Interval Estimation Procedures

We are now ready to estimate population values based on sample statistics. We will do this by constructing confidence intervals or statements that say that the parameter is within a certain range of values. Confidence intervals are constructed in three steps.

The first step is to decide on the risk that you are willing to take of being wrong. An estimate is wrong if it does not include the population parameter. This probability of error is called **alpha** (symbolized as α). The exact value of alpha will depend on the nature of the research situation, but a 0.05 probability is commonly used. Setting alpha equal to 0.05—also called using the 95% **confidence level**—means that over the long run, the researcher is willing to be wrong only 5% of the time. Or to put it another way, if an infinite number of intervals were constructed at this alpha level (and with all other things being equal), 95% of them would contain the population value and 5% would not. In reality, of course, only one interval is constructed, and by setting the probability of error very low, we are setting the odds in our favor that the interval will include the population value.

The second step is to picture the sampling distribution, divide the probability of error equally into the upper and lower tails of the distribution, and then find the corresponding Z score. For example, if we decided to set alpha equal to 0.05, we would place half (0.025) of this probability in the lower tail and half in the upper tail of the distribution. The sampling distribution would thus be divided as illustrated in Figure 6.7.

We need to find the Z score that marks the beginnings of the shaded areas in Figure 6.7. In Chapter 5, we learned how to first calculate a Z score and then find an area under the normal curve. Here, we will reverse that process. We need to find the Z score beyond which lies a proportion of .0250 of the total area. To do this, go down column c of Appendix A until you find this proportional value (.0250). The associated Z score is 1.96. Because the curve is symmetrical and we are interested in the upper and lower tails, we designate the Z score that corresponds to an alpha of .05 as ± 1.96 (see Figure 6.8).

We now know that 95% of all possible sample outcomes fall within ± 1.96 Z score units of the population value. In reality, of course, there is only one sample

FIGURE 6.7 The Sampling Distribution With Alpha (α) Equal to 0.05

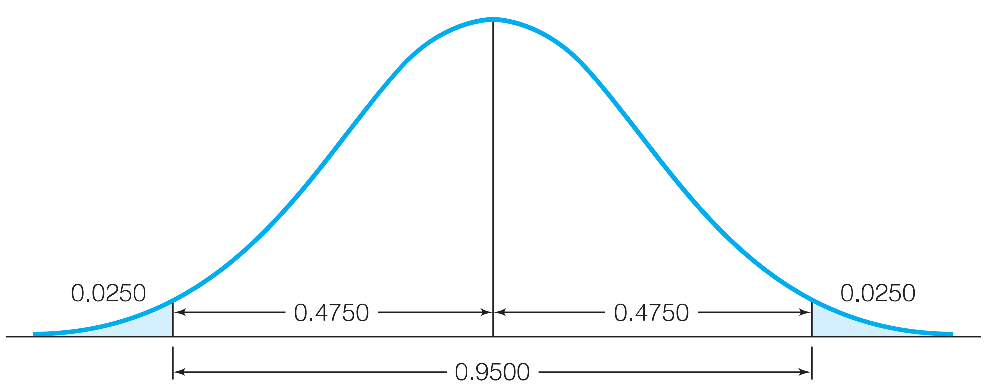
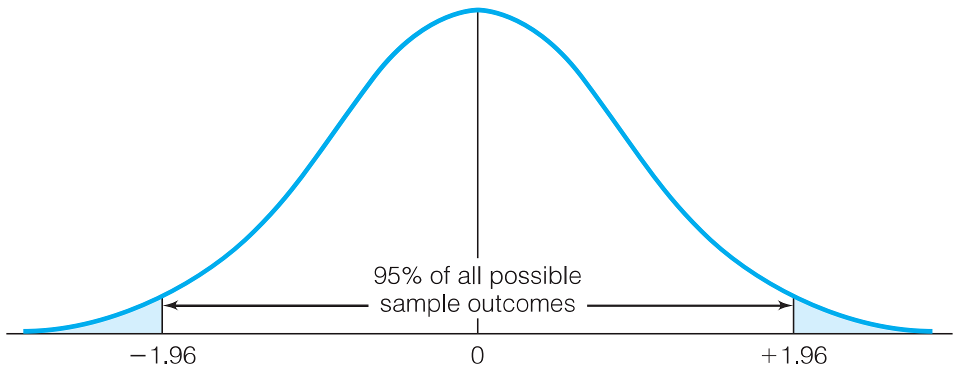


FIGURE 6.8 Finding the *Z* Score That Corresponds to an Alpha (α) of 0.05



outcome, but if we construct an interval estimate based on ±1.96 *Z*s, the probability is that 95% of all such intervals will trap the population value. Thus, we can be 95% confident that our interval contains the population value.

Besides the 95% level, there are four other commonly used confidence levels—all listed in the left-hand column of Table 6.2, along with the 95% level. Note the relationship between the confidence levels (expressed as percentages) and the alpha levels (expressed as proportions); they are different ways of saying the same thing. The alpha level expresses the probability that the confidence interval will be *wrong* and *will not* include the population value. Confidence levels express our confidence that the interval is *correct* and *will* include the parameter in which we are interested. To find the corresponding *Z* scores for any levels, follow the procedures outlined above for an alpha of 0.05. Table 6.2 summarizes all the information you will need.

You should turn to Appendix A and confirm for yourself that the *Z* scores in Table 6.2 do indeed correspond to these alpha levels. As you do, note that in the cases where alpha is set at 0.10 and 0.01, the precise areas we seek do not appear in the table. For example, with an alpha of 0.10, we would look in column c ("Area Beyond *Z*") for the area 0.0500. Instead, we find an area of 0.0505 ($Z = \pm 1.64$) and an area of 0.0495 ($Z = \pm 1.65$). The *Z* score we are seeking is somewhere between these two other scores. When this condition occurs, take the larger of the two scores as *Z*. This will make the interval as wide as possible under the circumstances and is thus the most conservative course of action. In the case of an alpha of 0.01, we encounter the same problem (the exact area .0050 is not in the table); resolve it the same way and take the larger score as *Z*. For the alpha of 0.001, we take the largest of the several Z scores listed for the area as our

TABLE 6.2 *Z* Scores for Various Levels of Alpha (α)

Confidence Level	Alpha (α)	$\alpha/2$	Z Score
90%	0.10	0.0500	±1.65
95%	0.05	0.0250	±1.96
99%	0.01	0.0050	±2.58
99.9%	0.001	0.0005	±3.32
99.99%	0.0001	0.00005	±3.90

Z score. Finally, for the lowest alpha of 0.0001, the table is not detailed enough to show exact areas, and we will use ± 3.90 as our Z score. *(For practice in finding Z scores for various levels of confidence, see problem 6.3.)*

The third step is to actually construct the confidence interval. In the sections that follow, we illustrate how to construct an interval estimate—first with sample means and then with sample proportions.

Interval Estimation Procedures for Sample Means (Large Samples)

The formula for constructing a confidence interval based on sample means is given in Formula 6.1:

FORMULA 6.1

$$c.i. = \overline{X} \pm Z\left(\frac{\sigma}{\sqrt{N}}\right)$$

Where: $c.i.$ = confidence interval
$\overline{X}$ = the sample mean
Z = the Z score as determined by the alpha level
$\frac{\sigma}{\sqrt{N}}$ = the standard deviation of the sampling distribution or the standard error of the mean

As an example, suppose you wanted to estimate the average IQ of a community and had randomly selected a sample of 200 residents, with a sample mean IQ of 105. Assume that the population standard deviation for IQ scores is about 15, so we can set σ equal to 15. If we are willing to run a 5% chance of being wrong and set alpha at 0.05, the corresponding Z score will be 1.96. These values can be directly substituted into Formula 6.1 and an interval can be constructed:

$$c.i. = \overline{X} \pm Z\left(\frac{\sigma}{\sqrt{N}}\right)$$

$$c.i. = 105 \pm 1.96\left(\frac{15}{\sqrt{200}}\right)$$

$$c.i. = 105 \pm 1.96\left(\frac{15}{14.14}\right)$$

$$c.i. = 105 \pm (1.96)(1.06)$$

$$c.i. = 105 \pm 2.08$$

That is, our estimate is that the average IQ for the population in question is somewhere between 102.92 (105 − 2.08) and 107.08 (105 + 2.08). Because 95% of all possible sample means are within ± 1.96 Zs (or 2.08 IQ units in this case) of the mean of the sampling distribution, the odds are very high that our interval will contain the population mean. In fact, even if the sample mean is as far off as ± 1.96 Zs (which is unlikely), our interval will still contain $\mu_{\overline{X}}$ and, thus, μ. Only if our sample mean is one of the few that is more than ± 1.96 Zs from the mean of the sampling distribution will we have failed to include the population mean.

Note that in this example the value of the population standard deviation was supplied. Needless to say, it is unusual to have such information about a population. In the great majority of cases, we will have no knowledge of σ. However, in such cases, we can estimate σ with s, the sample standard deviation. Unfortunately, s is a biased estimator of σ, and the formula must be changed slightly to correct for the bias. For larger samples, the bias of s will not affect the interval very much. The revised formula for cases in which σ is unknown is:

FORMULA 6.2

$$c.i. = \overline{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

In comparing this formula with 6.1, note that there are two changes. First, σ is replaced by s, and second, the denominator of the last term is the square root of $N - 1$ rather than the square root of N. The latter change is the correction for the fact that s is biased.

Let me stress here that the substitution of s for σ is permitted only for large samples (that is, samples with 100 or more cases). For smaller samples, when the value of the population standard deviation is unknown, the standardized normal distribution summarized in Appendix A cannot be used in the estimation process. To construct confidence intervals from sample means with samples smaller than 100, we must use a different theoretical distribution, called the Student's t distribution, to find areas under the sampling distribution. We will defer the presentation of the t distribution until Chapter 7 and confine our attention here to estimation procedures for large samples only.

We will close this section by working through a sample problem with Formula 6.2. Average income for a random sample of a particular community is \$45,000, with a standard deviation of \$200. What is the 95% interval estimate of the population mean, μ?

Given that

$$\overline{X} = \$45{,}000$$

$$s = \$200$$

$$N = 500$$

and using an alpha of 0.05, the interval can be constructed:

$$c.i. = \overline{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

$$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{\sqrt{499}}\right)$$

$$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{22.34}\right)$$

$$c.i. = 45{,}000 \pm (1.96)(8.95)$$

$$c.i. = 45{,}000 \pm 17.55$$

ONE STEP AT A TIME — Constructing Confidence Intervals for Sample Means by Using Formula 6.2

Step	***Operation***
1.	Select an alpha level and then find the associated *Z* score in Table 6.1. If you use the conventional alpha level of 0.05, the *Z* score is ±1.96.
2.	Substitute the sample values into Formula 6.2.

To Solve Formula 6.2

1.	Find the square root of $N - 1$.
2.	Divide the value you found in step 1 into *s*, the sample standard deviation.
3.	Multiply the value you found in step 2 by the value of *Z*.
4.	The value you found in step 3 is the width of the confidence interval. To find the lower and upper limits of the interval, subtract and add this value to the sample mean.

Interpreting the Confidence Interval

1. Express the confidence interval in a sentence or two that identifies each of these elements:
   a. The sample statistic (a mean in this case)
   b. The confidence interval
   c. The sample size (*N*)
   d. The population for which you are estimating
   e. The confidence level (e.g., 95%)

The confidence interval we constructed in this section could be expressed as: "The average income for this community is \$45,000 ± \$17.55. This estimate is based on a sample of 500 respondents, and we can be 95% confident that the interval estimate is correct."

Applying Statistics 6.1 Estimating a Population Mean

A study of the leisure activities of Americans was conducted on a sample of 1,000 households. The respondents identified television viewing as a major form of recreation. If the sample reported an average of 6.2 hours of television viewing a day, what is the estimate of the population mean? The information from the sample is:

$$\bar{X} = 6.2$$
$$s = 0.7$$
$$N = 1{,}000$$

If we set alpha at 0.05, the corresponding *Z* score will be ±1.96 and the 95% confidence interval will be:

$$c.i. = \bar{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

$$c.i. = 6.2 \pm 1.96\left(\frac{0.7}{\sqrt{1{,}000-1}}\right)$$

$$c.i. = 6.2 \pm 1.96\left(\frac{0.7}{31.61}\right)$$

$$c.i. = 6.2 \pm (1.96)(0.02)$$

$$c.i. = 6.2 \pm 0.04$$

Based on this result, we would estimate that the population spends an average of 6.2 ± .04 hours per day viewing television. The lower limit of our interval estimate (6.2 − 0.04) is 6.16 and the upper limit (6.2 + 0.04) is 6.24. Thus, another way to state the interval would be:

$$6.16 \leq \mu \geq 6.24$$

The population mean is greater than or equal to 6.16 and less than or equal to 6.24. Because alpha was set at the .05 level, this estimate has a 5% chance of being wrong (that is, of not containing the population mean).

The average income for the community as a whole is between \$44,982.45 (45,000 − 17.55) and \$45,017.55 (45,000 + 17.55). Remember that this interval has only a 5% chance of being wrong (that is, of not containing the population mean). See the "One Step at a Time" box for instructions on reporting the confidence interval clearly and completely. *(For practice in constructing and expressing confidence intervals for sample means, see problems 6.1, 6.4–6.7, and 6.18a–6.18c.)*

Interval Estimation Procedures for Sample Proportions (Large Samples)

Estimation procedures for sample proportions are essentially the same as those for sample means. The major difference is that because proportions are different statistics, we must use a different sampling distribution. In fact, again based on the central limit theorem, we know that sample proportions have sampling distributions that are normal in shape, with means (μ_p) equal to the population value (P_u) and standard deviations (σ_p) equal to $\sqrt{\frac{P_u(1 - P_u)}{N}}$. The formula for constructing confidence intervals based on sample proportions is:

FORMULA 6.3

$$c.i. = P_s \pm Z\sqrt{\frac{P_u(1 - P_u)}{N}}$$

The values for P_s and N come directly from the sample and the value of Z is determined by the confidence level, as was the case with sample means. This leaves one unknown in the formula: P_u—the same value we are trying to estimate. This dilemma can be resolved by setting the value of P_u at 0.5. Because the second term in the numerator under the radical $(1 - P_u)$ is the reciprocal of P_u, the entire expression will always have a value of 0.5×0.5, or 0.25, which is the maximum value this expression can attain. That is, if we set P_u at any value other than 0.5, the expression $P_u(1 - P_u)$ will decrease in value. For example, if we set P_u at 0.4, the second term $(1 - P_u)$ would be 0.6 and the value of the entire expression would decrease to 0.24. Setting P_u at 0.5 ensures that the expression $P_u(1 - P_u)$ will be at its maximum possible value, and consequently, the interval will be at maximum width. This is the most conservative solution possible to the dilemma posed by having to assign a value to P_u in the estimation equation.

To illustrate these procedures, assume you wish to estimate the proportion of students at your university who missed at least one day of classes because of illness last semester. Out of a random sample of 200 students, 60 reported that they had been sick enough to miss classes at least once during the previous semester. The sample proportion on which we will base our

STATISTICS
IN EVERYDAY LIFE

What are the happiest days of the year for Americans? The Gallup poll has been tracking people's happiness since October 2009 and found that during that period, the days on which the most Americans felt "happiness and enjoyment without stress and worry" were Christmas 2010 (65%), Easter 2011 (64%), Mother's Day 2010 (63%), and the 4th of July 2010 (63%). These results are based on daily samples of about 1,000 Americans, are accurate to within ±3%, and have a 95% confidence level.

How could these results be explained? Why would these particular holidays be the most enjoyable? What additional information would you like to have to begin to answer these questions?

The complete data set is available at http://www.gallup.com/poll/106915/Gallup-Daily-US-Mood.aspx.

estimate is thus 60/200, or 0.30. At the 95% level, the interval estimate will be

$$c.i. = P_s \pm Z\sqrt{\frac{P_u(1 - P_u)}{N}}$$
$$c.i. = 0.30 \pm 1.96\sqrt{\frac{(0.5)(0.5)}{200}}$$
$$c.i. = 0.30 \pm 1.96\sqrt{\frac{0.25}{200}}$$
$$c.i. = 0.30 \pm 1.96\sqrt{0.00125}$$
$$c.i. = 0.30 \pm (1.96)(0.035)$$
$$c.i. = 0.30 \pm 0.07$$

Based on the sample proportion of 0.30, you would estimate that the proportion of students who missed at least one day of classes because of illness was between 0.23 and 0.37. The estimate could, of course, also be phrased in percentages by reporting that between 23% and 37% of the student body was affected by illness at least once during the past semester.

As was the case with sample means, the final step in the process is to express the confidence interval in a way that is easy to understand and that includes all relevant information. See the "One Step at a Time" box for guidelines and an example. *(For practice with confidence intervals for sample proportions, see problems 6.2, 6.8–6.12, 6.16–6.17, and 6.18d–6.18g.)*

ONE STEP AT A TIME **Constructing Confidence Intervals for Sample Proportions**

Step	***Operation***
1.	Select an alpha level and then find the associated Z score in Table 6.1. If you use the conventional alpha level of 0.05, the Z score is ±1.96.

(*continued next page*)

ONE STEP AT A TIME *(continued)*

Step	***Operation***
	To Solve Formula 6.3
1.	Substitute the value 0.25 for the expression $P_u(1 - P_u)$ in the numerator of the fraction under the square root sign.
2.	Divide N into 0.25.
3.	Find the square root of the value you found in step 2.
4.	Multiply the value you found in step 3 by the value of Z.
5.	The value you found in step 4 is the width of the confidence interval. To find the lower and upper limits of the interval, subtract and add this value to the sample proportion.
	Interpreting the Confidence Interval
1.	Express the confidence interval in a sentence or two that identifies each of these elements: a. The sample statistic (a proportion in this case) b. The confidence interval c. The sample size (N) d. The population for which you are estimating e. The confidence level (e.g., 95%)

The confidence interval we constructed in this section could be expressed as: "On this campus, 30% ± 7% of students were sick enough to miss class at least once during the semester. This estimate is based on a sample of 200 respondents, and we can be 95% confident that the interval estimate is correct."

Applying Statistics 6.2 Estimating Population Proportions

A total of 2,164 adult Canadians and 1,249 residents of the United States were randomly selected to participate in a study of attitudes and values, including their ideas about marriage and the family. One item asked if they agreed or disagreed with the idea that "marriage is an outdated institution." The results, expressed in proportions, are:

	Canada	U.S.
N	2,164	1,249
Proportion Agreeing	0.22	0.13

For Canadians, the confidence interval estimate to the population at the 95% confidence level is:

$$c.i. = P_s \pm Z\sqrt{\frac{P_u(1 - P_u)}{N}}$$

$$c.i. = 0.22 \pm 1.96\sqrt{\frac{0.50(1 - 0.50)}{2164}}$$

$$c.i. = 0.22 \pm 1.96\sqrt{\frac{0.25}{2164}}$$

$$c.i. = 0.22 \pm 1.96\sqrt{0.0001}$$

$$c.i. = 0.22 \pm (1.96)(0.01)$$

$$c.i. = 0.22 \pm 0.02$$

Expressing these results in terms of percentages, we can conclude that between 20% and 24% of adult Canadians agree that marriage is an outdated institution. This estimate is based on a sample of 2,164 and is constructed at the 95% confidence level.

For residents of the U.S., the confidence interval estimate to the population at the 95% confidence level is:

$$c.i. = P_s \pm Z\sqrt{\frac{P_u(1 - P_u)}{N}}$$

$$c.i. = 0.13 \pm 1.96\sqrt{\frac{0.50(1 - 0.50)}{1249}}$$

Applying Statistics 6.2 *(continued)*

$$c.i. = 0.13 \pm 1.96\sqrt{\frac{0.25}{1249}}$$

$$c.i. = 0.13 \pm 1.96\sqrt{0.0002}$$

$$c.i. = 0.13 \pm (1.96)(0.014)$$

$$c.i. = 0.13 \pm 0.03$$

Again, expressing results in terms of percentages, we can conclude that between 10% and 16% of adult Americans agree that marriage is an outdated institution. This estimate is based on a sample of 1,249 and is constructed at the 95% confidence level.

Source: World Values Survey, http://worldvaluessurvey.com.

Computing Confidence Intervals: A Summary

To this point, we have covered the construction of confidence intervals for sample means and sample proportions. In both cases, the procedures assume large samples (N greater than 100). The procedures for constructing confidence intervals for small samples are not covered in this text. Table 6.3 presents the three formulas for confidence intervals organized by the situations in which they are used. For sample means, when the population standard deviation is known, use Formula 6.1. When the population standard deviation is unknown (which is the usual case), use Formula 6.2. For sample proportions, always use Formula 6.3.

Controlling the Width of Interval Estimates

The width of a confidence interval for either sample means or sample proportions can be partly controlled by manipulating two terms in the equation. First, the confidence level can be raised or lowered, and second, the interval can be widened or narrowed by gathering samples of different size. The researcher alone determines the risk he or she is willing to take of being wrong (that is, of not including the population value in the interval estimate). The exact confidence level (or alpha level) will depend, in part, on the purpose of the research. For example, if potentially harmful drugs were being tested, the researcher would naturally demand very high levels of confidence (99.99%

TABLE 6.3 Choosing Formulas for Confidence Intervals

If the sample statistic is a	and	use formula	
mean	the population standard deviation is known	6.1	$c.i. = \bar{X} \pm Z\left(\frac{\sigma}{\sqrt{N}}\right)$
mean	the population standard deviation is unknown	6.2	$c.i. = \bar{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$
proportion		6.3	$c.i. = P_s \pm Z\sqrt{\frac{P_u(1-P_u)}{N}}$

or even 99.999%). On the other hand, if intervals are being constructed only for loose "guesstimates," then much lower confidence levels can be tolerated (such as 90%).

The relationship between interval size and confidence level is that intervals widen as confidence levels increase. This relationship should make intuitive sense. Wider intervals are more likely to trap the population value; hence, more confidence can be placed in them.

To illustrate this relationship, let us return to the example where we estimated the average income for a community. In this problem, we were working with a sample of 500 residents, and the average income for this sample was \$45,000, with a standard deviation of \$200. We constructed the 95% confidence interval and found that it extended 17.55 around the sample mean (that is, the interval was \$45,000 ± 17.55).

If we had constructed the 90% confidence interval for these sample data (a lower confidence level), the Z score in the formula would have decreased to ± 1.65 and the interval would have been narrower:

$$c.i. = \bar{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

$$c.i. = 45{,}000 \pm 1.65\left(\frac{200}{\sqrt{499}}\right)$$

$$c.i. = 45{,}000 \pm 1.65(8.95)$$

$$c.i. = 45{,}000 \pm 14.77$$

On the other hand, if we had constructed the 99% confidence interval, the Z score would have increased to ± 2.58 and the interval would have been wider:

$$c.i. = \bar{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

$$c.i. = 45{,}000 \pm 2.58\left(\frac{200}{\sqrt{499}}\right)$$

$$c.i. = 45{,}000 \pm 2.58(8.95)$$

$$c.i. = 45{,}000 \pm 23.09$$

At the 99.9% confidence level, the Z score would be ± 3.32 and the interval would be wider still:

$$c.i. = \bar{X} \pm Z\left(\frac{s}{\sqrt{N-1}}\right)$$

$$c.i. = 45{,}000 \pm 3.32\left(\frac{200}{\sqrt{499}}\right)$$

$$c.i. = 45{,}000 \pm 3.32(8.95)$$

$$c.i. = 45{,}000 \pm 29.71$$

These four intervals are grouped together in Table 6.4 and the increase in interval size can be readily observed. Although sample means have been used to illustrate the relationship between interval width and confidence level, exactly the same relationships apply to sample proportions. (*To further explore the relationship between alpha and interval width, see problem 6.13.*)

TABLE 6.4 Interval Estimates for Four Confidence Levels ($\bar{X}$ = $45,000, s = $200, N = 500 Throughout)

Alpha	Confidence Level	Interval	Interval Width
0.10	90%	$45,000 ± 14.77	$29.54
0.05	95%	$45,000 ± 17.55	$35.10
0.01	99%	$45,000 ± 23.09	$46.18
0.001	99.9%	$45,000 ± 29.71	$59.42

The sample size bears the opposite relationship to interval width. As the sample size increases, interval width decreases. Larger samples give more precise (narrower) estimates. Again, an example should make this clearer. In Table 6.5, confidence intervals for four samples of various sizes are constructed and then grouped together for purposes of comparison. The sample data are the same as in Table 6.4 and the confidence level is 95% throughout. The relationships illustrated in Table 6.5 also hold true, of course, for sample proportions. *(To further explore the relationship between the sample size and the interval width, see problem 6.14.)*

Notice that the decrease in interval width (or increase in precision) does not bear a constant or linear relationship with the sample size. For example, sample 2 is five times larger than sample 1, but the interval constructed with the larger sample size is not five times as narrow. This is an important relationship because it means that N might have to be increased many times over to appreciably improve the accuracy of an estimate. Because the cost of a research project is directly related to the sample size, this relationship implies a point of diminishing returns in estimation procedures. A sample of 10,000 will cost about twice as much as a sample of 5,000, but estimates based on the larger sample will not be twice as precise.

TABLE 6.5 Interval Estimates for Four Different Samples ($\bar{X}$ = $45,000, s = $200, Alpha = 0.05 Throughout)

Sample 1 (N = 100)	Sample 2 (N = 500)
$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{\sqrt{99}}\right)$	$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{\sqrt{499}}\right)$
$c.i. = 45{,}000 \pm 39.40$	$c.i. = 45{,}000 \pm 17.55$

Sample 3 (N = 1,000)	Sample 4 (N = 10,000)
$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{\sqrt{999}}\right)$	$c.i. = 45{,}000 \pm 1.96\left(\frac{200}{\sqrt{9{,}999}}\right)$
$c.i. = 45{,}000 \pm 12.40$	$c.i. = 45{,}000 \pm 3.92$

Sample	N	Interval Width
1	100	$78.80
2	500	$35.10
3	1,000	$24.80
4	10,000	$ 7.84