Functional Analysis | **Spring 2020**

Lecture 6

Lecturer: Muhammad Yaseen | *Topic: Representation of Functionals*

Bessel Inequality

Theorem 6.1 ***Let $(e_k)_1^\infty$ be an orthonormal sequence in an inner product space then for every $x \in X$,***

$$\sum_{k=1}^{\infty} | < x, e_k > |^2 \leq \|x\|^2 \tag{1}$$

Proof: Let $Y_n = \text{Span}\{e_1, \cdots, e_n\}$. Then every $y \in Y_n$ can be uniquely written as

$$y = \sum_{k=1}^{n} \alpha_k e_k,$$

where $\alpha_j =< y, e_j >$.

. . . . .

Aside:

We have

$$y = \sum_{k=1}^{n} \alpha_k e_k$$

$$\Rightarrow < y, e_j > = \sum_{k=1}^{n} \alpha_k < e_k, e_j >$$

$$\Rightarrow < y, e_j > = \alpha_j$$

. . . . .

We claim that for a particular choice of $\alpha_k =< x, e_k >$, where $x \in X$ but $x \notin Y_n$, we can obtain a $y \in Y_n$, such that $z = (x - y) \perp y$. To see this first consider

$$\|y\|^2 =< y, y >$$

$$=< \sum_{k=1}^{n} \alpha_k e_k, \sum_{m=1}^{n} \alpha_m e_m >$$
$$=< \sum_{k=1}^{n} < x, e_k > e_k, \sum_{m=1}^{n} < x, e_m > e_m >$$
$$= \sum_{k=1}^{n} < x, e_k >< e_k, \sum_{m=1}^{n} < x, e_m > e_m >$$
$$= \sum_{k=1}^{n} \sum_{m=1}^{n} < x, e_k > \overline{< x, e_m >} < e_k, e_m >$$
$$= \sum_{k=1}^{n} < x, e_k > \overline{< x, e_k >}$$
$$= \sum_{k=1}^{n} | < x, e_k > |^2$$

That is

$$\|y\|^2 = \sum_{k=1}^{n} | < x, e_k > |^2. \tag{2}$$

Now consider

$$\begin{aligned} < z, y > &=< x - y, y > \\ &=< x, y > - < y, y > \\ &= \sum_{k=1}^{n} \overline{< x, e_k >} < x, e_k > - \|y\|^2 \\ &= \sum_{k=1}^{n} | < x, e_k > |^2 - \|y\|^2 \\ &= \|y\|^2 - \|y\|^2 \text{ (by (2))} \\ &= 0 \end{aligned}$$

This implies that $z = (x - y) \perp y$. Now from $z = x - y$, we have $x = z + y$ so that by Pathagorous theorem, we have

$$\begin{aligned} \|x\| &= \|z\|^2 + \|y\|^2 \\ \Rightarrow \|z| &= \|x\|^2 - \|y\|^2 \\ &= \|x\|^2 - \sum_{k=1}^{n} | < x, e_k > |^2 \geq 0 \end{aligned}$$

This implies that

$$\sum_{k=1}^{n} |< x, e_k >|^2 \leq \|x\|^2.$$

Since $n \in \mathbb{N}$ is arbitrary, therefore letting $n \to \infty$ we get

$$\sum_{k=1}^{\infty} |< x, e_k >|^2 \leq \|x\|^2.$$

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Lemma 6.2 *If $< v_1, w >=< v_2, w >$ for all w in an inner product space X, then $v_1 = v_2$. Also if $< v, w >= 0$ for all $w \in X$, then $v = 0$.*

Proof: Suppose that $< v_1, w >=< v_2, w >$ for all $w \in X$. Then for all $w \in X$, we have

$$< v_1, w > - < v_2, w > = 0$$

$$< v_1 - v_2, w > = 0$$

Choose $w = v_1 - v_2$ so that

$$< v_1 - v_2, v_1 - v_2 > = 0$$

$$\Rightarrow \|v_1 - v_2\|^2 = 0$$

$$\Rightarrow \|v_1 - v_2\| = 0$$

$$\Rightarrow v_1 - v_2 = 0$$

$$\Rightarrow v_1 = v_2.$$

Now suppose that $< v, w >= 0$ for all $w \in X$. Choose $w = v$ so that

$$< v, v > = 0$$

$$\Rightarrow \|v\|^2 = 0$$

$$\Rightarrow \|v\| = 0$$

$$\Rightarrow v = 0.$$

Definition 6.3 ***Let X and Y be vector spaces over the same field, then the mapping $h : X \times Y \to K$ is called a sesquilinear form if $\forall x, x_1, x_2 \in X, \quad y, y_1, y_2 \in Y$ and scalers α, β, the following conditions are satisfied:***

(i) $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

(ii) $h(\alpha x, y) = \alpha h(x, y)$

(iii) $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

(iv) $h(x, \beta y) = \overline{\beta} h(x, y)$

So h is linear in first argument and conjugate linaer in second argument. h is bounded if $|h(x, y)| \leq c\|x\|\|y\|$ or $|h(x, y)| \leq \|h\|\|x\|\|y\|$ and

$$\|h\| = \sup_{x,y \neq 0} \frac{|h(x, y)|}{\|x\|\|y\|}$$

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