Journal of Inequalities and Special Functions
ISSN: ..., URL: http://www.ilirias.com
Volume Issue (2013), Pages -.

A NOTE ON k-HYPERGEMETRIC DIFFERENTIAL EQUATIONS

SHAHID MUBEEN , MAMMONA NAZ , GAUHAR RAHMAN

Abstract. Mubeen [5] has recently introduced the second order linear differential equations of k-hypergeometric and confluent k-hypergeometric functions. The main objective of this present paper is to develop these k-hypergeometric and confluent k-hypergeometric differential equations by adopting the same method, used by Rainville in [6]. Also, we find out the solutions of the form $w =_2 F_{1,k}((\alpha, k),(\beta, k);(\gamma, k); z)$ and $w =_1 F_{1,k}((\alpha, k);(\beta, k); z)$ of k-hypergeometric and confluent k-hypergeometric differential equations respectively, in the same way.

1. Introduction

The hypergeometric function satisfies a linear second-order differential equation

$$z(1-z)w'' + [\gamma - (\alpha+\beta+1)z]w' - \alpha\beta w = 0 \tag{1.1}$$

which has three regular singular points 0, 1 and ∞. This equation was found by Euler [1] and was extensively studied by Gauss [2] and Kummer [3]. Riemann [7], introducing a more abstract approach, gave the generalization of the equation (1.1) to three arbitrary regular singular points. The confluent hypergeometric equation

$$zw'' + [\beta - z]w' - \alpha w = 0 \tag{1.2}$$

is obtained when we start with a linear second-order differential equation whose only singularities are regular singularities at 0, β and ∞; we let $\beta \to \infty$. The resulting equation has ∞ as an irregular singular point obtained from a confluence of two regular singularities. Thus the confluent equation can be derived from the hypergeometric equation by changing the independent variable z to z/β and letting $\beta \to \infty$. The solutions are ${}_1F_1$ functions. This equation was found by Kummer [4].

Submitted May 27, 2013. Published .

2. The k-Hypergeometric Differential Equation

In this section, we arise the k-hypergeometric differential equation by supposing ${}_2F_{1,k}\,((\alpha,k),(\beta,k);(\gamma,k);z)$ as one solution of the desired equation. The operator $\theta = \frac{d}{dz}$ is helpful in deriving a differential equation satisfied by k-hypergeometric function

$$w = {}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);z) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{z^n}{n!}. \tag{2.1}$$

From (2.1), we find that

$$\begin{aligned} k\theta(k\theta+\gamma-k)w &= k\theta(k\theta+\gamma-k)\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{z^n}{n!} \\ &= k\theta\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{(k\theta+\gamma-k)z^n}{n!}. \end{aligned}$$

Since $k\theta(z^n) = knz^n$ and $(k\theta+\gamma-k)z^n = (kn+\gamma-k)z^n$, therefore

$$\begin{aligned} k\theta(k\theta+\gamma-k)w &= k\theta\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{(kn+\gamma-k)z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}(kn+\gamma-k)k}{(\gamma)_{n,k}} \frac{\theta(z^n)}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}(kn+\gamma-k)}{(\gamma)_{n,k}} \frac{(knZ^n)}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n-1,k}} \frac{kz^n}{(n-1)!}. \end{aligned}$$

By shifting index, we obtain that

$$\begin{aligned} k\theta(k\theta+\gamma-k)w &= \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1,k}(\beta)_{n+1,k}}{(\gamma)_{n,k}} \frac{kz^{n+1}}{n!} \\ &= kz\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}(\alpha+nk)(\beta+nk)}{(\gamma)_{n,k}} \frac{z^n}{n!} \\ &= kz(\alpha+k\theta)(\beta+k\theta)\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{z^n}{n!}. \end{aligned}$$

This implies that

$$k\theta(k\theta+\gamma-k)w = kz(\alpha+k\theta)(\beta+k\theta)w.$$

As we have already shown that $w = {}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);z)$ is a solution of the k-differential equation, so the above equation can be written in the following form

$$[\theta(k\theta+\gamma-k) - z(k\theta+\alpha)(k\theta+\beta)]w = 0, \tag{2.2}$$

where $k \neq 0$ and $\theta = z\frac{d}{dz}$. Since

$$\begin{aligned} \theta(w) = \theta(w) &= zw' \\ \theta(\theta-1)w &= z^2w'' \end{aligned}$$

therefore the equation (2.2) can be rewritten in the form,

$$kz[kz(1-kz)w'' + [\gamma - (\alpha+\beta+k)kz]w' - \alpha\beta w] = 0$$

since $kz \neq 0$

$$kz(1-kz)w'' + [\gamma - (\alpha+\beta+k)kz]w' - \alpha\beta w = 0 \tag{2.3}$$

Conversely, We show that the $w =_2 F_{1,k}\left((\alpha,k),(\beta,k);(\gamma,k);z\right)$ is one solution of the k-hypergeometric differential equation (2.3). If $P_0(z) = -\alpha\beta$, $P_1(z) = \gamma - (\alpha + \beta + k)kz$ and $P_2(z) = kz(1-kz)$, then $P_2(0) = 0$. Hence, $z = 0$ is singular point. To see if it is regular,we study the following limits:

$$\lim_{z\longrightarrow a} \frac{(z-a)P_1(z)}{P_2(z)} = \lim_{z\longrightarrow 0} \frac{(z-0)(\gamma - (\alpha+\beta+kz)kz)}{kz(1-kz)} = \gamma/k$$

$$\lim_{z\longrightarrow a} \frac{(z-a)^2P_0(z)}{P_2(z)} = \lim_{z\longrightarrow 0} \frac{(z-0)^2(-\alpha\beta)}{kz(1-kz)} = 0.$$

Hence, both limits exist and $z = 0$ is a regular singular point.
Therefore,we assume the solution of the form

$$w = \sum_{n=0}^{\infty} d_n z^{n+r}$$

with $d_0 \neq 0$. This implies that

$$\begin{aligned} w' &= \sum_{n=0}^{\infty} d_n(n+r)z^{n+r-1} \\ w'' &= \sum_{n=0}^{\infty} d_n(n+r)(n+r-1)z^{n+r-2} \end{aligned}$$

Substituting these into the k-hypergeometric differential equation, we get the following

$$\begin{aligned} &k\sum_{n=0}^{\infty} d_n(n+r)(n+r-1)z^{n+r-1} - k^2\sum_{n=0}^{\infty} d_n(n+r)(n+r-1)z^{n+r} \\ &+\gamma\sum_{n=0}^{\infty} d_n(n+r)z^{n+r-1} - (\alpha+\beta+k)k\sum_{n=0}^{\infty} d_n(n+r)z^{n+r} - \alpha\beta\sum_{n=0}^{\infty} d_n z^{n+r} = 0. \end{aligned}$$

In order to simplify this equation, we need all powers to be the same (equal to $n + r - 1$), the smallest power. Hence, we switch the indices and obtain as follows:

$$\begin{aligned} &k\sum_{n=0}^{\infty} d_n(n+r)(n+r-1)z^{n+r-1} - k^2\sum_{n=1}^{\infty} d_{n-1}(n+r-1)(n+r-2)z^{n+r-1} \\ &+\gamma\sum_{n=0}^{\infty} d_n(n+r)z^{n+r-1} - (\alpha+\beta+k)k\sum_{n=1}^{\infty} d_{n-1}(n+r-1)z^{n+r-1} \\ &-\alpha\beta\sum_{n=1}^{\infty} d_{n-1}z^{n+r-1} = 0. \end{aligned}$$

Thus isolating the first term of the sums starting from 0, we get

$$d_0(kr(r-1)+\gamma r)z^{r-1}+k\sum_{n=1}^{\infty}d_n(n+r)(n+r-1)z^{n+r-1}$$
$$-k^2\sum_{n=1}^{\infty}d_{n-1}(n+r-1)(n+r-2)z^{n+r-1}+\gamma\sum_{n=1}^{\infty}d_n(n+r)z^{n+r-1}$$
$$-(\alpha+\beta+k)k\sum_{n=1}^{\infty}d_{n-1}(n+r-1)z^{n+r-1}-\alpha\beta\sum_{n=1}^{\infty}d_{n-1}z^{n+r-1}=0.$$

Now from the linear independence of all powers of z, that is, of the functions 1, z, z^2 etc., the coefficients of z^j vanish for all j. Hence, from the first term, we have

$$d_0(kr(r-1)+\gamma r)=0$$

which is the indicial equation.
Since $d_0\neq 0$, we have

$$kr(r-1)+\gamma r=0.$$

Hence,

$$r_1=0, r_2=1-\gamma/k$$

Also, from the rest of the terms, we have

$$kd_n(n+r)(n+r-1)-k^2d_{n-1}(n+r-1)(n+r-2)+\gamma d_n(n+r)$$
$$-(\alpha+\beta+k)kd_{n-1}(n+r-1)-\alpha\beta d_{n-1}=0$$
$$(n+r)(\gamma+k(n+r-1))d_n=(\alpha+k(n+r-1))(\beta+k(n+r-1))d_{n-1}$$
$$d_n=\frac{(\alpha+k(n+r-1))(\beta+k(n+r-1))}{(n+r)(\gamma+k(n+r-1))}d_{n-1}.$$

Hence, we get the recurrence relation,

$$d_n=\frac{(\alpha+k(n+r-1))(\beta+k(n+r-1))}{(n+r)(\gamma+k(n+r-1))}d_{n-1} \tag{2.4}$$

for $n\geq 1$. Let's now simplify this relation by giving d_n in terms of d_0 instead of d_{n-1}.
From the recurrence relation,

$$d_1=\frac{(\alpha+rk)(\beta+rk)}{(r+1)(\gamma+rk)}d_0$$
$$d_2=\frac{(\alpha+rk)_{2,k}(\beta+rk)_{2,k}}{(\gamma+rk)_{2,k}(r+1)_2}d_0$$

Similarly,

$$d_3=\frac{(\alpha+rk)_{3,k}(\beta+rk)_{3,k}}{(\gamma+rk)_{3,k}(r+1)_3}d_0$$

As we can see,

$$d_n=\frac{(\alpha+rk)_{n,k}(\beta+rk)_{n,k}}{(\gamma+rk)_{n,k}(r+1)_n}d_0 \tag{2.5}$$

for $n\geq 0$. Hence our assumed solution takes the form

$$w=d_0\sum_{n=0}^{\infty}\frac{(\alpha+rk)_{n,k}(\beta+rk)_{n,k}}{(\gamma+rk)_{n,k}}\frac{z^{n+r}}{(r+1)_n}. \tag{2.6}$$

By putting $r_1 = 0$ and $d_0 = 1$, we get w as

$$w =_2 F_{1,k}((\alpha, k), (\beta, k); (\gamma, k); z). \tag{2.7}$$

Remark: In this paper we conclude that if we take $k \to 1$, then we get Euler's hypergeometric differential equation for hypergeometric function ${}_2F_1$ given by Euler.

3. The Confluent k-Hypergeometric Differential Equation

The operator $\theta = \frac{d}{dz}$ is helpful in deriving a differential equation satisfied by confluent k-hypergeometric function

$$w = {}_1F_{1,k}((\alpha, k); (\beta, k); z) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n,k}} \frac{z^n}{n!}. \tag{3.1}$$

From (3.1), we find that

$$\begin{aligned} k\theta(k\theta + \beta - k)w &= k\theta \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n,k}} \frac{(kn + \beta - k)z^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n-1,k}} \frac{kz^n}{(n-1)!}. \end{aligned}$$

By shifting index, we obtain that

$$\begin{aligned} k\theta(k\theta + \beta - k)w &= \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1,k}}{(\beta)_{n,k}} \frac{kz^{n+1}}{n!} \\ &= kz(\alpha + k\theta) \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}}{(\beta)_{n,k}} \frac{z^n}{n!}. \end{aligned}$$

This gives the following form

$$k\theta(k\theta + \beta - k)w = kz(\alpha + k\theta)w.$$

As we have already shown that $w = {}_1F_{1,k}((\alpha, k); (\beta, k); z)$ is a solution of the differential equation

$$[\theta(k\theta + \beta - k) - z(k\theta + \alpha))]w = 0 \tag{3.2}$$

where $\theta = z\frac{d}{dz}$ and $k \neq 0$. Since

$$\begin{aligned} \theta(w) &= zw' \\ \theta(\theta - 1)w &= z^2 w'' \end{aligned}$$

Thus equation (3.2) can be rewritten in the form

$$z[kzw'' + (\beta - kz)w' - \alpha w] = 0.$$

Since $z \neq 0$, therefore we get the desired confluent k-hypergeometric differential equation

$$kzw'' + (\beta - kz)w' - \alpha w = 0 \tag{3.3}$$

Note: To find solution of the above confluent k-hypergeometric differential equation, we use the same proceeder as in section 2.
Remark: In this paper, we conclude that if we take $k \to 1$, then we get Kummer's equation for confluent hypergeometric function ${}_1F_1$.

References

[1] L. Euler, *Opera Omnia Ser*, 1(1769), 11-13.
[2] C. F. Gauss, Comm. Soc. Reg. Sci. II 3 (1812), 123-162.
[3] E. J. Kummer, Riene Ang. Math. 15 (1836), 39-83.
[4] E. E. Kummer, *De integralibus quibusdam definitis et seriebus infinitis*, für die reine und angewandte Mathematik (in Latin) 17 (1837), 28-242.
[5] S. Mubeen, *Solution of some integral equations involving confluent k-hypergeometric functions*, Applied Mathematics, 4 (2013), no. 7A, 9-11.
[6] E. D. Rainville, *Special Funtions*, Macmillan Company, New York, 1960.
[7] B. Riemann, K. Gess. Wiss. Goett. 7 (1857), 1-24.

Corresponding Author
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
E-mail address: smjhanda@gmail.com