Functional Analysis	**Spring 2020**

Lecture 3

Lecturer: Muhammad Yaseen — *Topic: Representation of Functionals*

Riesz Representation Theorem for Functionals

Theorem 3.1 ***Let H_1 and H_2 be Hilbert spaces and $h : H_1 \times H_2 \to K$ be a bounded sesquilinear form. Then h has representation***

$$h(x, y) =< Sx, y >,$$

Where $S : H_1 \to H_2$ is bounded linear operator uniquely determined by h and has the norm $\|S\| = \|h\|$.

Proof: (i) Consider $\overline{h(x, y)}$ with $x \in H_1$ fixed. This is linear in y because of the bar. Thus h is bounded linear functional. So by Riesz representation theorem for functionals, for fixed $x \in H_1$ we have

$$f(y) = \overline{h(x, y)} =< y, z >, \tag{1}$$

where $z \in H_2$ is uniquely determined by f with $\|z\| = \|f\| = \|\overline{h}\|$. Then (1) implies

$$h(x, y) =< z, y > . \tag{2}$$

Here $z \in H_2$ is unique but of course depends upon our choice of $x \in H_1$. It follows that (2) with variable x defines an operator $S : H_1 \to H_2$ given by $Sx = z$. Substituting $z = Sx$ in (2), we obtain

$$h(x, y) =< Sx, y > \tag{3}$$

(ii) S is Linear

From (3), we have

$< S(\alpha x_1 + \beta x_2), y > = h(\alpha x_1 + \beta x_2, y)$

$$= \alpha h(x_1, y) + \beta h(x_2, y) \text{ because } h \text{ is linear in first argument.}$$
$$= \alpha < Sx_1, y > +\beta < Sx_2, y >$$
$$< \alpha Sx_1 + \beta Sx_2, y > \text{ for all } y \in H_2$$

This implies that $S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2$ (because if $< v_1, w >=< v_2, w >$ for all w, then $v_1 = v_2$). This proves that S is linear.

(iii) To prove that S is bounded: If $S = 0$, then it is bounded. If $\neq 0$, then

$$\begin{aligned} \|h\| &= \sup_{x,y\neq 0} \frac{|h(x,y)|}{\|x\|\|y\|} \\ &= \sup_{x,y\neq 0} \frac{|< Sx, y >|}{\|x\|\|y\|} \\ &\geq \sup_{x\neq 0} \frac{|< Sx, Sx >|}{\|x\|\|Sx\|} \\ &= \sup_{x\neq 0} \frac{\|Sx\|^2}{\|x\|\|Sx\|} \\ &= \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|. \end{aligned}$$

That is

$$\|S\| \leq \|h\| < \infty \text{ because } h \text{ is bounded.} \tag{4}$$

Now consider

$$\begin{aligned} \|h\| &= \sup_{x,y\neq 0} \frac{|h(x,y)|}{\|x\|\|y\|} \\ &= \sup_{x,y\neq 0} \frac{|< Sx, y >|}{\|x\|\|y\|} \\ &\leq \sup_{x,y\neq 0} \frac{\|Sx\|\|y\|}{\|x\|\|y\|} \text{ by Schwarz inequality} \\ &= \sup_{x,y\neq 0} \frac{\|Sx\|}{\|x\|} \end{aligned}$$

This implies that

$$\|h\| \leq \|S\| \tag{5}$$

From (4) and (5), we obtain $\|S\| = \|h\|$.

(iv) S is unique

Suppose that there exists $T : H_1 \to H_2$ such that

$$h(x,y) = < Sx, y > = < Tx, y > \ \forall x \in H_1,\ y \in H_2$$
$$\Rightarrow Sx = Tx \ \forall x \in H_1$$
$$\Rightarrow S = T.$$

This completes the proof.

■ Question 1: Show that the function $< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $< x, y > = \sum_{i=1}^{n} \xi_i \eta_i$ is an inner product space on $\mathbb{R}^n$.

Solution:

1. $< x + y, z > = \sum_{i=1}^{n} (\xi_i + \eta_i)\zeta_i = \sum_{i=1}^{n} \xi_i \zeta_i + \sum_{i=1}^{n} \eta_i \zeta_i = < x, z > + < y, z >$
2. $< \alpha x, y > = \sum_{i=1}^{n} (\alpha \xi_i)\eta_i = \alpha \sum_{i=1}^{n} \xi_i \eta_i = \alpha < x, y >$.
3. $< x, y > = \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} \eta_i \xi_i = < y, x >$.
4. $< x, x > = \sum_{i=1}^{n} \xi_i^2 = 0 \ \Leftrightarrow \xi_i = 0 \forall i \ \Leftrightarrow x = 0.$

Question 2: Show that the function $< \cdot, \cdot >: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by $< x, y > = \sum_{i=1}^{n} \xi_i \overline{\eta_i}$ is an inner product space on $\mathbb{C}^n$.

Question 2: Let X be an $n-$dimensional vector space with basis $\{e_1, \cdots, e_n\}$. Let $x, y \in X$ have representaion of the form $x = \sum_{i=1}^{n} \lambda_i e_i$ and $y = \sum_{i=1}^{n} \mu_i e_i$. Show that the function $< \cdot, \cdot >: X \times X \to \mathbb{C}$ defined by $< x, y > = \sum_{i=1}^{n} \lambda_i \overline{\mu_i}$ is an inner product space on X.

Question 3: Show that the function $\| \cdot \| : X \to \mathbb{R}$ defined by $\|x\| = \sqrt{< x, x >}$ is a norm on X.

Question 4: Let X be an inner product space and let $(x_n)_1^\infty$ and $(y_n)_1^\infty$ be

convergent sequences in X with $x_n \xrightarrow{\|\cdot\|} x$ and $y_n \xrightarrow{\|\cdot\|} y$. Then how that $\lim_{n\to\infty} < x_n, y_n > = < x, y >$.

Solution: Consider

$$\begin{aligned}
| < x_n, y_n > - < x, y > | &= | < x_n, y_n > - < x_n, y > + < x_n, y > - < x, y > | \\
&\leq | < x_n, y_n > - < x_n, y > | + | < x_n, y > - < x, y > | \\
&\leq | < x_n, y_n - y > | + | < x_n - x, y > | \\
&\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \\
&\to 0 \text{ as } n \to \infty \text{ (because since } (x_n)_1^\infty \text{ is bounded, therefore}
\end{aligned}$$

$\|x_n\| < \infty$ and $x_n \to x, \; y_n \to y$.)

This implies that $\lim_{n\to\infty} < x_n, y_n > = < x, y >$.