Functional Analysis	**Spring 2020**

Lecture 1

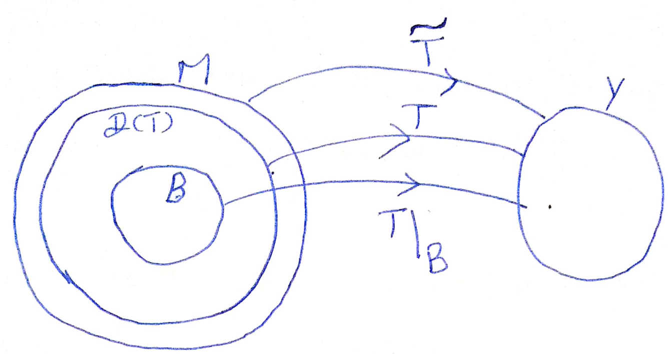
Lecturer: Muhammad Yaseen — *Topic: Bounded Linear Operators*

Definition 1.1 *The restriction of an operator $T : \mathcal{D}(T) \to Y$ to a subset $B \subset \mathcal{D}(T)$ is denoted by $T|_B : B \to Y$ and is defined by*

$$T|_B(x) = Tx, \quad \forall\ x \in B.$$

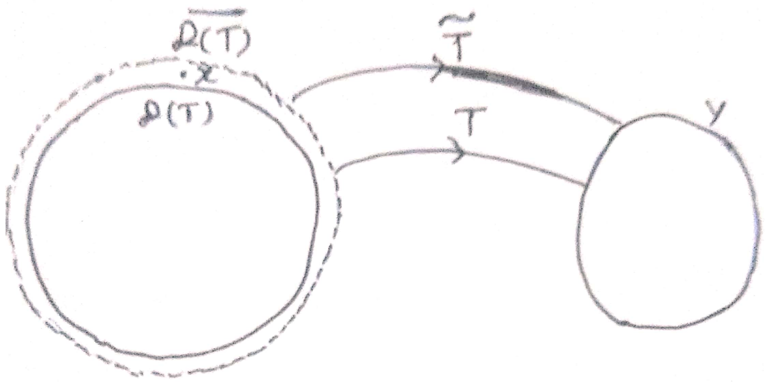
An extension of T to a set $M \supset \mathcal{D}(T)$ is the operator $\tilde{T} : M \to Y$ such that

$$\tilde{T}|_{\mathcal{D}(T)} = T \text{ i.e. } \tilde{T}|_{\mathcal{D}(T)}(x) = Tx\ \forall\ x \in \mathcal{D}(T).$$



Theorem 1.2 *Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator, where $\mathcal{D}(T)$ lies in a normed space and Y is a Banach space. Then T has an extension $\tilde{T} : \overline{\mathcal{D}(T)} \to Y$ such that $\tilde{T}$ is bounded and linear operator with $\|\tilde{T}\| = \|T\|$.*

Proof: First of all, we show the existence of $\tilde{T}$.

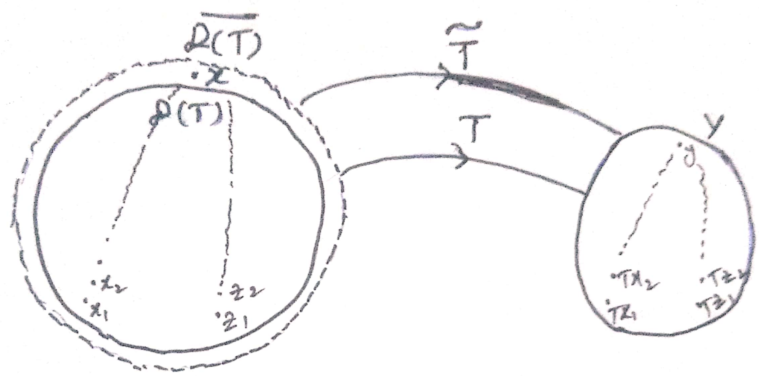


Let $x \in \overline{\mathcal{D}(T)}$. Then by a previous theorem there exists a sequence $(x_n)_1^\infty$ in $\mathcal{D}(T)$ such that $x_n \to x$. Since T is bounded and linear, therefore,

$$\begin{aligned} \|Tx_n - Tx_m\| &= \|T(x_n - x_m)\| \quad \text{(because } T \text{ is linear)} \\ &\leq \|T\|\|x_n - x_m\| \to 0 \text{ as } m, n \to 0 \quad \text{(because } T \text{ is bounded)} \end{aligned}$$

where we have used the fact that $(x_n)_1^\infty$ being convergent is Cauchy. So $(Tx_n)_1^\infty$ is a Cauchy sequence in Y. Since Y is complete, there exists a $y \in Y$ such that $Tx_n \to y$ i.e. $\lim\limits_{n\to\infty} Tx_n = y$. Using this y as an image of $x \in \overline{\mathcal{D}(T)}$, we can define $\tilde{T}$ as $\tilde{T}x = \lim\limits_{n\to\infty} Tx_n = y$, where $x_n \to x$. Clearly $\tilde{T}x = Tx, \ \forall\ x \in \mathcal{D}(T)$($\because$ if $x \in \mathcal{D}(T)$, then the sequence $x, x, \cdots$, is in $\mathcal{D}(T)$ and converges to x so that $\tilde{T}x = \lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Tx = Tx$).

Now we show that this definition of $\tilde{T}$ is independent of the choice of sequence in $\mathcal{D}(T)$ converging to x.



Suppose that $x_n \to x$ and $z_n \to x$. Since $(x_n)_1^\infty$ is Cauchy (because it is convergent), therefore $(Tx_n)_1^\infty$ is a Cauchy sequence in Y (as shown in the beginning of the proof). Since Y is complete, $(Tx_n)_1^\infty$ converges. As $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$, therefore, $\lim_{n\to\infty} (x_n - z_n) = 0$. Then

$$\|Tx_n - Tz_n\| = \|T(x_n - z_n)\| \leq \|T\|\|x_n - z_n\| \to 0 \text{ as } n \to \infty$$

so that $Tx_n \to Tz_n$ as $n \to \infty$ i.e.

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tz_n = \tilde{T}x = y.$$

Thus $\tilde{T}$ is an extension and is uniquely defined at each point of $\overline{\mathcal{D}(T)}$.

<u>To Prove that $\tilde{T}$ is linear</u>

Consider

$$\begin{aligned}
\tilde{T}(\alpha x + \beta y) &= \lim_{n\to\infty} T(\alpha x_n + \beta y_n) \text{ where } x_n \to x,\ y_n \to y \\
&= \lim_{n\to\infty} (\alpha T x_n + \beta T y_n) \text{ (because } T \text{ is linear)} \\
&= \alpha \lim_{n\to\infty} Tx_n + \beta \lim_{n\to\infty} Ty_n \\
&= \alpha \tilde{T}x + \beta \tilde{T}y.
\end{aligned}$$

So $\tilde{T}$ is linear.

<u>To Prove that $\tilde{T}$ is bounded</u>

Let $x_n \to x$ and consider

$$\begin{aligned}\|\tilde{T}x\| &= \|\lim_{n\to\infty} Tx_n\| = \lim_{n\to\infty}\|Tx_n\| \\ &\le \lim_{n\to\infty}\|T\|\|x_n\| = \|T\|\lim_{n\to\infty}\|x_n\| \text{ (because } T \text{ is bounded)} \\ &= \|T\|\|\lim_{n\to\infty} x_n\| = \|T\|\|x\|\end{aligned}$$

i.e.

$$\|\tilde{T}x\| \le \|T\|\|x\| \tag{1}$$

so that $\tilde{T}$ is bounded. Now (1) implies

$$\begin{aligned}\frac{\|\tilde{T}x\|}{\|x\|} &\le \|T\| \ \forall\ x \neq 0 \\ \Rightarrow \sup_{\substack{x\in\mathcal{D}(T)\\ x\neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} &\le \|T\| \\ \Rightarrow \|\tilde{T}\| &\le \|T\| \qquad (2)\end{aligned}$$

Since $\overline{\mathcal{D}(T)} \supset \mathcal{D}(T)$, therefore,

$$\|\tilde{T}\| \ge \|T\| \tag{3}$$

From (2) and (3), it follows that $\|\tilde{T}\| = \|T\|$. This completes the proof. ■

Definition 1.3 ***A linear functional f is a linear operator with domain a vector space and range in the scaler field K of the vector space X i.e.***

$$f : X \to K,$$

where $K = \mathbb{R}$ if X is real vector space and $K = \mathbb{C}$ if X is complex vector space.

Definition 1.4 ***A bounded linear functional f is a bounded linear operator with domain a vector space and range in the scaler field K. So if f is bounded, then there exists $c > 0$ such that***

$$|f(x)| \le c\|x\| \text{ or } |f(x)| \le \|f\|\|x\|.$$

In this case the norm of f exists and is defined as

$$\|f\| = \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in \mathcal{D}(f) \\ \|x\|=1}} |f(x)|.$$

Remark 1.5 ***The results that we proved for bounded linear operators continue to hold true for bounded linear functionals.***

Example 1.6 ***The norm $\|\cdot\| : X \to \mathbb{R}$ on a vector space X is a functional on X and it is nonlinear because***

$$\|x + y\| \leq \|x\| + \|y\|.$$

Example 1.7 ***Consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x) = x{\cdot}a$, where $x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $a = (a_1, a_2, a_3)$ is a fixed vector in $\mathbb{R}^3$. Show that f is bounded.***

Solution: f is bounded because

$$\begin{aligned} |f(x)| &= |x \cdot a| \quad \forall\ x \in \mathbb{R}^3 \\ &\leq \|x\|\|a\| \quad \forall\ x \in \mathbb{R}^3 \\ \Rightarrow |f(x)| &\leq \|a\|\|x\| \quad \forall\ x \in \mathbb{R}^3 \end{aligned} \tag{4}$$

Now (4) implies that

$$\begin{aligned} \frac{|f(x)|}{\|x\|} &\leq \|a\| \quad \forall\ x \neq 0 \\ \Rightarrow \sup_{x \in \mathcal{D}(f)} \frac{|f(x)|}{\|x\|} &\leq \|a\| \quad \forall\ x \neq 0 \\ \Rightarrow \|f\| &\leq \|a\| \end{aligned} \tag{5}$$

Now

$$\begin{aligned} \|f\| &= \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{x \in \mathcal{D}(f)} \frac{|x \cdot a|}{\|x\|} \\ &\geq \frac{|a.a|}{\|a\|} = \frac{\|a\|^2}{\|a\|} \\ \Rightarrow \|f\| &\geq \|a\| \end{aligned} \tag{6}$$

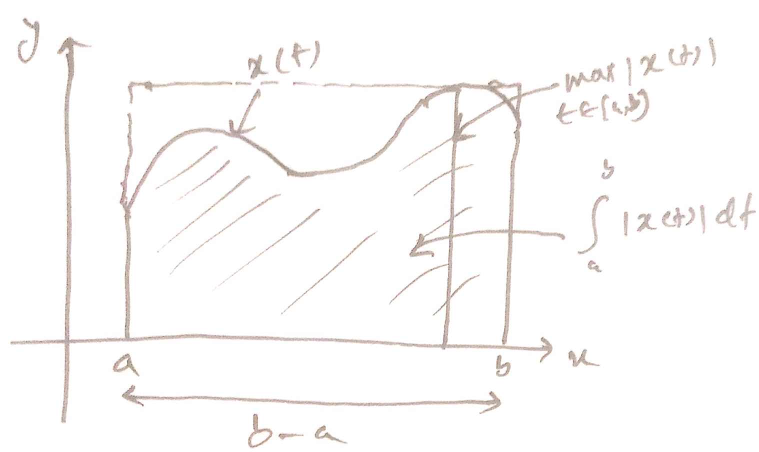
From (5) and (6), we see that $\|f\| = \|a\|$. ■

Example 1.8 ***Consider $f : c[a,b] \to \mathbb{R}$ defined by $f(x) = \int_a^b x(t)dt \ \ \forall\, x \in c[a,b]$. Show that f is bounded.***

Solution: f is bounded because

$$
\begin{aligned}
|f(x)| &= \left|\int_a^b x(t)dt\right| \ \ \forall\, x \in c[a,b] \\
&\le \int_a^b |x(t)|dt \ \ \forall\, x \in c[a,b] \\
&\le (b-a) \max_{x\in[a,b]} |x(t)| \ \ \forall\, x \in c[a,b]
\end{aligned}
$$

$$\Rightarrow |f(x)| \le (b-a)\|x\| \ \ \forall\, x \in c[a,b]. \tag{7}$$



Now (7) implies that

$$
\begin{aligned}
\frac{|f(x)|}{\|x\|} &\le (b-a) \ \ \forall\, x \ne 0 \\
\Rightarrow \sup_{\substack{x\in\mathcal{D}(f)\\ x\ne 0}} \frac{|f(x)|}{\|x\|} &\le (b-a) \\
\Rightarrow \|f\| &\le (b-a)
\end{aligned}
\tag{8}
$$

Also

$$\|f\| = \sup_{\substack{x\in\mathcal{D}(f)\\ x\neq 0}} \frac{|f(x)|}{\|x\|}$$

$$\geq \frac{|f(1)|}{\|1\|} = \frac{\int_a^b 1dt}{\max_{t\in[a,b]}(1)} = (b-a)$$

$$\Rightarrow \|f\| \geq (b-a). \tag{9}$$

From (8) and (9) it follows that $\|f\| = (b-a)$. ■

Example 1.9 *Consider the space l^2 and choose a fixed sequence $a = (\alpha_i)_1^\infty \in l^2$. Define a functional on l^2 by $f(x) = \sum_1^\infty \xi_i \alpha_i \quad \forall\ x = (\xi_i)_1^\infty \in l^2$. Show that f is linear and bounded.*

Solution: f is linear because

$$f(\alpha x + \beta y) = \sum_1^\infty (\alpha\xi_i + \beta\eta_i)\alpha_i$$

$$= \alpha \sum_1^\infty \xi_i\alpha_i + \beta \sum_1^\infty \eta_i\alpha_i$$

$$= \alpha f(x) + \beta f(y)$$

f is also bounded because

$$|f(x)| = |\sum_1^\infty \xi_i\alpha_i|$$

$$\leq (\sum_1^\infty |\xi_i|^2)^{1/2}(\sum_1^\infty |\alpha_i|^2)^{1/2}(\text{M. I. for} p = q = 2)$$

$$\Rightarrow |f(x)| \leq \|x\|\|a\| \tag{10}$$

Now (10) implies that

$$\frac{|f(x)|}{\|x\|} \le \|a\|$$
$$\Rightarrow \sup_{\substack{x\in\mathcal{D}(f)\\ x\neq 0}} \frac{|f(x)|}{\|x\|} \le \|a\|$$
$$\Rightarrow \|f\| \le \|a\| \tag{11}$$

Now consider

$$\begin{aligned}\|f\| &= \sup_{\substack{x\in\mathcal{D}(f)\\ x\neq 0}} \frac{|f(x)|}{\|x\|} \\ &\ge \frac{|f(a)|}{\|a\|} = \frac{\sum_1^\infty |\alpha_i|^2}{\|a\|} \\ &= \frac{\|a\|^2}{\|a\|} = \|a\| \end{aligned}$$
$$\Rightarrow \|f\| \ge \|a\| \tag{12}$$

From (11) and (12), we have $\|f\| = \|a\|$ ∎

Definition 1.10 *Let X be a vector space. Then the space of all linear functionals on X is denoted by X^* and is called the algebraic dual space of X. It is easy to see that X^* forms a vector space under the operations*

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \forall\ f_1, f_2 \in X^*$$
$$(\alpha f)(x) = \alpha f(x) \quad \forall\ f \in X^* \text{ and } \forall\ \alpha \in K.$$

We may go one step further and define linear functionals on X^*. The set of all linear functionals on X^* is denoted by X^{**} and is called the second algebraic dual space of X.

Definition 1.11 *Let X and Y be normed spaces over the same field. Then $B(X, Y)$ is the set of all bounded linear operators from X into Y. $B(X, Y)$*

forms a vector space under the operations

$$\begin{aligned}(T_1 + T_2)(x) &= T_1) + T_2x \quad \forall\ T_1, T_2 \in B(X, Y)\\ (\alpha T)(x) &= \alpha T x \quad \forall\ T \in B(X, Y)\ \textit{and}\ \forall\ \alpha \in K.\end{aligned} \tag{13}$$

Also $B(X, Y)$ forms a normed space under the norm defined by

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\|.$$

Theorem 1.12 *If Y is a Banach space then $B(X, Y)$ is a Banach space.*

Proof: Let $(T_n)_1^\infty$ be a Cauchy sequence in $B(X, Y)$. Then for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|T_n - T_m\| < \epsilon \quad \forall\ n, m > N$$

For any $x \in X$, the sequence $(T_n x)_1^\infty$ is in Y and consider

$$\begin{aligned}\|T_n x - T_m x\| &= \|(T_n - T_m)x\|\\ &\leq \|T_n - T_m\|\|x\| \quad (\because T_n,\ T_m \in B(X, Y),\ \ \therefore T_n - T_m \in B(X, Y)\\ &\qquad\qquad \text{so that } T_n - T_m \text{ is bounded})\\ &\leq \epsilon\|x\| \forall\ m, n > N\end{aligned} \tag{14}$$

This implies that $(T_n x)_1^\infty$ is a Cauchy sequence in Y. Since Y is complete, there exists a $y \in Y$ such that $T_n x \to y$ as $n \to \infty$ (i.e. $\lim_{n\to\infty} T_n x = y$). Clearly, the limit y depends upon our choice of $x \in X$. This defines an operator $T : X \to Y$ by $Tx = \lim_{n\to\infty} T_n x = y$. To prove that $B(X, Y)$ is complete, we have to show that $T \in B(X, Y)$ and $T_n \xrightarrow{\|x\|} T$

T is linear

$$\begin{aligned}T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y)\\ &= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)\\ &= \alpha T(x) + \beta T(y).\end{aligned}$$

T is bounded

From (14), we have

$$\|T_n x - T_m x\| \leq \epsilon \|x\| \ \forall \ m, n > N.$$

Letting $m \to \infty$ in it, we obtain

$$\begin{aligned} \|T_n x - Tx\| &\leq \epsilon \|x\| \ \forall \ n > N \\ \Rightarrow \|(T_n - T)x\| &\leq \epsilon \|x\| \ \forall \ n > N \end{aligned} \tag{15}$$

$\Rightarrow T_n - T$ is bounded so that $T = T_n - (T_n - T)$ is also bounded. Thus $T \in B(X, Y)$.

Now from (14), we have

$$\begin{aligned} \frac{\|T_n x - Tx\|}{\|x\|} &\leq \epsilon \ \forall \ n > N \\ \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|T_n x - Tx\|}{\|x\|} &\leq \epsilon \ \forall \ n > N \\ \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|(T_n - T)x\|}{\|x\|} &\leq \epsilon \ \forall \ n > N \\ \Rightarrow \|T_n - T\| &\leq \epsilon \ \forall \ n > N \\ \Rightarrow T_n &\xrightarrow{\|x\|} T \end{aligned}$$

Hence $B(X, Y)$ is a normed space. ∎