Functional Analysis	Spring 2020

Lecture 5

Lecturer: Muhammad Yaseen — *Topic:Ries Representation theorem*

Definition 5.1 ***A vector space X is said to be the direct sum of two subspaces Y and Z of x written***

$$X = Y \oplus Y$$

if each $x \in X$ has unique representation of the form $x = y + z$, where $y \in Y$ and $z \in Z$.

Remark 5.2 ***In case of a general Hilbert space H, we are mainly interested in representing H as a direct sum of a closed subspace Y of H and its orthogonal complement,***

$$Y^{\perp} = \{z \in H | \ z \perp Y\} = \text{the set of all vectors orthogonal to } Y.$$

Theorem 5.3 ***Let Y be any closed subspace of a Hilbert space H, then***

$$H = Y \oplus Y^{\perp}.$$

Riesz Representation theorem for functionals

Theorem 5.4 ***Every bounded linear functional f on a Hilbert space H can be represented by means of inner product as***

$$f(x) = < x, z >, \quad \forall x \in H, \tag{1}$$

where z which depends upon f is unique and has the norm

$$\|z\| = \|f\|. \tag{2}$$

Proof: We need to prove that

(a) f has representation (1).

(b) z in (1) is unique.

(c) (2) hold.

(a): If $f = 0$, then (1) and (2) hold trivially if we choose $z = 0$. Let $f \neq 0$. To motivate the idea of the proof, let us investigate what properties z must have such that (1) holds. First of all $z \neq 0$ because otherwise $f = 0$, which is a contradiction. If for some $x \in H$, $f(x) =< x, z >= 0$, then $x \in \mathcal{N}(f)$ and $z \perp \mathcal{N}(f)^{\perp}$. This suggests that we must consider $\mathcal{N}(f)$ and $\mathcal{N}(f)^{\perp}$. Since $\mathcal{N}(f)$ is closed, therefore by previous theorem, we have

$$H = \mathcal{N}(f) \oplus \mathcal{N}(f)^{\perp}$$

Since $f \neq 0$, therefore, $H \neq \mathcal{N}(f)$ so that $\mathcal{N}(f) \neq \{0\}$. So choose $0 \neq z_0 \in \mathcal{N}(f)^{\perp}$ and define

$$v = f(x)z_0 - f(z_0)x$$
$$\Rightarrow f(v) = f(x)f(z_0) - f(z_0)f(x) = 0,$$

so that $v \in \mathcal{N}(f)$. Since $z_0 \in \mathcal{N}(f)^{\perp}$, therefore,

$$\begin{aligned} 0 &=< v, z_0 > \\ &=< f(x)z_0 - f(z_0)x, z_0 > \\ &= f(x) < z_0, z_0 > -f(z_0) < x, z_0 > \end{aligned}$$
$$\Rightarrow f(x) < z_0, z_0 > = f(z_0) < x, z_0 >$$
$$\Rightarrow f(x) = \frac{f(z_0)}{< z_0, z_0 >} < x, z_0 >=< x, \overline{\frac{f(z_0)}{< z_0, z_0 >}} z_0 >$$
$$\Rightarrow f(x) =< x, z >, \text{ where } z = \overline{\frac{f(z_0)}{< z_0, z_0 >}} z_0.$$

(b): Suppose that for all $x \in X$,

$$f(x) =< x, z >=< x, z_1 >$$

$$\Rightarrow < x, z > - < x, z_1 > = 0$$
$$\Rightarrow < x, z - z_1 > = 0$$

Choose $x = z - z_1$ so that

$$
\begin{aligned}
< z - z_1, z - z_1 > &= 0 \\
\Rightarrow \|z - z_1\|^2 &= 0 \\
\Rightarrow \|z - z_1\| &= 0 \\
\Rightarrow z - z_1 &= 0 \\
\Rightarrow z &= z_1.
\end{aligned}
$$

So z in (1) is unique.

(C): If $f = 0$ and $z = 0$, then (2) holds trivially. Let $f \neq 0$ and $z \neq 0$ and replace x with z in (1) to get

$$
\begin{aligned}
f(z) &= < z, z > = \|z\|^2 \\
\Rightarrow \|z\|^2 = f(z) &\leq |f(z)| \quad \text{(because a number is less equal its absolute value)} \\
&\leq \|f\|\|z\| \quad \text{(because } f \text{ is bounded)} \\
\Rightarrow \|z\| &\leq \|f\| \qquad (3)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
f(x) &= < x, z > \\
\Rightarrow |f(x)| &= |< x, z >| \\
&\leq \|x\|\|z\| \quad \text{(by Schwarz Inequality)} \\
\Rightarrow \frac{|f(x)|}{\|x\|} &\leq \|z\| \\
\Rightarrow \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} &\leq \|z\| \\
\Rightarrow \|f\| &\leq \|z\|. \qquad (4)
\end{aligned}
$$

From (3) and (4), we obtain $\|f\| = \|z\|$. This completes the proof. ■