Functional Analysis **Spring 2020**

Lecture 3

Lecturer: Muhammad Yaseen *Topic: Inner Product Space and Hilbert Space*

Definition 3.1 ***An inner product on a vector space X is a mapping from $X \times X$ into a scaler filed K of X i.e. it is a mapping $< \cdot, \cdot >: X \times X \to K$ such that for all $x, y, z \in X$ and for all scalers α we have***

(i) $< x + y, z > = < x, z > + < y, z >$

(ii) $< \alpha x, y > = \alpha < x, y >$

(iii) $< x, y > = \overline{< y, x >}$

(iv) $< x, x > = 0 \Leftrightarrow x = 0$

The pair $(< \cdot, \cdot >, X)$ is called the inner product space. A complete inner product space is called a Hilbert space.

Remark 3.2 ***From (i) and (ii), we have***

$$
\begin{aligned}
< \alpha x + \beta y, z > &= < \alpha x, z > + < \beta y, z > \\
&= \alpha < x, z > + \beta < y, z >
\end{aligned}
$$

and from (iii), we have

$$
\begin{aligned}
< x, \beta y > &= \overline{< \beta y, x >} = \overline{\beta < y, x >} \\
&= \overline{\beta} \quad \overline{< y, x >} = \overline{\beta} < x, y > \\
i.e. < x, \beta y > &= \overline{\beta} < x, y >
\end{aligned}
$$

Definition 3.3 ***An inner product defines a norm on a vector space X by***

$$\|x\| = \sqrt{< x, x >}$$

and metric on X is given by

$$d(x,y) = \|x-y\| = \sqrt{<x-y, x-y>}.$$

Remark 3.4 *Every inner product space is a normed space but converse is not true in general.*

Theorem 3.5 *Every inner product space X satisfies the parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in X.$$

Proof:

$$\begin{aligned}
\text{L.H.S.} &= \|x+y\|^2 + \|x-y\|^2 \\
&= <x+y, x+y> + <x-y, x-y> \\
&= <x, x+y> + <y, x+y> + <x, x-y> - <y, x-y> \\
&= <x,x> + <x,y> + <y,x> + <y,y> + <x,x> - <x,y> \\
&- <y,x> + <y,y> \\
&= 2(<x,x> + <y,y>) \\
&= 2(\|x\|^2 + \|y\|^2) \\
&= \text{R.H.S.}
\end{aligned}$$

■

Example 3.6 *Consider $X = c[a,b]$ with $\|x\| = \max_{t \in [a,b]} |x(t)|$. We know that $(c[a,b], \|x\|)$ is a normed space. Show that $c[a,b]$ is not an inner product space.*

Solution: Consider $x(t) = 1$ and $y(t) = \frac{t-a}{b-a}$. Then

$$(x+y)(t) = x(t) + y(t) = 1 + \frac{t-a}{b-a}$$
$$(x-y)(t) = x(t) - y(t) = 1 - \frac{t-a}{b-a}$$

Note that $y'(t) = \frac{1}{b-a} > 0$ so that y is increasing. Then

$$\|x\| = \max_{t\in[a,b]} |x(t)| = 1,$$
$$\|y\| = \max_{t\in[a,b]} |y(t)| = \max_{t\in[a,b]} |\frac{t-a}{b-a}| = \frac{b-a}{b-a} = 1,$$
$$\|x+y\| = \max_{t\in[a,b]} |x(t)+y(t)| = \max_{t\in[a,b]} |1+\frac{t-a}{b-a}| = 2,$$
$$\|x-y\| = \max_{t\in[a,b]} |x(t)-y(t)| = \max_{t\in[a,b]} |1-\frac{t-a}{b-a}| = 1.$$

So

$$\|x+y\|^2 + \|x-y\|^2 = (2)^2 + (1)^2 = 5,$$
$$2(\|x\|^2 + \|y\|^2) = 2(1+1) = 4$$
$$\Rightarrow \|x+y\|^2 + \|x-y\|^2 \neq 2(\|x\|^2 + \|y\|^2)$$

So $c[a,b]$ is not an inner product space. ■

Example 3.7 *Consider $X = l^p$, $1 \leq p \leq \infty$ with $\|x\| = (\sum_{i=1}^{\infty} |\xi_i|^p)^{(\frac{1}{p})}$. Show that l^p is not an inner product space except when $p = 2$.*

Solution: Choose $x = (1,1,0,0,\cdots)$ and $y = (1,-1,0,0,\cdots)$, then $x+y = (2,0,0,\cdots)$. Now

$$\|x\| = 2^{\frac{1}{p}},$$
$$\|y\| = 2^{\frac{1}{p}},$$
$$\|x+y\| = 2,$$
$$\|x-y\| = 2.$$

Then

$$\|x+y\|^2 + \|x-y\|^2 = 4+4 = 8,$$
$$2(\|x\|^2 + \|y\|^2) = 2(2^{\frac{2}{p}} + 2^{\frac{2}{p}}) = 4 \times 2^{\frac{2}{p}}$$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ if and only if $8 = 4 \times 2^{\frac{2}{p}}$ i.e. if and only if $p = 2$. If $p \neq 2$, then parallelogram law is not satisfied. So l^p is not an inner product space when $p \neq 2$. ■

Example 3.8 *Consider $X = l^2$ with $<\cdot,\cdot>: l^2 \times l^2 \to K$ by $< x, y >= \sum_1^\infty \xi_i \overline{\eta_i}$, where $x = (\xi_i)_1^\infty$ and $y = (\eta_i)_1^\infty \in l^2$. Show that l^2 is an inner product space.*

Solution:

$$
\begin{aligned}
(i)\ \ < x_1 + x_2, y > &= \sum_1^\infty (\xi_i^{(1)} + \xi_i^{(2)}) \overline{\eta_i}, \\
&= \sum_1^\infty \xi_i^{(1)} \overline{\eta_i} + \sum_1^\infty \xi_i^{(2)} \overline{\eta_i}, \\
&= < x_1, y > + < x_2, y >, \\
(ii)\ \ < \alpha x, y > &= \sum_1^\infty (\alpha \xi_i) \overline{\eta_i}, \\
&= \alpha \sum_1^\infty \xi_i \overline{\eta_i}, \\
&= \alpha < x, y >, \\
(iii)\ \ < x, y > &= \sum_1^\infty \xi_i \overline{\eta_i}, \\
&= \overline{\sum_1^\infty \eta_i \overline{\xi_i}}, \\
&= \overline{< y, x >}, \\
(iv)\ \ < x, x > &= 0 \\
\Leftrightarrow \sum_1^\infty \xi_i \overline{\xi_i} &= 0 \\
\Leftrightarrow \sum_1^\infty |\xi_i|^2 &= 0 \\
\Leftrightarrow \xi_i &= 0
\end{aligned}
$$

$$\Leftrightarrow x = 0$$

So l^2 is an inner product space. ■

Lemma 3.9 *An inner product on an inner product space X and the corresponding norm satisfy*

$$< x, y > \leq \|x\|\|y\|, \quad \textit{(Schwarz Inequality)} \tag{1}$$

where equality holds if and only if $\{x, y\}$ is linearly dependent.

Proof: If $y = 0$, then (1) is trivially true because in this case $\|y\| = 0$ and $< x, 0 > = < x, 0 \times x > = 0\times < x, x > = 0$. If $y \neq 0$, then for any scaler α, we have

$$\begin{aligned} 0 \leq \|x - \alpha y\|^2 &= < x - \alpha y, x - \alpha y > \\ &= < x, x - \alpha y > -\alpha < y, x - \alpha y > \\ &= < x, x > -\overline{\alpha} < x, y > -\alpha < y, x > +\alpha\overline{\alpha} < y, y > \\ &= < x, x > -\overline{\alpha} < x, y > -\alpha[< y, x > -\overline{\alpha} < y, y >] \end{aligned}$$

Choose α such that $\overline{\alpha} = \frac{<y,x>}{<y,y>}$ in above expression, we obtain

$$\begin{aligned} 0 \leq \|x - \alpha y\|^2 &= < x, x > -\overline{\alpha} < x, y > \\ &= < x, x > -\frac{< y, x >}{< y, y >} < x, y > \\ &= < x, x > -\frac{< y, x >< x, y >}{< y, y >} \\ &= < x, x > -\frac{\overline{< x, y >} < x, y >}{< y, y >} \\ &= < x, x > -\frac{| < x, y > |^2}{< y, y >} \\ < x, x > &\geq \frac{| < x, y > |^2}{\|y\|^2} \\ \|x\|^2 &\geq \frac{| < x, y > |^2}{\|y\|^2} \end{aligned}$$

$$< x, y > |^2 \leq \|x\|^2 \|y\|^2$$
$$< x, y > | \leq \|x\| \|y\|.$$

The equality holds in above discussion if and only if

$$0 = \|x - \alpha y\|^2$$
$$\Leftrightarrow 0 = \|x - \alpha y\|$$
$$\Leftrightarrow 0 = x - \alpha y$$
$$\Leftrightarrow 0x = \alpha y$$

so that $\{x, y\}$ is linearly dependent. ■