}{2}-\frac{a}{k}-\frac{b}{k}}F_k\left(\frac{k}{2}+a-b, \frac{k}{2}+b-a; \frac{3k}{2}-a-b; x\right)$$

as well as the solution $y=F_k(a,b;a+b+\frac{k}{2};4x(1-kx))$

$$F_k\left(a,b;a+b+\frac{k}{2};4x(1-kx)\right) = AF_k\left(2a,2b;a+b+\frac{k}{2};x\right)$$
$$+Bx^{\frac{1}{2}-\frac{a}{k}-\frac{b}{k}}F_k\left(\frac{k}{2}+a-b, \frac{k}{2}+b-a; \frac{3k}{2}-a-b; x\right)$$

In equation (3.3.15) the left member and the first term on the right are analytic at x=0, but the last term is not analytic at x=0 because of the factor $x^{\frac{1}{2}-\frac{a}{k}-\frac{b}{k}}$.Hence B=0 and A is easily determined by using x=0 in the resultant identity

$$F_k\left(a,b;a+b+\frac{k}{2};4x(1-kx)\right) = AF_k\left(2a,2b;a+b+\frac{k}{2};x\right)$$

Thus A=1

$$F_k\left(a,b;a+b+\frac{k}{2};4x(1-kx)\right) = F_k\left(2a,2b;a+b+\frac{k}{2};x\right)$$

Theorem 3.3.6. *If c is neither zero nor a negative integer, and if both $|x| < 1$ and $|4x(1-kx)| < 1, k > 0$,*

$$F_k(a, k-a; c; x) = (1-kx)^{\frac{c}{k}-1} F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; c; 4x(1-kx))$$

Proof:-*In the identity of theorem (3.3.12) replace a by $\frac{1}{2}c - \frac{1}{2}a$ and b by $\frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}$*

$$\begin{aligned} &F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; \frac{1}{2}c - \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2} + \frac{k}{2}; 4x(1-kx)) \\ &= F_k(2(\frac{1}{2}c - \frac{1}{2}a), 2(\frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}); \frac{1}{2}c - \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2} + \frac{k}{2}; x) \end{aligned}$$

$$F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; c; 4x(1-kx)) = F_k(c-a, c+a-k; c; x)$$

using theorem

$$F_k\left(a, b; c; z\right) = (1-kz)^{\frac{c-a-b}{k}} F_k\left(c-a, c-b; c; z\right)$$

on R.H.S

$$\begin{aligned} F_k(c-a, c+a-k; c; x) &= (1-kx)^{\frac{c-c+a-c-a+k}{k}} F_k(c-c+a, c-c-a+k; c; x) \\ &= (1-kx)^{1-\frac{c}{k}} F_k(a, k-a; c; x) \end{aligned}$$

so,

$$F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; c; 4x(1-kx)) = (1-kx)^{1-\frac{c}{k}} F_k(a, k-a; c; x)$$

$$F_k(a, k-a; c; x) = (1-kx)^{\frac{c}{k}-1} F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; c; 4x(1-kx))$$

Theorem 3.3.7. *Show that,*

$$F_k\left[(-nk, k), (k-b-nk, k); (a, k); \frac{1}{k}\right] = \frac{(a+b-k)_{2n,k}}{(a)_{n,k}(a+b-1)_{n,k}}$$

Solution:-*Since*

$${}_2F_{1k}\left((a, k), (b, k); (c, k); \frac{1}{k}\right) = \frac{\Gamma_k(c)\Gamma_k(c-b-a)}{\Gamma_k(c-b)\Gamma_k(c-a)}$$

$$\begin{aligned}
F_k\left[(-nk,k),(k-b-nk,k);(a,k);\frac{1}{k}\right] &= \frac{\Gamma_k(a)\Gamma_k(a+nk-k+b+nk)}{\Gamma_k(a+nk)\Gamma_k(a-k+b+nk)} \\
&= \frac{\Gamma_k(a)\Gamma_k(a+b+2nk-k)}{\Gamma_k(a+nk)\Gamma_k(a+b+nk-k)} \\
&= \frac{\Gamma_k(a)}{\Gamma_k(a+nk)} \quad \frac{\Gamma_k(a+b+2nk-k)}{\Gamma_k(a+b+nk-k)} \quad \frac{\Gamma_k(a+b-k)}{\Gamma_k(a+b-k)} \\
&= \frac{1}{(a)_{n,k}} \quad \frac{1}{(a+b-k)_{n,k}} \quad (a+b-k)_{2n,k} \\
&= \frac{(a+b-k)_{2n,k}}{(a)_{n,k}(a+b-k)_{n,k}}
\end{aligned}$$

Theorem 3.3.8.

$$F_k(a,b;\frac{1}{2}(a+b+k);\frac{1}{2k}) = \frac{\Gamma_k(\frac{k}{2})\Gamma_k[\frac{1}{2}(k+a+b)]}{\Gamma_k[\frac{1}{2}(k+a)]\Gamma_k[\frac{1}{2}(k+b)]}$$

Proof:-*Since*

$$F_k(a,b;a+b+\frac{k}{2};4x(1-kx)) = F_k(2a,2b;a+b+\frac{k}{2};x)$$

Replace a by $\frac{a}{2}$ and b by $\frac{b}{2}$

$$F_k(\frac{a}{2},\frac{b}{2};\frac{a}{2}+\frac{b}{2}+\frac{k}{2};4x(1-kx)) = F(a,b;\frac{a}{2}+\frac{b}{2}+\frac{k}{2};x)$$

put $x = \frac{1}{2k}$

$$\begin{aligned}
F_k(a,b;\frac{a}{2}+\frac{b}{2}+\frac{k}{2};\frac{1}{2k}) &= F_k(\frac{a}{2},\frac{b}{2};\frac{a}{2}+\frac{b}{2}+\frac{k}{2};\frac{1}{k}) \\
&= \frac{\Gamma_k(\frac{a}{2}+\frac{b}{2}+\frac{k}{2})\Gamma_k(\frac{a}{2}+\frac{b}{2}+\frac{k}{2}-\frac{a}{2}-\frac{b}{2})}{\Gamma_k(\frac{a}{2}+\frac{b}{2}+\frac{k}{2}-\frac{a}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{k}{2}-\frac{b}{2})} \\
&= \frac{\Gamma_k(\frac{k}{2})\Gamma_k[\frac{1}{2}(k+a+b)]}{\Gamma_k[\frac{1}{2}(k+a)]\Gamma_k[\frac{1}{2}(k+b)]}
\end{aligned}$$

Theorem 3.3.9.

$$F_k(a,k-a;c;\frac{1}{2k}) = \frac{\Gamma_k(\frac{1}{2}c)\Gamma_k[\frac{1}{2}(c+k)]}{\Gamma_k[\frac{1}{2}(a+c)]\Gamma_k[\frac{1}{2}(k+c-a)]}$$

Proof:-*Since*

$$F_k(a,k-a;c;x) = (1-kx)^{\frac{c}{k}-1}F_k(\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c+\frac{1}{2}a-\frac{k}{2};c;4x(1-kx))$$

put $\quad x = \frac{1}{2k}$

$$\begin{aligned} F_k(a, k-a; c; \frac{1}{2k}) &= (1-\frac{k}{2k})^{\frac{c}{k}-1} F_k(\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{k}{2}; c; \frac{1}{k}) \\ &= \frac{(\frac{1}{2})^{\frac{c}{k}-1}\Gamma_k(c)\Gamma_k(c - \frac{1}{2}c + \frac{1}{2}a - \frac{1}{2}c - \frac{1}{2}a + \frac{k}{2})}{\Gamma_k(c - \frac{1}{2}c + \frac{1}{2}a)\Gamma_k(c - \frac{1}{2}c - \frac{1}{2}a + \frac{k}{2})} \\ &= \frac{2^{1-\frac{c}{k}}\Gamma_k(c)\Gamma_k(\frac{k}{2})}{\Gamma_k(\frac{1}{2}c + \frac{1}{2}a)\Gamma_k(\frac{1}{2}c - \frac{1}{2}a + \frac{k}{2})} \end{aligned}$$

since

$$2^{2\frac{z}{k}-1}\Gamma_k(z)\Gamma_k(z + \frac{k}{2}) = \sqrt{\frac{\pi}{k}}\Gamma_k(2z)$$

put $z = \frac{c}{2}$ *in above equation*

$$2^{\frac{2}{k}(\frac{c}{2})-1}\Gamma_k(\frac{c}{2})\Gamma_k[\frac{1}{2}(c+k)] = \sqrt{\frac{\pi}{k}}\Gamma_k(c)$$

$$2^{\frac{c}{k}-1}\Gamma_k(\frac{c}{2})\Gamma_k[\frac{1}{2}(c+k)] = \sqrt{\frac{\pi}{k}}\Gamma_k(c)$$

$$\frac{2^{\frac{c}{k}-1}\Gamma_k(\frac{c}{2})\Gamma_k[\frac{1}{2}(c+k)]}{\sqrt{\frac{\pi}{k}}} = \Gamma_k(c)$$

since $\Gamma_k(\frac{k}{2}) = \sqrt{\frac{\pi}{k}}$
so

$$\begin{aligned} F_k(a, k-a; c; \frac{1}{2k}) &= \frac{2^{1-\frac{c}{k}}2^{\frac{c}{k}-1}\Gamma_k(\frac{c}{2})\Gamma_k[\frac{1}{2}(c+k)]\sqrt{\frac{\pi}{k}}}{\sqrt{\frac{\pi}{k}}\Gamma_k(\frac{1}{2}c + \frac{1}{2}a)\Gamma_k(\frac{1}{2}c - \frac{1}{2}a + \frac{k}{2})} \\ &= \frac{\Gamma_k(\frac{1}{2}c)\Gamma_k[\frac{1}{2}(c+k)]}{\Gamma_k[\frac{1}{2}(a+c)]\Gamma_k[\frac{1}{2}(k+c-a)]} \end{aligned}$$

Theorem 3.3.10.

$$\frac{d^n}{dx^n}\left[x^{\frac{a}{k}-1+n}F_k(a, b; c; x)\right] = \frac{(a)_{n,k}}{k^n}x^{\frac{a}{k}-1}F_k(a + nk, b; c; x)$$

Proof:-

$$
\begin{aligned}
L.H.S &= \frac{d^n}{dx^n}\left[x^{\frac{a}{k}-1+n}F_k(a,b;c;x)\right] \\
&= \sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}m!}\frac{d^n}{dx^n}x^{m+n+\frac{a}{k}-1} \\
&= \sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}m!}(n+m+\frac{a}{k}-1)(n+m+\frac{a}{k}-2)...(n+m+\frac{a}{k}-n)x^{n+m+\frac{a}{k}-n-1} \\
&= \sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}m!}(a+(m+n-1)k)(a+(m+n-2)k)...(a+mk)x^{m+\frac{a}{k}-1} \\
&= \sum_{m=0}^{\infty}\frac{(a)_{m,k}(a)_{n+m,k}(b)_{m,k}x^{m+\frac{a}{k}-1}}{k^n(a)_{m,k}(c)_{m,k}m!} \\
&= \sum_{m=0}^{\infty}\frac{(a+nk)_{m,k}(a)_{n,k}(b)_{m,k}x^{m+\frac{a}{k}-1}}{k^n(c)_{m,k}m!} \\
&= \frac{(a)_{n,k}}{k}x^{\frac{a}{k}-1}F_k(a+nk,b;c;x)
\end{aligned}
$$

Kummer's k-Theorem

Theorem 3.3.11. *If (k+a-b) is neither zero nor a negative integer,and* $Re(b) < 1, k > 0$ *for convergence,*

$$F_k(a,b;k+a-b;\frac{-1}{k}) = \frac{\Gamma_k(k+a-b)\Gamma_k(k+\frac{1}{2}a)}{\Gamma_k(k+\frac{1}{2}a-b)\Gamma(k+a)}$$

Proof:-*Since*

$$(1-kz)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{k}{2}+\frac{1}{2}a-b;k+a-b;\frac{-4z}{(1-kz)^2}\right) = F_k(a,b;k+a-b;z)$$

put $z = \frac{-1}{k}$ *then we have*

$$(1+1)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{k}{2}+\frac{1}{2}a-b;k+a-b;\frac{\frac{4}{k}}{(1+1)^2}\right) = F_k(a,b;k+a-b;\frac{-1}{k})$$

$$(2)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{k}{2}+\frac{1}{2}a-b;k+a-b;\frac{1}{k}\right) = F_k(a,b;k+a-b;\frac{-1}{k})$$

$$\frac{\Gamma_k(k+a-b)\Gamma_k(k+a-b-\frac{1}{2}a-\frac{1}{2}a-\frac{k}{2}+b)}{2^{\frac{a}{k}}\Gamma_k(k+a-b-\frac{1}{2}a)\Gamma_k(k+a-b-\frac{k}{2}-\frac{1}{2}a)+b}=F_k(a,b;k+a-b;\frac{-1}{k})$$

$$\frac{\Gamma_k(k+a-b)\Gamma_k(\frac{k}{2})}{2^{\frac{a}{k}}\Gamma_k(k+\frac{1}{2}a-b)\Gamma_k(\frac{k}{2}+\frac{1}{2}a)}=F_k(a,b;k+a-b;\frac{-1}{k})$$

By Legendre's Duplication formula

$$\Gamma_k(z)\Gamma_k(z+\frac{k}{2})=\Gamma_k(\frac{k}{2})2^{1-\frac{2z}{k}}\Gamma_k(2z)$$

put $z=\frac{a+k}{2}$ *in above equation*

$$\Gamma_k(\frac{k}{2}+\frac{1}{2}a)\Gamma_k(\frac{k}{2}+\frac{1}{2}a+\frac{k}{2})=\Gamma_k(\frac{k}{2})2^{1-\frac{2}{k}(\frac{a+k}{2})}\Gamma_k(k+a)$$

$$\Gamma_k(\frac{k}{2}+\frac{1}{2}a)+\Gamma_k(k+\frac{1}{2}a)=2^{\frac{-a}{k}}\Gamma_k(\frac{k}{2})\Gamma_k(k+a)$$

$$\frac{2^{\frac{a}{k}}\Gamma_k(\frac{k}{2}+\frac{1}{2}a)+\Gamma_k(k+\frac{1}{2}a)}{\Gamma_k(k+a)}=\Gamma_k(\frac{k}{2})$$

so

$$F_k(a,b;k+a-b;\frac{-1}{k})=\frac{\Gamma_k(k+a-b)\Gamma_k(k+\frac{1}{2}a)}{\Gamma_k(k+\frac{1}{2}a-b)\Gamma_k(k+a)}$$

Theorem 3.3.12.

$$(1-kx)^{1-\frac{c}{k}}F_k(a,k-a;c;x)=(1-2kx)^{\frac{a-c}{k}}F_k\left[\frac{1}{2}c-\frac{1}{2}a,\frac{1}{2}c-\frac{1}{2}a+\frac{k}{2};c;\frac{4x(kx-1)}{(1-2kx)^2}\right]$$

Proof:-*Since*

$$(1-kx)^{1-\frac{c}{k}}F_k(a,k-a;c;x)=F_k\left(\frac{c-a}{2},\frac{c+a-k}{2};c;4x(1-kx)\right)$$

since

$$F_k(a,b;c;z)=(1-kz)^{\frac{-a}{k}}F_k(a,c-b;c;\frac{-z}{1-kz})$$

therefore

$$\begin{aligned}(1-kx)^{1-\frac{c}{k}}F_k(a,k-a;c;x) &= (1-4kx(1-kx))^{-\frac{c-a}{2k}}F_k\left(\frac{c-a}{2},c-\frac{c+a-k}{2};c;\right.\\ &\left.\frac{-4x(1-kx)}{1-4kx(1-kx)}\right)\\ &= (1-4kx(1-kx))^{-\frac{c-a}{2k}}F_k\left(\frac{c-a}{2},c-\frac{c+a-k}{2};c;\right.\\ &\left.\frac{-4x(1-kx)}{1-4kx(1-kx)}\right)\\ &= (1-2kx)^{2\frac{a-c}{2k}}F_k\left(\frac{c-a}{2},\frac{c-a+k}{2};c;\frac{4x(kx-1)}{(1-2kx)^2}\right)\\ &= (1-2kx)^{\frac{a-c}{k}}F_k\left(\frac{c-a}{2},\frac{c-a+k}{2};c;\frac{4x(kx-1)}{(1-2kx)^2}\right)\end{aligned}$$

Theorem 3.3.13.

$$F_k(2a,2b;a+b+\frac{k}{2};\frac{1}{2k})=\frac{\Gamma_k(a+b+\frac{k}{2})\Gamma_k(\frac{k}{2})}{\Gamma_k(a+\frac{k}{2})\Gamma_k(b+\frac{k}{2})}$$

Proof:-*Since*

$$F_k(2a,2b;a+b+\frac{k}{2};x)=F_k(a,b;a+b+\frac{k}{2};4x(1-kx))$$

let $x=\frac{1}{2k}$

$$\begin{aligned}F_k(2a,2b;a+b+\frac{k}{2};\frac{1}{2k}) &= F_k(a,b;a+b+\frac{k}{2};\frac{1}{k})\\ &= \frac{\Gamma_k(a+b+\frac{k}{2})\Gamma_k(a+b+\frac{k}{2}-a-b)}{\Gamma_k(a+b+\frac{k}{2}-a)\Gamma_k(a+b+\frac{k}{2}-b)}\\ &= \frac{\Gamma_k(a+b+\frac{k}{2})\Gamma_k(\frac{k}{2})}{\Gamma_k(a+\frac{k}{2})\Gamma_k(b+\frac{k}{2})}\end{aligned}$$

Theorem 3.3.14.

$$F_k(-nk,a+nk;c;\frac{1}{k})=\frac{(-1)^n(k+a-c)_{n,k}}{(c)_{n,k}}$$

Proof:-*Since*

$$F_k(a,b;c;\frac{1}{k}) = \frac{\Gamma_k(c)\Gamma_k(c-a-b)}{\Gamma_k(c-a)\Gamma_k(c-b)}$$

therefore

$$\begin{aligned} F(-nk,a+nk;c;\frac{1}{k}) &= \frac{\Gamma_k(c)\Gamma_k(c+nk-a-nk)}{\Gamma_k(c+nk)\Gamma_k(c-a-nk)} \\ &= \frac{\Gamma_k(c)\Gamma_k(c-a)}{\Gamma_k(c+nk)\Gamma_k(c-a-nk)} \\ &= \frac{\Gamma_k(c-a)}{(c)_{n,k}\Gamma_k(c-a-nk)} \end{aligned}$$

since

$$\frac{\Gamma_k(k-\alpha-nk)}{\Gamma_k(k-\alpha)} = \frac{(-1)^n}{(\alpha)_{n,k}}$$

Let $\alpha = k-c+a$
then

$$\frac{\Gamma_k(k-k+c-a-nk)}{\Gamma_k(k-k+c-a)} = \frac{(-1)^n}{(k-c+a)_{n,k}}$$

$$\frac{\Gamma_k(c-a-nk)}{\Gamma_k(c-a)} = \frac{(-1)^n}{(k-c+a)_{n,k}}$$

therefore

$$F_k(-nk,a+nk;c;\frac{1}{k}) = \frac{(k-c+a)_{n,k}}{(c)_{n,k}(-1)^n} = \frac{(-1)^n(k-c+a)_{n,k}}{(c)_{n,k}}$$

Theorem 3.3.15. *If* $p \leq q+1$, *if* $Re(b_1) > Re(a_1) > 0$, *if no one of* $b_1, b_2 ... b_q$ *is zero or a negative integer,*$k > 0$, *and if* $|z| < 1$,

$${}_pF_{q,k}\left(a_1,a_2,...a_p;b_1,b_2,...b_q;z\right) = \frac{\Gamma_k(b_1)}{k\Gamma_k(a_1)\Gamma_k(b_1-a_1)}$$

$$\int_0^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}{}_{p-1}F_{q-1,k}\left(a_2,...a_p;b_2,...b_q;zt\right)dt$$

if $p \leq q$, *the condition* $|z| < 1$ *may be omitted.*

Proof:-

$$
\begin{aligned}
{}_pF_{q,k}\left(a_1,a_2,...a_p;b_1,b_2,...b_q;z\right) &= \sum_{n=0}^{\infty}\frac{(a_1)_{n,k}(a_2)_{n,k}...(a_p)_{n,k}z^n}{(b_1)_{n,k}(b_2)_{n,k}...(b_q)_{n,k}n!} \\
&= \frac{\Gamma_k(b_1)}{\Gamma_k(b_1-a_1)\Gamma_k(a_1)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a_1+nk)\Gamma_k(b_1-a_1)}{\Gamma_k(b_1+nk)} \\
&\frac{(a_2)_{n,k}...(a_p)_{n,k}z^n}{(b_2)_{n,k}...(b_q)_{n,k}n!} \\
&= \frac{\Gamma_k(b_1)}{\Gamma_k(b_1-a_1)\Gamma_k(a_1)}\sum_{n=0}^{\infty}B_k(a_1+nk,b_1-a_1)\frac{(a_2)_{n,k}...(a_p)_{n,k}z^n}{(b_2)_{n,k}...(b_q)_{n,k}n!} \\
&= \frac{\Gamma_k(b_1)}{\Gamma_k(b_1-a_1)\Gamma_k(a_1)}\sum_{n=0}^{\infty}\frac{1}{k}\int_0^1 t^{\frac{a_1+nk}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1} \\
&\frac{(a_2)_{n,k}...(a_p)_{n,k}z^n}{(b_2)_{n,k}...(b_q)_{n,k}n!}dt
\end{aligned}
$$

$$
\begin{aligned}
{}_pF_{q,k}\left(a_1,a_2,...a_p;b_1,b_2,...b_q;z\right) &= \frac{\Gamma_k(b_1)}{k\Gamma_k(b_1-a_1)\Gamma_k(a_1)}\int_0^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}\sum_{n=0}^{\infty} \\
&\frac{(a_2)_{n,k}...(a_p)_{n,k}}{(b_2)_{n,k}...(b_q)_{n,k}}\frac{(tz)^n}{n!}dt \\
&= \frac{\Gamma_k(b_1)}{k\Gamma_k(b_1-a_1)\Gamma_k(a_1)}\int_0^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1} \\
&{}_{p-1}F_{q-1,k}\left(a_2,...a_p;b_2,...b_q;zt\right)dt
\end{aligned}
$$

Saalschutz'k-theorem

Theorem 3.3.16. *If n is a non negative integer and if a,b,c are independent of n,k > 0,*

$$
{}_3F_{2,k}\left((-nk,k),(a,k),(b,k);(c,k),(k-c+a+b-nk,k);\frac{1}{k}\right)=\frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}(c-a-b)_{n,k}}
$$

Proof:-*Since*

$$
F_k\left(a,b;c;z\right)=(1-kz)^{\frac{c-a-b}{k}}F_k\left(c-a,c-b;c;z\right)
$$

Replace a by c-a and b by c-b

$$
F_k\left(c-a,c-b;c;z\right)=(1-kz)^{\frac{c-(c-a)-(c-b)}{k}}F_k\left(a,b;c;z\right)
$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} = (1-kz)^{\frac{-(c-a-b)}{k}} F_k\left(a,b;c;z\right)$$

$$\because \qquad (1-ky)^{\frac{-a}{k}} = \sum_{n=0}^{\infty} \frac{(a)_{n,k} y^n}{n!}$$

$$\begin{aligned} \therefore \qquad \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \frac{(c-a-b)_{n,k} z^n}{n!} \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k} z^m}{(c)_{m,k} m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(c-a-b)_{n,k}(a)_{m,k}(b)_{m,k} z^{n+m}}{(c)_{m,k} n! m!} \end{aligned}$$

as we know that,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_k(m,n-m)$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(c-a-b)_{n-m,k}(a)_{m,k}(b)_{m,k} z^n}{(c)_{m,k}(n-m)! m!}$$

since

$$(n-m)! = \frac{(-1)^m k^m (n!)}{(-nk)_{m,k}}$$

and

$$(a)_{n-m,k} = \frac{(-1)^m (a)_{n,k}}{(k-a-nk)_{m,k}}$$

$$\begin{aligned} \therefore \qquad \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-nk)_{m,k}(c-a-b)_{n-m,k}(a)_{m,k}(b)_{m,k} z^n}{(c)_{m,k} k^m (-1)^m n! m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-nk)_{m,k}(c-a-b)_{n,k}(-1)^m (a)_{m,k}(b)_{m,k} z^n}{(c)_{m,k} k^m (-1)^m (k-c+a+b-nk)_{m,k} n! m!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(-nk)_{m,k}(a)_{m,k}(b)_{m,k}}{(c)_{m,k}(k-c+a+b-nk)_{m,k} k^m m!} \right] \frac{z^n (c-a-b)_{n,k}}{n!}$$

comparing the coefficients of like terms

$$\frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}} = {}_3F_{2,k}\left((-nk,k),(a,k),(b,k);(c,k),(k-c+a+b-nk,k);\frac{1}{k}\right)(c-a-b)_{n,k}$$

$$\Rightarrow \quad {}_3F_{2,k}\left((-nk,k),(a,k),(b,k);(c,k),(k-c+a+b-nk,k);\frac{1}{k}\right) = \frac{(c-a)_{n,k}(c-b)_{n,k}}{(c)_{n,k}(c-a-b)_{n,k}}$$

Theorem 3.3.17. *If n is non negative integer and if a and b are independent of n,$k>0$,*

$${}_3F_{2,k}\left((-nk,k),(a+nk,k),(\frac{k}{2}+\frac{1}{2}a-b,k);(k+a-b,k),(\frac{1}{2}a+\frac{k}{2},k);\frac{1}{k}\right) = \frac{(b)_{n,k}}{(k+a-b)_{n,k}}$$

Proof:-

$$Since F_k\left(a,b;k+a-b;z\right) = (1-kz)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a,\frac{k}{2}+\frac{1}{2}a-b;k+a-b;\frac{-4z}{(1-kz)^2}\right)$$

then

$$\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}z^n}{(k+a-b)_{n,k}n!} = \sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_{m,k}(\frac{k}{2}+\frac{1}{2}a-b)_{m,k}}{(k+a-b)_{m,k}m!}\frac{(-4z)^m}{(1-kz)^{2m+\frac{a}{k}}}$$

$$\because \qquad (1-ky)^{\frac{-a}{k}} = \sum_{n=0}^{\infty}\frac{(a)_{n,k}y^n}{n!}$$

$$\therefore \qquad \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}z^n}{(k+a-b)_{n,k}n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m 2^{2m} z^{m+n}(a+2mk)_{n,k}(\frac{a}{2})_{m,k}(\frac{a+k}{2}-b)_{m,k}}{(k+a-b)_{m,k}m!n!}$$

$$\because \qquad (a)_{m,k}(a+mk)_{n,k} = (a)_{m+n,k}$$

$$\therefore \qquad \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}z^n}{(k+a-b)_{n,k}n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m 2^{2m} z^{m+n}(a)_{2m+n,k}(\frac{a}{2})_{m,k}(\frac{a+k}{2}-b)_{m,k}}{(k+a-b)_{m,k}(a)_{2m,k}m!n!}$$

as we know that,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}A_k(m,n-m)$$

$$\therefore \quad \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}z^n}{(k+a-b)_{n,k}n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m 2^{2m} z^n (a)_{m+n,k}(\frac{a}{2})_{m,k}(\frac{a+k}{2}-b)_{m,k}}{(k+a-b)_{m,k}(a)_{2m,k}m!(n-m)!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m 2^{2m} z^n (a)_{m+n,k}(\frac{a}{2})_{m,k}(\frac{a+k}{2}-b)_{m,k}}{(k+a-b)_{m,k}m!2^{2m}(\frac{a}{2})_{m,k}(\frac{a+k}{2})_{m,k}(n-m)!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m z^n (a)_{m+n,k}(\frac{a+k}{2}-b)_{m,k}}{(k+a-b)_{m,k}m!(\frac{a+k}{2})_{m,k}(n-m)!}$$

since

$$(n-m)! = \frac{(-1)^m k^m (n!)}{(-nk)_{m,k}}$$

$$\therefore \quad \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}z^n}{(k+a-b)_{n,k}n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m z^n (a)_{m+n,k}(\frac{a+k}{2}-b)_{m,k}(-nk)_{m,k}}{(k+a-b)_{m,k}k^m m!(\frac{a+k}{2})_{m,k}(-1)^m n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{z^n (a)_{n,k}(a+nk)_{m,k}(\frac{a+k}{2}-b)_{m,k}(-nk)_{m,k}}{(k+a-b)_{m,k}m!k^m(\frac{a+k}{2})_{m,k}n!}$$

$$= \sum_{n=0}^{\infty} {}_3F_{2,k}\left((-nk,k),(a+nk,k),(\frac{k}{2}+\frac{1}{2}a-b,k);(k+a-b,k),(\frac{1}{2}a+\frac{k}{2},k);\frac{1}{k}\right)\frac{z^n(a)_n}{n!}$$

comparing the coefficients of like terms

$$\frac{(b)_{n,k}}{(k+a-b)_{n,k}} = {}_3F_{2,k}\left((-nk,k),(a+nk,k),(\frac{k}{2}+\frac{1}{2}a-b,k);(k+a-b,k),(\frac{1}{2}a+\frac{k}{2},k);\frac{1}{k}\right)$$

Whipple's k-theorem

Theorem 3.3.18. *If neither (a-b) nor (a-c) nor a is a negative integer,* $k > 0$,

$$F_k(a,b,c;k+a-b,k+a-c;\frac{x}{k}) = (1-kx)^{\frac{-a}{k}} F_k\left(\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2},k+a-b-c;k+a-b,k+a-c;\frac{-4x}{(1-kx)^2}\right)$$

Proof:-

$$F_k(a,b,c;k+a-b,k+a-c;\frac{x}{k}) = \sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{(k+a-b)_{m,k}(k+a-c)_{m,k}}\frac{x^m}{m!k^m}$$

$$= \sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b+mk)\Gamma_k(k+a-c+mk)}\frac{x^m}{m!k^m}$$

multiplying and dividing by $\Gamma_k(k+a)\Gamma_k(k+a-b-c)$

$$= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_m(c)_{m,k}\Gamma_k(k+a+2mk)\Gamma_k(k+a-b-c)}{\Gamma_k(k+a-b+mk)\Gamma_k(k+a-c+mk)(k+a)_{2m,k}} \frac{x^m}{m!k^m}$$

since

$$F_k(a,b;c;\frac{1}{k}) = \frac{\Gamma_k(c)\Gamma_k(c-a-b)}{\Gamma_k(c-a)\Gamma_k(c-b)}$$

therefore

$$\begin{aligned}
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{m=0}^{\infty} F_k(c+mk, b+mk; k+a+2mk; \frac{1}{k}) \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{(k+a)_{2m,k}} \frac{x^m}{m!} \\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(c+mk)_{n,k}(b+mk)_{n,k}}{(k+a+2mk)_{n,k}n!k^n} \right] \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{(k+a)_{2m,k}} \frac{x^m}{m!}
\end{aligned}$$

since

$$(a)_{m,k}(a+mk)_{n,k} = (a)_{m+n,k}$$

therefore

$$\begin{aligned}
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m,k}(b)_{m+n,k}(c)_{m+n,k}}{(k+a)_{2m+n,k}} \frac{x^m}{m!n!k^n} \\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_{m,k}(b)_{n,k}(c)_{n,k}}{(k+a)_{m+n,k}} \frac{x^m}{m!k^{n-m}(n-m)!}
\end{aligned}$$

since

$$(n-m)! = \frac{(-1)^m k^m n!}{(-nk)_{m,k}}$$

therefore

$$\begin{aligned}
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_{m,k}(b)_{n,k}(c)_{n,k}(-nk)_{m,k}(-1)^m}{(k+a+nk)_{m,k}(k+a)_{n,k}n!k^n} \frac{x^m}{m!} \\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(a)_{m,k}(-nk)_{m,k}(-x)^m}{(k+a+nk)_{m,k}m!} \right] \frac{(b)_{n,k}(c)_{n,k}}{(k+a)_{n,k}n!k^n} \\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)} \sum_{n=0}^{\infty} F_k(-nk, a; k+a+nk; -x) \frac{(b)_{n,k}(c)_{n,k}}{(k+a)_{n,k}n!k^n}
\end{aligned}$$

since

$$F_k(a,b;k+a-b;z)=(1+kz)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2};k+a-b;\frac{4z}{(1+kz)^2}\right)$$

put $z=-x, \qquad b=-nk$
then

$$F_k(a,b;k+a+nk;-x)=(1-kx)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2};k+a+nk;\frac{-4x}{(1-kx)^2}\right)$$

$$=\frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{n=0}^{\infty}(1-kx)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2};k+a+nk;\frac{-4x}{(1-kx)^2}\right)\frac{(b)_{n,k}(c)_{n,k}}{(k+a)_{n,k}n!k^n}$$

$$\begin{aligned}
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_{m,k}(\frac{1}{2}a+\frac{k}{2})_{m,k}(-1)^m2^{2m}x^m}{(k+a+nk)_{m,k}(1-kx)^{\frac{a}{k}+2m}(k+a)_{n,k}}\frac{(b)_{n,k}(c)_{n,k}}{n!m!k^n}\\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_{m,k}(\frac{1}{2}a+\frac{k}{2})_{m,k}(-1)^m2^{2m}x^m}{(k+a)_{m+n,k}(1-kx)^{\frac{a}{k}+2m}}\frac{(b)_{n,k}(c)_{n,k}}{n!m!k^n}\\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m,k}(-1)^mx^m(b)_{n,k}(c)_{n,k}}{(k+a)_{m+n,k}(1-kx)^{\frac{a}{k}+2m}m!n!k^n}\\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(b)_{n,k}(c)_{n,k}}{n!(k+a+mk)_{n,k}k^n}\right]\frac{(a)_{2m,k}(-1)^mx^m}{m!(k+a)_{m,k}(1-kx)^{\frac{a}{k}+2m}}\\
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{m=0}^{\infty}F_k(b,c;k+a+mk;\frac{1}{k})\frac{(a)_{2m,k}(-1)^mx^m}{m!(k+a)_{m,k}(1-kx)^{\frac{a}{k}+2m}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b-c)\Gamma_k(k+a)}\sum_{m=0}^{\infty}\frac{\Gamma_k(k+a+mk)\Gamma_k(k+a+mk-b-c)}{\Gamma_k(k+a+mk-b)\Gamma_k(k+a+mk-c)}\\
&\qquad\qquad\frac{(a)_{2m,k}(-1)^mx^m}{m!(k+a)_{m,k}(1-kx)^{\frac{a}{k}+2m}}
\end{aligned}$$

$$\begin{aligned}
&= (1-kx)^{\frac{-a}{k}}\sum_{m=0}^{\infty}\frac{(-1)^m(a)_{2m,k}(k+a-b-c)_{m,k}(k+a)_{m,k}x^m}{m!(k+a-b)_{m,k}(k+a-c)_{m,k}(1-kx)^{2m}(k+a)_{m,k}}\\
&= (1-kx)^{\frac{-a}{k}}F_k\left(\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2},k+a-b-c;k+a-b,k+a-c;\frac{-4x}{(1-kx)^2}\right)
\end{aligned}$$

Dixon's k-theorem

Theorem 3.3.19.

$$F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{\Gamma_k(k+\frac{1}{2}a)\Gamma_k(k+a-b)\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k+a)\Gamma_k(k+\frac{1}{2}a-b)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

Proof:-

$$\begin{aligned} F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) &= \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{(k+a-b)_{m,k}(k+a-c)_{m,k}m!k^{2m}} \\ &= \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a-b+mk)\Gamma_k(k+a-c+mk)m!k^{2m}} \end{aligned}$$

$$= \Gamma_k(k+a-b)\Gamma_k(k+a-c)\sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{\Gamma_k(k+a-b+mk)\Gamma_k(k+a-c+mk)m!k^{2m}}$$

since

$$(k+a)_{2m,k} = \frac{\Gamma_k(k+a+2mk)}{\Gamma_k(k+a)}$$

multiplying and dividing by $\Gamma_k(k+a-b-c)(k+a)_{2m,k}$

$$\begin{aligned} &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}\Gamma_k(k+a+2mk)\Gamma_k(k+a-b-c)}{\Gamma_k(k+a-b+mk)\Gamma_k(k+a-c+mk)m!k^{2m}(k+a)_{2m,k}} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{m=0}^{\infty} \frac{(a)_{m,k}(b)_{m,k}(c)_{m,k}}{m!k^m(k+a)_{2m,k}} F_k(c+mk,b+mk;k+a+2mk;\frac{1}{k}) \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c+mk)_{n,k}(b+mk)_{n,k}(a)_{m,k}(b)_{m,k}(c)_{m,k}}{(k+a+2mk)_{n,k}m!k^mk^n(k+a)_{2m,k}n!} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c)_{m+n,k}(b)_{m+n,k}(a)_{m,k}}{m!k^nk^m(k+a)_{2m+n,k}n!} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(c)_{n,k}(b)_{n,k}(a)_{m,k}}{k^mm!(k+a)_{m+n,k}(n-m)!k^{n-m}} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(c)_{n,k}(b)_{n,k}(a)_{m,k}(-1)^m(-nk)_{m,k}}{m!(k+a+nk)_{m,k}(k+a)_{n,k}n!k^nk^m} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)} \sum_{n=0}^{\infty} F_k(a,-nk;k+a+nk;\frac{-1}{k})\frac{(b)_{n,k}(c)_{n,k}}{(k+a)_{n,k}k^nn!} \end{aligned}$$

since

$$F_k(a,b;k+a+b;\frac{-1}{k}) = \frac{\Gamma_k(k+a-b)\Gamma_k(k+\frac{a}{2})}{\Gamma_k(k+\frac{a}{2}-b)\Gamma_k(k+a)}$$

put $b=-nk$

$$F_k(a,-nk;k+a-nk;\frac{-1}{k}) = \frac{\Gamma_k(k+a+nk)\Gamma_k(k+\frac{a}{2})}{\Gamma_k(k+\frac{a}{2}+nk)\Gamma_k(k+a)} = \frac{(k+a)_{n,k}}{(k+\frac{a}{2})_{n,k}}$$

therefore

$$\begin{aligned} F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k}) &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)}\sum_{n=0}^{\infty}\frac{(k+a)_{n,k}}{(k+\frac{a}{2})_{n,k}}\frac{(b)_{n,k}(c)_{n,k}}{(k+a)_{n,k}n!k^n} \\ &= \frac{\Gamma_k(k+a-b)\Gamma_k(k+a-c)}{\Gamma_k(k+a)\Gamma_k(k+a-b-c)}F_k(b,c;k+\frac{a}{2};\frac{1}{k}) \\ &= \frac{\Gamma_k(k+\frac{1}{2}a)\Gamma_k(k+a-b)\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k+a)\Gamma_k(k+\frac{1}{2}a-b)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)} \end{aligned}$$

Theorem 3.3.20. *If* $Re(\alpha)>0, Re(\beta)>0$, *and if* m *and* s *are positive integers,*$k>0$, *then inside the region of convergence of the resultant series*

$$\int_0^t x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;cx^m(t-x)^s\right)dx = kB_k(\alpha,\beta)t^{\frac{\alpha+\beta}{k}-1}$$

$${}_{p+m+s}F_{q+m+s,k}\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+k}{m},...,\frac{\alpha+(m-1)k}{m},\frac{\beta}{s},\frac{\beta+k}{s},...,\frac{\beta+(s-1)k}{s}\right.$$

$$\left.;b_1,...,b_q,\frac{\alpha+\beta}{m+s},\frac{\alpha+\beta+k}{m+s},...\frac{\alpha+\beta+(m+s-1)k}{m+s};\frac{m^m s^s c t^{m+s}}{m+s^{m+s}}\right)$$

Proof:-

$$L.H.S = \int_0^t x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;cx^m(t-x)^s\right)dx$$

Let $x=tv \Rightarrow \quad \frac{dx}{dv}=t$ *when* $x=0 \Rightarrow \quad v=0$ *when* $x=t \Rightarrow \quad v=1$

$$\begin{aligned} &= \int_0^1 (tv)^{\frac{\alpha}{k}-1}(t-tv)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1,a_2,...a_p;b_1,b_2,...,b_q;ct^{m+s}v^m(1-v)^s\right)tdv \\ &= \int_0^1 t^{\frac{\alpha}{k}-1}v^{\frac{\alpha}{k}-1}t^{\frac{\beta}{k}-1}(1-v)^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{n(m+s)}v^{mn}(1-v)^{sn}}{n!}tdv \\ &= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{n(m+s)}}{n!}\frac{1}{k}\int_0^1 v^{\frac{\alpha}{k}+mn-1}(1-v)^{\frac{\beta}{k}+sn-1}dv \end{aligned}$$

since

$$B_k(p,q)=\frac{1}{k}\int_0^1 t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}dt$$

therefore

$$\begin{aligned}
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{n(m+s)}}{n!}B_k(\alpha+mnk,\beta+snk)\\
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{n(m+s)}}{n!}\frac{\Gamma_k(\alpha+mnk)\Gamma_k(\beta+snk)}{\Gamma_k(\alpha+mnk+\beta+snk)}\\
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{n(m+s)}}{n!}\frac{(\alpha)_{mn,k}(\beta)_{sn,k}\Gamma_k(\alpha)\Gamma_k(\beta)}{(\alpha+\beta)_{(m+s)n,k}\Gamma_k(\alpha+\beta)}
\end{aligned}$$

since

$$(\alpha)_{mn,k}=m^{nm}(\frac{\alpha}{m})_{n,k}(\frac{\alpha+k}{m})_{n,k}...(\frac{\alpha+(m-1)k}{m})_{n,k}$$

therefore

$$= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k},...,(a_p)_{n,k}}{(b_1)_{n,k},...,(b_q)_{n,k}}\frac{c^n t^{n(m+s)}s^{ns}m^{nm}}{n!}$$

$$\frac{(\frac{\alpha}{m})_{n,k}(\frac{\alpha+k}{m})_{n,k},...,(\frac{\alpha+(m-1)k}{m})_{n,k}(\frac{\beta}{s})_{n,k}(\frac{\beta+k}{s})_{n,k},...,(\frac{\beta+(s-1)k}{s})_{n,k}}{(m+s)^{n(m+s)}(\frac{\alpha+\beta}{m+s})_{n,k}(\frac{\alpha+\beta+k}{m+s})_{n,k},...,(\frac{\alpha+\beta+(m+s-1)k}{m+s})_{n,k}}\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$$

$$= kB_k(\alpha,\beta)t^{\frac{\alpha+\beta}{k}-1}{}_{p+m+s}F_{q+m+s,k}$$

$$\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+k}{m},...,\frac{\alpha+(m-1)k}{m},\frac{\beta}{s},\frac{\beta+k}{s},...,\frac{\beta+(s-1)k}{s}\right.$$

$$\left.;b_1,...,b_q,\frac{\alpha+\beta}{m+s},\frac{\alpha+\beta+k}{m+s},...,\frac{\alpha+\beta+(m+s-1)k}{m+s};\frac{m^m s^s ct^{m+s}}{m+s^{m+s}}\right)$$

Theorem 3.3.21. *If $Re(\alpha)>0, Re(\beta)>0$, and if m and s are positive integers,$k>0$, then inside the region of convergence of the resultant series*

$$\int_0^t x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1,a_2,...a_p;b_1,b_2,...,b_q;cx^m\right)dx = kB_k(\alpha,\beta)t^{\frac{\alpha+\beta}{k}-1}{}_{p+m}F_{q+m,k}$$

$$\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+k}{m},...,\frac{\alpha+(m-1)k}{m};b_1,...,b_q,\frac{\alpha+\beta}{m},\frac{\alpha+\beta+k}{m},...,\frac{\alpha+\beta+(m-1)k}{m};ct^m\right)$$

Proof:-

$$L.H.S = \int_0^t x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1, a_2, ...a_p; b_1, b_2, ...b_q; cx^m\right)dx$$

Let $x = tv \Rightarrow \quad \frac{dx}{dv} = t$ *when* $x = 0 \Rightarrow \quad v = 0$ *when* $x = t \Rightarrow \quad v = 1$

$$\begin{aligned}
&= \int_0^1 (tv)^{\frac{\alpha}{k}-1}(t-tv)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1, a_2, ...a_p; b_1, b_2, ...b_q; ct^m v^m\right)tdv \\
&= \int_0^1 t^{\frac{\alpha}{k}-1}v^{\frac{\alpha}{k}-1}t^{\frac{\beta}{k}-1}(1-v)^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{mn}v^{mn}}{n!}tdv \\
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{mn}}{n!}\frac{1}{k}\int_0^1 v^{\frac{\alpha}{k}+mn-1}(1-v)^{\frac{\beta}{k}-1}dv
\end{aligned}$$

since

$$B_k(p,q) = \frac{1}{k}\int_0^1 t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}dt$$

therefore

$$\begin{aligned}
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{mn}}{n!}B_k(\alpha+mnk,\beta) \\
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{mn}}{n!}\frac{\Gamma_k(\alpha+mnk)\Gamma_k(\beta)}{\Gamma_k(\alpha+mnk+\beta)} \\
&= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{c^n t^{mn}}{n!}\frac{(\alpha)_{mn,k}\Gamma_k(\alpha)\Gamma_k(\beta)}{(\alpha+\beta)_{mn,k}\Gamma_k(\alpha+\beta)}
\end{aligned}$$

since

$$(\alpha)_{mn,k} = m^{nm}(\frac{\alpha}{m})_{n,k}(\frac{\alpha+k}{m})_{n,k}...(\frac{\alpha+(m-1)k}{m})_{n,k}$$

therefore

$$= kt^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(a_1)_{n,k},...,(a_p)_{n,k}}{(b_1)_{n,k},...,(b_q)_{n,k}}\frac{c^n t^{mn}m^{nm}}{n!}\frac{(\frac{\alpha}{m})_{n,k}(\frac{\alpha+k}{m})_{n,k},...,(\frac{\alpha+(m-1)k}{m})_{n,k}}{m^{nm}(\frac{\alpha+\beta}{m})_{n,k}(\frac{\alpha+\beta+k}{m})_{n,k},...,(\frac{\alpha+\beta+(m-1)k}{m})_{n,k}}\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$$

$$= kB_k(\alpha,\beta)t^{\frac{\alpha+\beta}{k}-1}{}_{p+m}F_{q+m,k}\left(a_1,a_2,...a_p,\frac{\alpha}{m},\frac{\alpha+k}{m},...\frac{\alpha+(m-1)k}{m}\right.$$
$$\left.;b_1,...b_q,\frac{\alpha+\beta}{m},\frac{\alpha+\beta+k}{m},...\frac{\alpha+\beta+(m-1)k}{m};ct^m\right)$$

Question:- show that

$$\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}\left[1-kx^2(t-x)^2\right]^{\frac{-1}{2}}dx=\frac{1}{2}\pi tF_k\left(\frac{k}{4},\frac{3k}{4};k;\frac{t^4}{16}\right)$$

Solution:-

$$L.H.S=\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}\left[1-kx^2(t-x)^2\right]^{\frac{-1}{2}}dx$$

as we know that

$$(1-ky)^{\frac{-a}{k}}=\sum_{n=0}^{\infty}\frac{(a)_{n,k}y^n}{n!}$$

$$\therefore\qquad \left[1-kx^2(t-x)^2\right]^{\frac{-k}{2k}}=\sum_{n=0}^{\infty}(\frac{k}{2})_{n,k}\left(\frac{x^2(t-x)^2}{n!}\right)^n={}_1F_{0,k}\left(\frac{k}{2};-;x^2(t-x)^2\right)$$

so,

$$L.H.S=\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}{}_1F_{0,k}\left(\frac{k}{2};-;x^2(t-x)^2\right)dx$$

now using the result

$$\int_0^t x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;cx^m(t-x)^s\right)dx=kB_k(\alpha,\beta)t^{\frac{\alpha+\beta}{k}-1}$$
$${}_{p+m+s}F_{q+m+s,k}\left(a_1,a_2,...,a_p,\frac{\alpha}{m},\frac{\alpha+k}{m},...,\frac{\alpha+(m-1)k}{m},\frac{\beta}{s},\frac{\beta+k}{s},...,\frac{\beta+(s-1)k}{s}\right.$$
$$\left.;b_1,...,b_q,\frac{\alpha+\beta}{m+s},\frac{\alpha+\beta+k}{m+s},...\frac{\alpha+\beta+(m+s-1)k}{m+s};\frac{m^ms^sct^{m+s}}{m+s^{m+s}}\right)$$

and comparing L.H.S of result with the above L.H.S

$\frac{\alpha}{k}-1=\frac{1}{2} \quad \Rightarrow \quad \alpha=\frac{3k}{2}$, $\frac{\beta}{k}-1=\frac{-1}{2} \quad \Rightarrow \beta=\frac{k}{2}$ $p=1, q=0, k=2, s=2, c=1$ so,

$$\int_0^t x^{\frac{1}{2}}(t-x)^{\frac{-1}{2}}{}_1F_{0,k}\left(\frac{k}{2};-;x^2(t-x)^2\right)dx = kB_k\left(\frac{3k}{2},\frac{k}{2}\right)t^{\frac{\frac{3k}{2}+\frac{k}{2}}{k}-1}{}_5F_{4,k}$$

$$\left(\frac{k}{2},\frac{\frac{3k}{2}}{2},\frac{\frac{3k}{2}+k}{2},\frac{\frac{k}{2}}{2},\frac{\frac{k}{2}+k}{2};-,\frac{\frac{3k}{2}+\frac{k}{2}}{4},\frac{\frac{3k}{2}+\frac{k}{2}+k}{4},\frac{\frac{3k}{2}+\frac{k}{2}+2k}{4},\frac{\frac{3k}{2}+\frac{k}{2}+3k}{4};\frac{2^2 2^2 t^4}{(2+2)^4}\right)$$

$$= \frac{\Gamma_k(\frac{3k}{2})\Gamma_k(\frac{k}{2})}{\Gamma_k(\frac{3k}{2}+\frac{k}{2})}kt{}_5F_{4,k}\left(\frac{k}{2},\frac{3k}{4},\frac{5k}{4},\frac{k}{4},\frac{3k}{4};-,\frac{k}{2},\frac{3k}{4},k,\frac{5k}{4};\frac{t^4}{16}\right)$$

$$= \frac{\Gamma_k(\frac{k}{2}+k)\Gamma_k(\frac{k}{2})}{\Gamma_k(2k)}kt\sum_{n=0}^{\infty}\frac{(\frac{k}{2})_{n,k}(\frac{3k}{4})_{n,k}(\frac{5k}{4})_{n,k}(\frac{k}{4})_{n,k}(\frac{3k}{4})_{n,k}}{(\frac{k}{2})_{n,k}(\frac{3k}{4})_{n,k}(k)_{n,k}(\frac{5k}{4})_{n,k}n!}(\frac{t^4}{16})^n$$

since

$\Gamma(2k)=1, \qquad \Gamma_k(k+z)=z\Gamma_k(z) \qquad \Gamma_k(\frac{k}{2})=\sqrt{\frac{\pi}{k}}$

therefore

$$\begin{aligned}
&= \frac{k}{2}\Gamma_k(\frac{k}{2})\Gamma_k(\frac{k}{2})kt\sum_{n=0}^{\infty}\frac{(\frac{k}{4})_{n,k}(\frac{3k}{4})_{n,k}}{(k)_{n,k}n!}(\frac{t^4}{16})^n \\
&= \frac{k}{2}\sqrt{\frac{\pi}{k}}\sqrt{\frac{\pi}{k}}tF_k\left(\frac{k}{4},\frac{3k}{4};k;\frac{t^4}{16}\right) \\
&= \frac{k}{2}\pi tF_k\left(\frac{k}{4},\frac{3k}{4};k;\frac{t^4}{16}\right)
\end{aligned}$$

Theorem 3.3.22. *Show that*

$$\frac{d}{dz}{}_pF_{q,k}\left((a_1,k),...,(a_p,k);(b_1,k),...,(b_q,k);z\right) = \frac{\prod_{m=1}^{p}a_m}{\prod_{j=1}^{q}b_j}$$

$${}_pF_{q,k}\left(a_1+k,...,a_p+k;b_1+k,...,b_q+k;z\right)$$

Proof:-

$$\begin{aligned} L.H.S &= \frac{d}{dz}{}_pF_{q,k}\left((a_1,k),...,(a_p,k);(b_1,k),...,(b_q,k);z\right) \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{d}{dz}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{nz^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{z^{n-1}}{(n-1)!} \end{aligned}$$

replace n by n+1

$$= \sum_{n=0}^{\infty}\frac{(a_1)_{n+1,k}...(a_p)_{n+1,k}}{(b_1)_{n+1,k}...(b_q)_{n+1,k}}\frac{z^n}{n!}$$

since

$$\begin{aligned} (a_1)_{n+1,k} &= a_1(a_1+k)(a_1+2k)...(a_1+(n+1-1)k) \\ &= a_1(a_1+k)(a_1+2k)...(a_1+nk) \\ &= a_1(a_1+k)_{n,k} \end{aligned}$$

similarly

$$(b_1)_{n+1,k} = b_1(b_1+k)_{n,k}$$

$$\therefore \qquad = \frac{\prod_{m=1}^{p} a_m}{\prod_{j=1}^{q} b_j}{}_pF_{q,k}\left(a_1+k,...,a_p+k;b_1+k,...,b_q+k;z\right)$$

Theorem 3.3.23.

$$\frac{d^n}{dz^n}{}_pF_{q,k}\left(a_1,...,a_p;b_1,...,b_q;z\right) = \frac{\prod_{m=1}^{p}(a_m)_{n,k}}{\prod_{j=1}^{q}(b_j)_{n,k}}{}_pF_{q,k}\left(a_1+nk,...,a_p+nk;b_1+nk,...,b_q+nk;z\right)$$

Proof:-

$$\begin{aligned} L.H.S &= \frac{d^n}{dz^n}{}_pF_{q,k}\left(a_1,...,a_p;b_1,...,b_q;z\right) \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{d^n}{dz^n}\frac{z^n}{n!} \\ &= \sum_{n=n}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{z^{n-n}n!}{n!} \\ &= \sum_{n=n}^{\infty}\frac{(a_1)_{n,k}...(a_p)_{n,k}}{(b_1)_{n,k}...(b_q)_{n,k}}\frac{z^{n-n}}{(n-n)!} \end{aligned}$$

replace n by n+n

$$= \sum_{n=0}^{\infty}\frac{(a_1)_{n+n,k}...(a_p)_{n+n,k}}{(b_1)_{n+n,k}...(b_q)_{n+n,k}}\frac{z^n}{n!}$$

since

$$\begin{aligned} (a_1)_{n,k}(a_1+nk)_{n,k} &= a_1(a_1+k)(a_1+2k)...(a_1+(n-1)k)(a_1+nk) \\ (a_1+nk+k)...(a_1+nk+(n-1)k) & \\ &= a_1(a_1+k)...(a_1+nk+nk-k) \\ &= (a_1)_{n+n,k} \end{aligned}$$

therefore

$$= \frac{\prod_{m=1}^{p}(a_m)_{n,k}}{\prod_{j=1}^{q}(b_j)_{n,k}}{}_pF_{q,k}\left(a_1+nk,...,a_p+nk;b_1+nk,...,b_q+nk;z\right)$$

Theorem 3.3.24. *Show that*

$${}_0F_{1,k}\left(-;a;x\right){}_0F_{1,k}\left(-;b;x\right) = {}_2F_{3,k}\left(\frac{1}{2}a+\frac{1}{2}b,\frac{1}{2}a+\frac{1}{2}b-\frac{k}{2};a,b,a+b-k;4x\right)$$

Proof:-

$$\begin{aligned} L.H.S &= {}_0F_{1,k}\left(-;a;x\right){}_0F_{1,k}\left(-;b;x\right) \\ &= \sum_{n=0}^{\infty}\frac{x^n}{(a)_{n,k}n!}\sum_{m=0}^{\infty}\frac{x^m}{(b)_{m,k}m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^{m+n}}{(a)_{n,k}(b)_{m,k}m!n!} \end{aligned}$$

since

$$(n-m)! = \frac{(-1)^m k^m (n!)}{(-nk)_{m,k}}$$

and

$$(a)_{n-m,k} = \frac{(-1)^m (a)_{n,k}}{(k-a-nk)_{m,k}}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_k(m,n-m)$$

so,

$$\begin{aligned} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^n}{(a)_{n-m,k}(b)_{m,k}m!(n-m)!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^n(k-a-nk)_{m,k}(-nk)_{m,k}}{(a)_{n,k}(b)_{m,k}k^m(-1)^m m!(-1)^m n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^n(k-a-n)_{m,k}(-nk)_{m,k}}{(a)_{n,k}(b)_{m,k}k^m m!n!} \\ &= \sum_{n=0}^{\infty} F_k\left((-nk,k),(k-a-nk,k);(b,k);\frac{1}{k}\right)\frac{x^n}{(a)_{n,k}n!} \end{aligned}$$

since

$$F_k\left((-nk,k),(k-a-nk,k);(b,k);\frac{1}{k}\right) = \frac{(b+a-k)_{2n,k}}{(b)_{n,k}(b+a-k)_{n,k}}$$

and

$$(a)_{2m,k} = 2^{2m}(\frac{a}{2})_{m,k}(\frac{a+k}{2})_{m,k}$$

$$\therefore \qquad = \sum_{n=0}^{\infty} \frac{(b+a-k)_{2n,k}}{(b)_{n,k}(b+a-k)_{n,k}} \frac{x^n}{(a)_{n,k}n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(\frac{b+a-k}{2})_{n,k}(\frac{b+a}{2})_{n,k}x^n}{(b)_{n,k}(b+a-k)_{n,k}(a)_{n,k}n!}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a+\frac{1}{2}b)_{n,k}(\frac{1}{2}a+\frac{1}{2}b-\frac{k}{2})_{n,k}}{(b)_{n,k}(b+a-k)_{n,k}(a)_{n,k}} \frac{(4x)^n}{n!}$$

$$= {}_2F_{3,k}\left(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}a+\frac{1}{2}b-\frac{k}{2}; a, b, a+b-k; 4x\right)$$

Theorem 3.3.25.

$$\frac{\Gamma_k(k+\frac{1}{2}a)}{\Gamma_k(k+a)} = \frac{cos\frac{\pi a}{2k}\Gamma_k(k-a)}{\Gamma_k(k-\frac{1}{2}a)}$$

and that

$$\frac{\Gamma_k(k-b+a)}{\Gamma_k(k-b+\frac{a}{2})} = \frac{sin\pi(\frac{b}{k}-\frac{a}{2k})\Gamma_k(b-\frac{a}{2})}{sin\pi(\frac{b-a}{k})\Gamma_k(b-a)}$$

and

$$F(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{cos\frac{\pi a}{2k}sin\pi(\frac{b}{k}-\frac{a}{2k})}{sin\pi(\frac{b-a}{k})}$$
$$\frac{\Gamma_k(k-a)\Gamma_k(b-\frac{a}{2})\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k-\frac{1}{2}a)\Gamma_k(b-a)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

Proof:-*Firstly we show that*

$$\frac{\Gamma_k(k+\frac{a}{2k})}{\Gamma_k(k+a)} = \frac{cos\frac{\pi a}{2k}\Gamma_k(k-a)}{\Gamma_k(k-\frac{1}{2}a)}$$

$$R.H.S = \frac{cos\frac{\pi a}{2k}\Gamma_k(k-a)}{\Gamma_k(k-\frac{1}{2}a)}$$

multiplying and dividing by $\Gamma_k(a)\Gamma_k(\frac{a}{2})$
since

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{ksin\frac{\pi z}{k}}$$

$$
\begin{aligned}
&= \frac{cos\frac{\pi a}{2k}\Gamma_k(k-a)\Gamma_k(a)\Gamma_k(\frac{a}{2})}{\Gamma_k(a)\Gamma_k(k-\frac{1}{2}a)\Gamma_k(\frac{a}{2})} \\
&= \frac{cos\frac{\pi a}{2k}\frac{\pi}{ksin\frac{\pi a}{k}}\Gamma_k(\frac{a}{2})}{\Gamma_k(a)\frac{\pi}{ksin\frac{\pi a}{2k}}} \\
&= \frac{\Gamma_k(\frac{a}{2})2sin\frac{\pi a}{2k}cos\frac{\pi a}{2k}}{\Gamma_k(a)2sin\frac{\pi a}{k}}
\end{aligned}
$$

since

$$2sinacosa = sin2a$$

$$\therefore \qquad = \frac{\Gamma_k(\frac{a}{2})sin2(\frac{\pi}{2k})}{\Gamma_k(a)2sin\frac{\pi a}{k}}$$

since

$$\Gamma_k(k+z) = z\Gamma_k(z)$$

$$
\begin{aligned}
\therefore \qquad &= \frac{1}{2}\frac{\Gamma_k(k+\frac{a}{2})}{\frac{a}{2}}\frac{a}{\Gamma_k(k+a)} \\
&= \frac{\Gamma_k(k+\frac{a}{2})}{\Gamma_k(k+a)} \\
&= L.H.S
\end{aligned}
$$

Now we show

$$\frac{\Gamma_k(k+a-b)}{\Gamma_k(k+\frac{1}{2}a-b)} = \frac{sin\pi(\frac{b}{k}-\frac{a}{2k})\Gamma_k(b-\frac{a}{2})}{sin\pi(\frac{b-a}{k})\Gamma_k(b-a)}$$

$$
\begin{aligned}
L.H.S &= \frac{sin\pi(\frac{b}{k}-\frac{a}{2k})\Gamma_k(b-\frac{a}{2})}{sin\pi(\frac{b-a}{k})\Gamma_k(b-a)} \\
&= \frac{\frac{\pi}{ksin\pi(\frac{b-a}{k})}}{\frac{\pi}{ksin\pi(\frac{b}{k}-\frac{a}{2k})}}\frac{\Gamma_k(b-\frac{a}{2})}{\Gamma_k(b-a)}
\end{aligned}
$$

since

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{ksin\frac{\pi z}{k}}$$

$$\begin{aligned}\therefore \qquad &= \frac{\Gamma_k(b-a)\Gamma_k(k-b+a)}{\Gamma_k(b-\frac{a}{2})\Gamma_k(k-b+\frac{a}{2})}\frac{\Gamma_k(b-\frac{a}{2})}{\Gamma_k(b-a)}\\ &= \frac{\Gamma_k(k+a-b)}{\Gamma_k(k+\frac{a}{2}-b)}\\ &= R.H.S\end{aligned}$$

now using Dixon's theorem

$$F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{\Gamma_k(k+\frac{1}{2}a)\Gamma_k(k+a-b)\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k+a)\Gamma_k(k+\frac{1}{2}a-b)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

since

$$\frac{\Gamma_k(k+\frac{1}{2}a)}{\Gamma_k(k+a)} = \frac{cos\frac{\pi a}{2k}\Gamma_k(k-a)}{\Gamma_k(k-\frac{1}{2}a)}$$

and

$$\frac{\Gamma_k(k-b+a)}{\Gamma_k(k-b+\frac{a}{2})} = \frac{sin\pi(\frac{b}{k}-\frac{a}{2k})\Gamma_k(b-\frac{a}{2})}{sin\pi(\frac{b-a}{k})\Gamma_k(b-a)}$$

therefore

$$F(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{cos\frac{\pi a}{2k}sin\pi(\frac{b}{k}-\frac{a}{2k})}{sin\pi(\frac{b-a}{k})}$$
$$\frac{\Gamma_k(k-a)\Gamma_k(b-\frac{a}{2})\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k-\frac{1}{2}a)\Gamma_k(b-a)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

Theorem 3.3.26.

$$F_k(-2nk,\alpha,k-\beta-2nk;k-\alpha-2nk,\beta;\frac{1}{k^2}) = \frac{k^n(2n)!(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{n!(\alpha)_{2n,k}(\beta)_{n,k}}$$

Proof:-*Consider*

$$F_k(-2nk,\alpha,k-\beta-2nk;k-\alpha-2nk,\beta;\frac{1}{k^2})$$

since

$$F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{cos\frac{\pi a}{2k}sin\pi(\frac{b}{k}-\frac{a}{2k})}{sin\pi(\frac{b-a}{k})}$$
$$\frac{\Gamma_k(k-a)\Gamma_k(b-\frac{a}{2})\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k-\frac{1}{2}a)\Gamma_k(b-a)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

Note that

$$a = -2nk$$

$$b = \alpha$$

$$c = k - \beta - 2nk$$

therefore

$$F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{cos(-\pi n)sin\pi(\frac{\alpha}{k}+n)\Gamma_k(k+2nk)\Gamma_k(\alpha+nk)\Gamma_k(\beta)}{sin\pi(\frac{\alpha}{k}+2n)\Gamma_k(k+nk)\Gamma_k(\alpha+2nk)\Gamma_k(\beta+nk)} \frac{\Gamma_k(\beta-\alpha+nk)}{\Gamma_k(\beta-\alpha)}$$

since

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}$$

and

$$\Gamma_k(nk) = k^{n-1}(n-1)!, k > 0, n \in N$$

therefore

$$\begin{aligned} F_k(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) &= \frac{(-1)^n sin\pi(\frac{\alpha+nk}{k})(2n)!k^{2n}(\alpha)_{n,k}\Gamma_k(\alpha)\Gamma_k(\beta)}{sin\pi(\frac{\alpha+2nk}{k})k^n n!(\alpha)_{2n,k}\Gamma_k(\alpha)(\beta)_{n,k}} \\ \frac{\Gamma_k(\beta-\alpha+nk)(\beta-\alpha)_{n,k}}{\Gamma_k(\beta)\Gamma_k(\beta-\alpha+nk)} & \\ &= \frac{(-1)^n sin\pi(\frac{\alpha+nk}{k})(2n)!k^{2n}(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{sin\pi(\frac{\alpha+2nk}{k})k^n n!(\alpha)_{2n,k}(\beta)_{n,k}} \\ &= \frac{(-1)^n \frac{\pi}{ksin\pi(\frac{\alpha+2nk}{k})}k^{2n}(2n)!(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{\frac{\pi}{ksin\pi(\frac{\alpha+nk}{k})}k^n n!(\alpha)_{2n,k}(\beta)_{n,k}} \end{aligned}$$

since

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{ksin\frac{\pi z}{k}}$$

therefore

$$\begin{aligned} &= \frac{(-1)^n\Gamma_k(\alpha+2nk)\Gamma_k(k-\alpha-2nk)(2n)!k^n(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{\Gamma_k(\alpha+nk)\Gamma_k(k-\alpha-nk)n!(\alpha)_{2n,k}(\beta)_{n,k}} \\ &= \frac{(-1)^n(\alpha)_{2n,k}\Gamma_k(\alpha)\Gamma_k(k-\alpha-2nk)(2n)!k^n(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{(\alpha)_{n,k}\Gamma_k(\alpha)\Gamma_k(k-\alpha-nk)n!(\alpha)_{2n,k}(\beta)_{n,k}} \end{aligned}$$

since

$$\frac{\Gamma_k(k-\alpha-nk)}{\Gamma_k(k-\alpha)} = \frac{(-1)^n}{(\alpha)_{n,k}}$$

$$= \frac{(-1)^n(\alpha)_{2n,k}\Gamma_k(\alpha)(\alpha)_{n,k}(-1)^{2n}\Gamma_k(k-\alpha)(2n)!k^n(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{(\alpha)_{n,k}\Gamma_k(\alpha)\Gamma_k(k-\alpha)(-1)^n(\alpha)_{2n,k}n!(\alpha)_{2n,k}(\beta)_{n,k}}$$
$$= \frac{k^n(2n)!(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{n!(\alpha)_{2n,k}(\beta)_{n,k}}$$

Ramanujan's k-theorem

Theorem 3.3.27.

$$F_k(\alpha;\beta;x)F_k(\alpha;\beta;\frac{-x}{k}) = F_k(\alpha,\beta-\alpha;\beta,\frac{1}{2}\beta,\frac{1}{2}\beta+\frac{k}{2};\frac{kx^2}{4})$$

Proof:-*since*

$$(n-m)! = \frac{(-1)^m k^m n!}{(-nk)_{m,k}}$$
$$(a)_{n-m,k} = \frac{(-1)^m (a)_{n,k}}{(k-a-nk)_{m,k}}$$

therefore

$$\begin{aligned}
F_k(\alpha;\beta;x)F_k(\alpha;\beta;\frac{-x}{k}) &= \sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}x^n}{(\beta)_{n,k}n!}\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(-x)^m}{(\beta)_{m,k}m!k^m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{n,k}(\alpha)_{m,k}(-1)^m x^{n+m}}{(\beta)_{n,k}(\beta)_{m,k}n!m!k^m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_{n-m,k}(\alpha)_{m,k}(-1)^m x^n}{(\beta)_{n-m,k}(\beta)_{m,k}(n-m)!m!k^m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(\alpha)_{m,k}(\alpha)_{n,k}(-1)^m(k-\beta-nk)_{m,k}(-nk)_{m,k}x^n}{(k-\alpha-nk)_{m,k}(\beta)_{m,k}(-1)^m(\beta)_{n,k}(-1)^m k^m n!m!k^m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_{m,k}(\alpha)_{n,k}(k-\beta-nk)_{m,k}(-nk)_{m,k}x^n}{(k-\alpha-nk)_{m,k}(\beta)_{m,k}(\beta)_{n,k}n!m!k^{2m}} \\
&= \sum_{n=0}^{\infty}F_k(-nk,\alpha,k-\beta-nk;\beta,k-\alpha-nk;\frac{1}{k^2})\frac{(\alpha)_{n,k}x^n}{(\beta)_{n,k}n!}
\end{aligned}$$

since

$$F_k(-2nk,\alpha,k-\beta-2nk;k-\alpha-2nk,\beta;\frac{1}{k^2}) = \frac{k^n(2n)!(\alpha)_{n,k}(\beta-\alpha)_{n,k}}{n!(\alpha)_{2n,k}(\beta)_{n,k}}$$

therefore

$$= \sum_{\frac{n}{2}=0}^{\infty}\frac{k^{\frac{n}{2}}(n)!(\alpha)_{\frac{n}{2},k}(\beta-\alpha)_{\frac{n}{2},k}}{(\frac{n}{2})!(\alpha)_{n,k}(\beta)_{\frac{n}{2},k}}\frac{(\alpha)_{n,k}x^n}{(\beta)_{n,k}n!}$$

$$= \sum_{\frac{n}{2}=0}^{\infty} \frac{k^{\frac{n}{2}}(\alpha)_{\frac{n}{2},k}(\beta-\alpha)_{\frac{n}{2},k}}{(\frac{n}{2})!(\beta)_{\frac{n}{2},k}} \frac{x^n}{(\beta)_{n,k}}$$

since

$$(\alpha)_{2n,k} = 2^{2n}(\frac{\alpha}{2})_{n,k}(\frac{\alpha+k}{2})_{n,k}$$

and

$$(\beta)_{2(\frac{n}{2}),k} = 2^{2(\frac{n}{2})}(\frac{\beta}{2})_{\frac{n}{2},k}(\frac{\beta+k}{2})_{\frac{n}{2},k} = 4^{\frac{n}{2}}(\frac{\beta}{2})_{\frac{n}{2},k}(\frac{\beta+k}{2})_{\frac{n}{2},k}$$

therefore

$$= \sum_{\frac{n}{2}=0}^{\infty} \frac{(\alpha)_{\frac{n}{2},k}(\beta-\alpha)_{\frac{n}{2},k}(x^2)^{\frac{n}{2}}k^{\frac{n}{2}}}{(\frac{n}{2})!(\beta)_{\frac{n}{2},k}4^{\frac{n}{2}}(\frac{\beta}{2})_{\frac{n}{2},k}(\frac{\beta+k}{2})_{\frac{n}{2},k}}$$

$$= \sum_{\frac{n}{2}=0}^{\infty} \frac{(\alpha)_{\frac{n}{2},k}(\beta-\alpha)_{\frac{n}{2},k}(\frac{x^2}{4})^{\frac{n}{2}}k^{\frac{n}{2}}}{(\frac{n}{2})!(\beta)_{\frac{n}{2},k}(\frac{\beta}{2})_{\frac{n}{2},k}(\frac{\beta+k}{2})_{\frac{n}{2},k}}$$

replacing $\frac{n}{2}$ *by* r

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_{r,k}(\beta-\alpha)_{r,k}(\frac{x^2}{4})^r k^r}{r!(\beta)_{r,k}(\frac{\beta}{2})_{r,k}(\frac{\beta+k}{2})_{r,k}}$$

$$= F_k(\alpha, \beta-\alpha; \beta, \frac{1}{2}\beta, \frac{1}{2}\beta + \frac{k}{2}; \frac{kx^2}{4})$$

Question:-Let

$$\Upsilon_{n,k} = F_k(-nk, k-a-nk, k-b-nk; a, b; \frac{1}{k^2})$$

show that

$$\Upsilon_{2n,k} = \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}}{n!(a)_{n,k}(b)_{n,k}(a+b-k)_{2n,k}}$$

Solution:-

$$\Upsilon_{2n,k} = F_k(-2nk, k-a-2nk, k-b-2nk; a, b; \frac{1}{k^2})$$

Since

$$F(a,b,c;k+a-b,k+a-c;\frac{1}{k^2}) = \frac{cos\frac{\pi a}{2k} sin\pi(\frac{b}{k}-\frac{a}{2k})}{sin\pi(\frac{b-a}{k})}$$

$$\frac{\Gamma_k(k-a)\Gamma_k(b-\frac{a}{2})\Gamma_k(k+a-c)\Gamma_k(k+\frac{1}{2}a-b-c)}{\Gamma_k(k-\frac{1}{2}a)\Gamma_k(b-a)\Gamma_k(k+\frac{1}{2}a-c)\Gamma_k(k+a-b-c)}$$

Note that

$$a = -2nk$$

$$b = k - a - 2nk$$

$$c = k - b - 2nk$$

therefore

$$F_k(-2nk, k-a-2nk, k-b-2nk; a, b; \frac{1}{k^2}) = \frac{cos\frac{\pi}{2k}(-2nk)sin\pi(\frac{k-a-2nk}{k}+n)}{sin\pi(\frac{k-a-2nk}{k}+2n)}$$
$$\frac{\Gamma_k(k+2nk)\Gamma_k(k-a-2nk+nk)\Gamma_k(k-2nk-k+b+2nk)}{\Gamma_k(k+nk)\Gamma_k(k-a-2nk+2nk)\Gamma_k(k-nk-k+b+2nk)}$$
$$\frac{\Gamma_k(k-nk-k+a+2nk-k+b+2nk)}{\Gamma_k(k-2nk-k+a+2nk-k+b+2nk)}$$

since

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}$$

and

$$\Gamma_k(nk) = k^{n-1}(n-1)!, k > 0, n \in N$$

therefore

$$\begin{aligned}
\Upsilon_{2n,k} &= \frac{cosn\pi sin\pi(\frac{k-a-nk}{k})k^{2n}(2n)!\Gamma_k(k-a-nk)\Gamma_k(b)\Gamma_k(a+b-k+3nk)\Gamma_k(a+b-k)}{k^n n! sin\pi(\frac{k-a}{k})\Gamma_k(k-a)\Gamma_k(b+nk)\Gamma_k(a+b-k+2nk)\Gamma_k(a+b-k)} \\
&= \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}\frac{\pi}{sin\pi(\frac{k-a}{k})}\Gamma_k(k-a-nk)}{n!(b)_{n,k}(a+b-k)_{2n,k}\frac{\pi}{sin\pi(\frac{k-a-nk}{k})}\Gamma_k(k-a)} \\
&= \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}\Gamma_k(k-a)\Gamma_k(k-k+a)\Gamma_k(k-a-nk)}{n!(b)_{n,k}(a+b-k)_{2n,k}\Gamma_k(k-a-nk)\Gamma_k(a+nk)\Gamma_k(k-a)} \\
&= \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}\Gamma_k(a)}{n!(b)_{n,k}(a+b-k)_{2n,k}\Gamma_k(a+nk)} \\
&= \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}}{n!(b)_{n,k}(a+b-k)_{2n,k}(a)_{n,k}}
\end{aligned}$$

Theorem 3.3.28.

$$F_k(-;a,b;t)F_k(-;a,b;\frac{-t}{k}) =_3 F_{8,k}\left(\frac{1}{3}(a+b-k),\frac{1}{3}(a+b),\frac{1}{3}(a+b+k);a,b,\frac{1}{2}a,\frac{1}{2}a+\frac{k}{2},\right.$$
$$\left.\frac{1}{2}b,\frac{1}{2}b+\frac{k}{2},\frac{1}{2}(a+b-k),\frac{1}{2}(a+b);\frac{-27kt^2}{64}\right)$$

Proof:-

$$\begin{aligned} F_k(-;a,b;t)F_k(-;a,b;\frac{-t}{k}) &= \sum_{n=0}^{\infty}\frac{t^n}{(a)_{n,k}(b)_{n,k}n!}\sum_{m=0}^{\infty}\frac{(-t^m)}{(a)_{m,k}(b)_{m,k}k^m m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m t^{n+m}}{(a)_{n,k}(b)_{n,k}(a)_{m,k}(b)_{m,k}k^m n!m!} \end{aligned}$$

since

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_k(m,n-m)$$

$$\therefore \qquad =\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m t^n}{(a)_{n-m,k}(b)_{n-m,k}(a)_{m,k}(b)_{m,k}(n-m)!k^m m!}$$

since

$$(n-m)!=\frac{(-1)^m k^m n!}{(-nk)_{m,k}}$$

and

$$(a)_{n-m,k}=\frac{(-1)^m(a)_{n,k}}{(k-a-nk)_{m,k}}$$

$$\begin{aligned} \therefore \qquad &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m t^n(k-a-nk)_{m,k}(k-b-nk)_{m,k}(-nk)_{m,k}}{(-1)^m(a)_{n,k}(-1)^m(b)_{n,k}(a)_{m,k}(b)_{m,k}(-1)^m k^m n!k^m m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{t^n(k-a-nk)_{m,k}(k-b-nk)_{m,k}(-nk)_{m,k}}{(a)_{n,k}(b)_{n,k}(a)_{m,k}(b)_{m,k}k^{2m}n!m!} \\ &= \sum_{n=0}^{\infty}F_k(-nk,k-a-nk,k-b-nk;a,b;\frac{1}{k^2})\frac{t^n}{n!(a)_{n,k}(b)_{n,k}} \end{aligned}$$

$$\because \qquad F_k(-2nk,k-a-2nk,k-b-2nk;a,b;\frac{1}{k^2})=\frac{(-1)^n k^n(2n)!(a+b-k)_{3n,k}}{n!(b)_{n,k}(a+b-k)_{2n,k}(a)_{n,k}}$$

$$\begin{aligned}
\therefore \qquad &= \sum_{n=0}^{\infty} \frac{(-1)^n k^n (2n)!(a+b-k)_{3n,k}}{n!(b)_{n,k}(a+b-k)_{2n,k}(a)_{n,k}} \frac{t^{2n}}{(2n)!(a)_{2n,k}(b)_{2n,k}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n k^n 3^{3n} (\frac{a+b-k}{3})_{n,k} (\frac{a+b}{3})_{n,k} (\frac{a+b+k}{3})_{n,k}}{n!(a)_{n,k}(b)_{n,k} 2^{2n} (\frac{1}{2}a)_{n,k} (\frac{1}{2}a+\frac{k}{2})_{n,k}} \frac{(t^2)^n}{2^{2n}(\frac{1}{2}b)_{n,k}(\frac{1}{2}b+\frac{k}{2})_{n,k} 2^{2n} (\frac{a+b-k}{2})_{n,k} (\frac{a+b}{2})_{n,k}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (27)^n k^n (\frac{a+b-k}{3})_{n,k} (\frac{a+b}{3})_{n,k} (\frac{a+b+k}{3})_{n,k}}{n!(a)_{n,k}(b)_{n,k}(\frac{1}{2}a)_{n,k}(\frac{1}{2}a+\frac{k}{2})_{n,k}} \frac{(t^2)^n}{(\frac{1}{2}b)_{n,k}(\frac{1}{2}b+\frac{k}{2})_{n,k}(\frac{a+b-k}{2})_{n,k}(\frac{a+b}{2})_{n,k}(64)^n} \\
&= \sum_{n=0}^{\infty} \frac{(\frac{a+b-k}{3})_{n,k}(\frac{a+b}{3})_{n,k}(\frac{a+b+k}{3})_{n,k}}{n!(a)_{n,k}(b)_{n,k}(\frac{1}{2}a)_{n,k}(\frac{1}{2}a+\frac{k}{2})_{n,k}} \frac{(-27kt^2)^n}{(\frac{1}{2}b)_{n,k}(\frac{1}{2}b+\frac{k}{2})_{n,k}(\frac{a+b-k}{2})_{n,k}(\frac{a+b}{2})_{n,k}(64)^n}
\end{aligned}$$

$$= {}_3F_{8,k}\left(\frac{1}{3}(a+b-k), \frac{1}{3}(a+b), \frac{1}{3}(a+b+k); a, b, \frac{1}{2}a, \frac{1}{2}a+\frac{k}{2}, \frac{1}{2}b, \frac{1}{2}b+\frac{k}{2}, \frac{1}{2}(a+b-k), \frac{1}{2}(a+b); \frac{-27kt^2}{64}\right)$$

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