# TENSOR ANALYSIS

Introduction and properties

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# Tensor

- Tensor are independent under coordinate transformation
- Magnitude but two or three direction.
- Describe the mathematical form of suitable natural law w.r.t frame of references.
- Tensor are arrays of numbers which transform in certain ways under coordinate transformation.
- Tensor is a multi-dimensional array of numbers.
- Scalar has zero tensor, vector is 2 rank and matrix is 2 rank of tensor .

**Introduction:**

Tensor Analysis is a generalization form of vector analysis. Its allows complex mathematical and physical relationship can be expressed in a compact way. Its is the great use of mechanics, fluid dynamics, elasticity, differential geometry, general relativity theory and many others fields of science and engineering. In this course we will discuss about Cartesian tensors.

**Cartesian tensor:** tensor which are expressed in terms of components transformed to rectangular Cartesian coordinate system. The notation of Cartesian tensor is

$$a_1x_1 + a_2x_2 + a_3x_3$$

Which can be written as $\sum_{j=1}^{3} a_j x_j$

**Dummy and Free Indices:**

An index which is repeated in a given expression so that the summation convention applies, is called a dummy index, while an index occurring only once in a given expression is called a free index and not imply any summation

**FOR EXAMPLE** :

In the expression $A_k B_{j,k}$ k is dummy index while j is a free index .

EAXAMPLE 01: Write each of the following using summation convention.

*i.* $a_{11}x_1 + a_{12}x_2 + a_{13}x_3$.

ii. $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13}$.

iii. $(x_1)^2+(x_2)^2+(x_3)^2$.

iv.

Solution:

*1.* $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = a_{1i}x_i$.

2. $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} = a_{1i}b_{1i}$.

3. $(x_1)^2+(x_2)^2+(x_3)^2= x_1x_1+x_2x_2+x_3x_3= x_ix_i$.

*4.* $\frac{\partial\varphi}{\partial x_1}dx_1+\frac{\partial\varphi}{\partial x_2}dx_2+\frac{\partial\varphi}{\partial x_3}dx_3 = \frac{\partial\varphi}{\partial x_i}dx_i$

- **Double Sums:**

An expression can involve more than one summation indices. For example $a_{ij}x_ix_j$ and taking summation on both $i$ and $j$ simultaneously. If an expression has two summation indices, there will be a total of $3^2$ terms in the sum; if there are three indices , there will be a total of $3^2$ $and\ so\ on$ .

**EXAMPLE 03: Write the terms in the expression $\boldsymbol{a_{ij}x_ix_j; i\ , j = 1, 2, 3}$**

Sol:

$$a_{ij}x_ix_j \quad = a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j$$
$$= a_{11}x_1x_1 + a_{21}x_2x_1 + a_{31}x_3x_1 + a_{12}x_1x_2 + a_{22}x_2x_2 + a_{32}x_3x_2$$
$$+ a_{13}x_1x_3 + a_{23}x_2x_3 + a_{33}x_3x_3.$$

Rewrite the system

$$a_{ij}x_ix_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3$$
$$+ a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3.$$

**EXAMPLE 04:** Write the following expression using summation convention.

$$\boldsymbol{a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} + a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} + a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33}}$$

**Sol:**

The given expression can be written as:

$=(a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13}) + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23}) + (a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33})$

$= a_{i1}b_{1i} + a_{i2}b_{2i} + a_{i3}b_{3i}$

$= a_{ij}b_{ji}$

- **SUBSTITUTION:**

consider the equation Q = $b_{ij}y_i x_j$.

Suppose it is required to substitute $y_i = a_{ir}x_r$ *in the* given equation

Q = $b_{ij}a_{ir}x_r\, x_j$. = $a_{ir}b_{ij}x_r\, x_j$

**ALGEBRA AND THE SUMMATION CONVENTION:**

Algebraic manipulations in tensors can be easily justified by properties of ordinary
There are several valid identities.

1. $a_{ij}(x_j + y_j) \equiv a_{ij}x_j + a_{ij}y_j$
2. $a_{ij}x_iy_j \equiv a_{ij}y_jx_i$
3. $a_{ij}x_ix_j \equiv a_{ji}x_ix_j$
4. $(a_{ij}+a_{ji})\,x_ix_j \equiv 2\,a_{ji}\,x_ix_j$
5. $(a_{ij} - a_{ji})\,x_ix_j \equiv 0$
6.