Functional Analysis	Spring 2020
Lecture 4	
Lecturer: Muhammad Yaseen	*Topic:Inner Product Space and Hilbert Space*

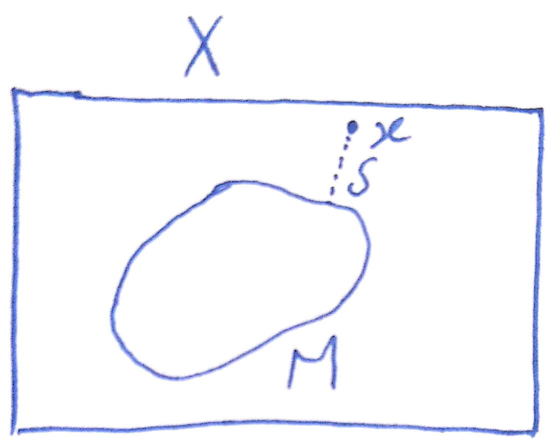
Definition 4.1 ***An element x of an inner product space X is said to be orthogonal to an element yinX if $< x, y >= 0$ and we write $x \perp y$. If $A, B \subseteq X$, then we write $x \perp A$ if $x \perp a, \quad \forall \quad a \in A$. We say that $A \perp B$ if $a \perp b \; \forall \quad a \in A$ and $\forall \quad b \in B$.***

Existence and Uniqueness Problem

Let X be a metric space and $M \subset X$. Then the distance δ from an element $x \in X$ to M is defined as

$$\delta = \inf_{\overline{y} \in M} d(x, \overline{y}).$$

If X is normed space, then

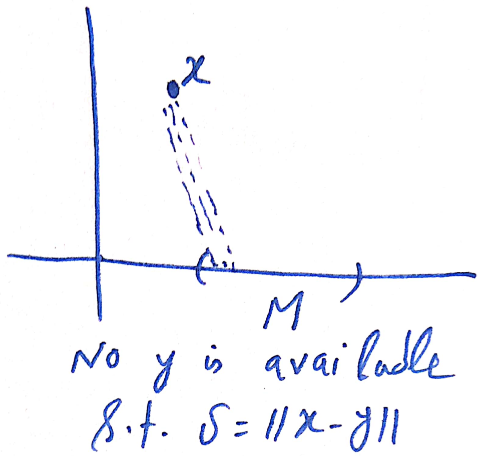


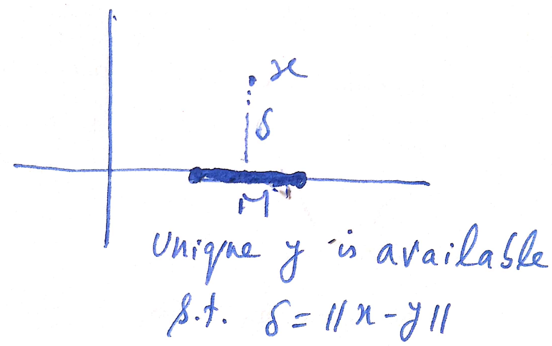
$$\delta = \inf_{\overline{y} \in M} \|x - \overline{y}\|.$$

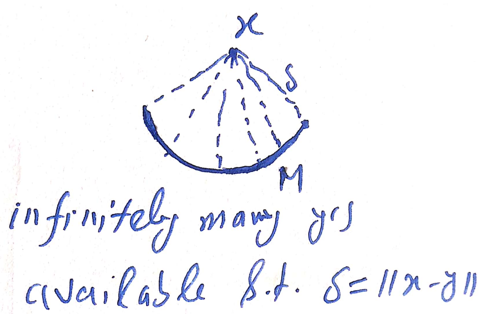
It is important to know that whether there exists a $y \in M$ such that

$$\delta = \|x - y\|.$$

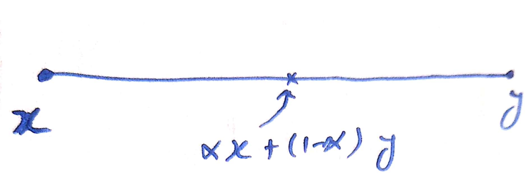
If such an element exists, whether it is unique?



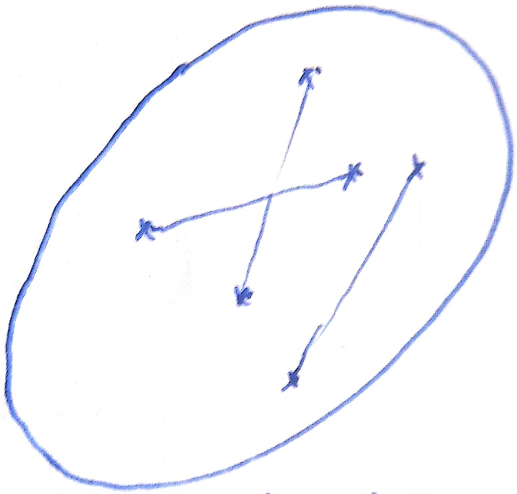




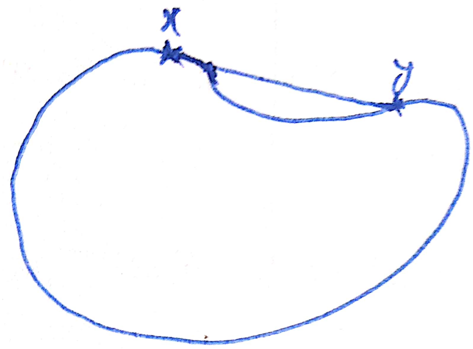
Definition 4.2 ***A segment joining two given elements x and y of a vector space X is defined to be $\alpha x + (1-\alpha)y, \quad \alpha \in [0,1]$.***



Definition 4.3 ***A subset M of X is said to be convex if for every $x, y \in M$, the line segment joining x and y is contained in M that is for all $x, y \in$***



Convex



Not Convex

$M, \quad \alpha x + (1-\alpha) y, \quad \alpha \in [0,1].$

Remark 4.4 *Every subspace of a vector space is convex set.*

Theorem 4.5 *Let X be an inner product space and $M \neq \phi$ a convex subset of X which is complete (under the metric induced by the inner product). Then for every $x \in X$, there exists a unique $y \in M$ such that*

$$\delta = \inf_{\overline{y} \in M} \|x - \overline{y}\|.$$

Proof: Existence of y.

We have

$$\delta = \inf_{\overline{y} \in M} \|x - \overline{y}\|$$

. Then by definition of infimum , there exists a sequence $(y_n)_1^\infty$ in M such that $delta_n = \|x - y_n\| \to \delta$ as $n \to \infty$. Let $v_n = x - y_n$. Then $\|v_n\| = \|x - y_n\| = \delta_n$. Now

$$\begin{aligned} \|v_n + v_m\| &= \|(x - y_n) + (x - y_m)\| \\ &= \|2x - (y_n + y_m)\| \end{aligned}$$

$$= 2\|x - \frac{1}{2}(y_n + y_m)\|$$

$$= 2\|x - (\frac{1}{2}y_n + (1 - \frac{1}{2})y_m)\|, \text{ (where } \frac{1}{2}y_n + (1-\frac{1}{2})y_m \in M \text{ because } M$$

is convex)

$$\geq 2\delta \quad (\because \delta = \inf_{\overline{y} \in M} \|x - \overline{y}\|, \quad \therefore \delta \leq \|x - \overline{y}\| \ \ \forall \overline{y} \in M)$$

i.e.

$$\|v_n + v_m\| \geq 2\delta \tag{1}$$

Now consider

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|(x - v_n) + (x - v_m)\| \\
&= \|v_n - v_m\|^2 \\
&= -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2) \\
&\leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \\
&\to -4\delta^2 + 2(\delta^2 + \delta^2) = 0 \text{ as } n, m \to \infty
\end{aligned}$$

i.e. $\|y_n - y_m\|^2 \to 0$ as $n, m \to \infty$. So $(y_n)_1^\infty$ is a Cauchy sequence in M. Since M is complete, there exists a $y \in M$ such that $y_n \to y$ as $n \to \infty$. Since $y \in M$, therefore by definition of infimum

$$\|x - y\| \geq \delta \tag{2}$$

Now

$$\begin{aligned}
\|x - y\| &\leq \|x - y_n\| + \|y_n - y\| \\
&= \delta_n + \|y_n - y\| \\
&\to \delta + 0 \ (\because \ \delta_n \to \delta \text{ and } y_n \to y \text{ as } n \to \infty)
\end{aligned}$$

. i.e.

$$\|x - y\| \leq \delta \tag{3}$$

Then (2) and (3) imply $\|x-y\|=\delta$

Uniqueness of y. **Suppose that there exist $y, y_0 \in M$ such that $\delta=\|x-y\|$ and $1\delta=\|x-y_0\|$. Then**

$$\begin{aligned}
\|y-y_0\|^2 &= \|(y-x)-(y_0-x)\|^2 \\
&= -\|(y-x)+(y_0-x)\|^2+2(\|(y-x)\|^2+\|(y_0-x)\|^2) \\
&= -\|-2x+(y+y_0)\|^2+2(\delta^2+\delta^2) \\
&= -4\|x-\frac{1}{2}(y+y_0)\|^2+4\delta^2 \\
&= -4\|x-(\frac{1}{2}y+(1-\frac{1}{2})y_0)\|^2+4\delta^2
\end{aligned}$$

$\because$ **M is convex** $\therefore$ $\frac{1}{2}y+(1-\frac{1}{2})y_0 \in M$ **so that** $\|x-(\frac{1}{2}y+(1-\frac{1}{2})y_0)\| \leq \delta$

$$\Rightarrow \|y-y_0\|^2 \leq -4\delta^2++4\delta^2=0$$

i.e. $\|y-y_0\| \leq 0$. But by definition, $\|y-y_0\| \geq 0$ so that $\|y-y_0\|=0$. This implies that $y-y_0=0$, that is $y=y_0$. This proves that y is unique. ■