
Student Solutions Manual

for use with

Complex Variables and Applications

Seventh Edition

Selected Solutions to Exercises in Chapters 1-7

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Chapter 1

SECTION 2

1. (a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = \sqrt{2} - i - i - \sqrt{2} = -2i;$

(b) $(2, -3)(-2, 1) = (-4 + 3, 6 + 2) = (-1, 8);$

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (10, 0)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1).$

2. (a) $\operatorname{Re}(iz) = \operatorname{Re}[i(x + iy)] = \operatorname{Re}(-y + ix) = -y = -\operatorname{Im} z;$

(b) $\operatorname{Im}(iz) = \operatorname{Im}[i(x + iy)] = \operatorname{Im}(-y + ix) = x = \operatorname{Re} z.$

3. $(1 + z)^2 = (1 + z)(1 + z) = (1 + z) \cdot 1 + (1 + z)z = 1 \cdot (1 + z) + z(1 + z)$

$$= 1 + z + z + z^2 = 1 + 2z + z^2.$$

4. If $z = 1 \pm i$, then $z^2 - 2z + 2 = (1 \pm i)^2 - 2(1 \pm i) + 2 = \pm 2i - 2 \mp 2i + 2 = 0.$

5. To prove that multiplication is commutative, write

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) \\ &= (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1) = (x_2, y_2)(x_1, y_1) = z_2 z_1. \end{aligned}$$

6. (a) To verify the associative law for addition, write

$$\begin{aligned} (z_1 + z_2) + z_3 &= [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\ &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) \\ &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] \\ &= z_1 + (z_2 + z_3). \end{aligned}$$

(b) To verify the distributive law, write

$$\begin{aligned}
 z(z_1 + z_2) &= (x, y)[(x_1, y_1) + (x_2, y_2)] = (x, y)(x_1 + x_2, y_1 + y_2) \\
 &= (xx_1 + xx_2 - yy_1 - yy_2, yx_1 + yx_2 + xy_1 + xy_2) \\
 &= (xx_1 - yy_1 + xx_2 - yy_2, yx_1 + xy_1 + yx_2 + xy_2) \\
 &= (xx_1 - yy_1, yx_1 + xy_1) + (xx_2 - yy_2, yx_2 + xy_2) \\
 &= (x, y)(x_1, y_1) + (x, y)(x_2, y_2) = zz_1 + zz_2.
 \end{aligned}$$

10. The problem here is to solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0).$$

Since

$$(x^2 - y^2 + x + 1, 2xy + y) = (0, 0),$$

it follows that

$$x^2 - y^2 + x + 1 = 0 \quad \text{and} \quad 2xy + y = 0.$$

By writing the second of these equations as $(2x + 1)y = 0$, we see that either $2x + 1 = 0$ or $y = 0$. If $y = 0$, the first equation becomes $x^2 + x + 1 = 0$, which has no real roots (according to the quadratic formula). Hence $2x + 1 = 0$, or $x = -1/2$. In that case, the first equation reveals that $y^2 = 3/4$, or $y = \pm\sqrt{3}/2$. Thus

$$z = (x, y) = \left(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right).$$

SECTION 3

$$1. \quad (a) \quad \frac{1+2i}{3-4i} + \frac{2-i}{5i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)(-5i)}{(5i)(-5i)} = \frac{-5+10i}{25} + \frac{-5-10i}{25} = -\frac{2}{5};$$

$$(b) \quad \frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(1-3i)(3-i)} = \frac{5i}{-10i} = -\frac{1}{2};$$

$$(c) \quad (1-i)^4 = [(1-i)(1-i)]^2 = (-2i)^2 = -4.$$

$$2. \quad (a) \quad (-1)z = -z \quad \text{since} \quad z + (-1)z = z[1 + (-1)] = z \cdot 0 = 0;$$

$$(b) \quad \frac{1}{1/z} = \frac{1}{z^{-1}} \cdot \frac{z}{z} = \frac{z}{1} = z \quad (z \neq 0).$$

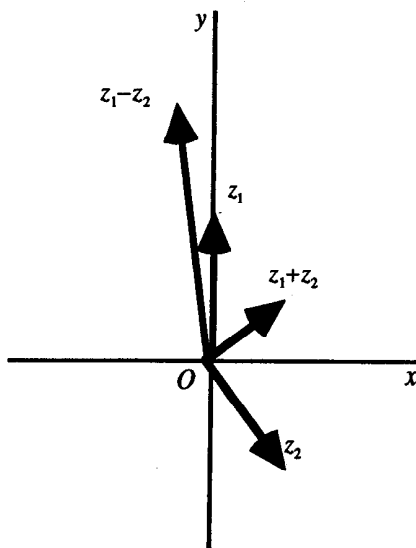
$$3. (z_1 z_2)(z_3 z_4) = z_1 [z_2 (z_3 z_4)] = z_1 [(z_2 z_3) z_4] = z_1 [(z_3 z_2) z_4] = z_1 [z_3 (z_2 z_4)] = (z_1 z_3)(z_2 z_4).$$

$$6. \frac{z_1 z_2}{z_3 z_4} = z_1 z_2 \left(\frac{1}{z_3 z_4} \right) = z_1 z_2 \left(\frac{1}{z_3} \right) \left(\frac{1}{z_4} \right) = z_1 \left(\frac{1}{z_3} \right) z_2 \left(\frac{1}{z_4} \right) = \left(\frac{z_1}{z_3} \right) \left(\frac{z_2}{z_4} \right) \quad (z_3 \neq 0, z_4 \neq 0).$$

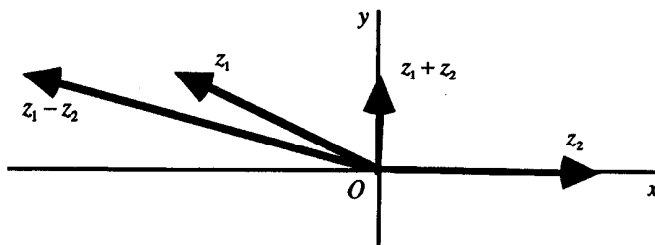
$$7. \frac{z_1 z}{z_2 z} = \left(\frac{z_1}{z_2} \right) \left(\frac{z}{z} \right) = \left(\frac{z_1}{z_2} \right) z \left(\frac{1}{z} \right) = \left(\frac{z_1}{z_2} \right) (z z^{-1}) = \left(\frac{z_1}{z_2} \right) \cdot 1 = \frac{z_1}{z_2} \quad (z_2 \neq 0, z \neq 0).$$

SECTION 4

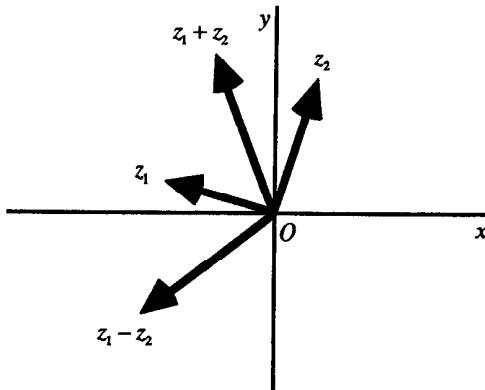
$$1. (a) \quad z_1 = 2i, \quad z_2 = \frac{2}{3} - i$$



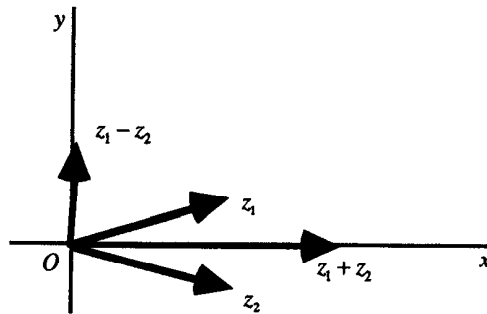
$$(b) \quad z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$$



(c) $z_1 = (-3, 1)$, $z_2 = (1, 4)$



(d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$



2. Inequalities (3), Sec. 4, are

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z| \quad \text{and} \quad \operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|.$$

These are obvious if we write them as

$$x \leq |x| \leq \sqrt{x^2 + y^2} \quad \text{and} \quad y \leq |y| \leq \sqrt{x^2 + y^2}.$$

3. In order to verify the inequality $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$, we rewrite it in the following ways:

$$\sqrt{2}\sqrt{x^2 + y^2} \geq |x| + |y|,$$

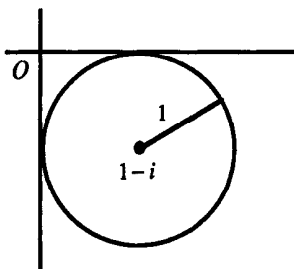
$$2(x^2 + y^2) \geq |x|^2 + 2|x||y| + |y|^2,$$

$$|x|^2 - 2|x||y| + |y|^2 \geq 0,$$

$$(|x| - |y|)^2 \geq 0.$$

This last form of the inequality to be verified is obviously true since the left-hand side is a perfect square.

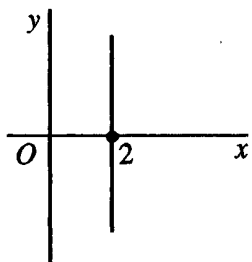
4. (a) Rewrite $|z - 1 + i| = 1$ as $|z - (1 - i)| = 1$. This is the circle centered at $1 - i$ with radius 1. It is shown below.



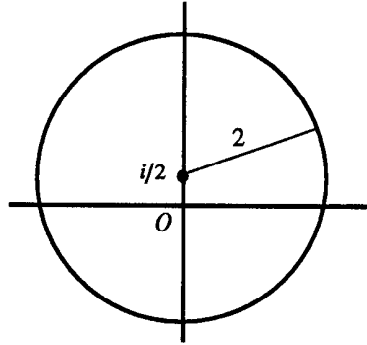
5. (a) Write $|z - 4i| + |z + 4i| = 10$ as $|z - 4i| + |z - (-4i)| = 10$ to see that this is the locus of all points z such that the sum of the distances from z to $4i$ and $-4i$ is a constant. Such a curve is an ellipse with foci $\pm 4i$.
- (b) Write $|z - 1| = |z + i|$ as $|z - 1| = |z - (-i)|$ to see that this is the locus of all points z such that the distance from z to 1 is always the same as the distance to $-i$. The curve is, then, the perpendicular bisector of the line segment from 1 to $-i$.

SECTION 5

1. (a) $\overline{\bar{z} + 3i} = \bar{\bar{z}} + \bar{3i} = z - 3i$;
- (b) $\overline{\bar{iz}} = \bar{i}\bar{z} = -i\bar{z}$;
- (c) $\overline{(2+i)^2} = \overline{(2+i)^2} = (2-i)^2 = 4 - 4i + i^2 = 4 - 4i - 1 = 3 - 4i$;
- (d) $|(\overline{2z + 5})(\sqrt{2} - i)| = |2z + 5||\sqrt{2} - i| = |2z + 5|\sqrt{2+1} = \sqrt{3}|2z + 5|$.
2. (a) Rewrite $\operatorname{Re}(\bar{z} - i) = 2$ as $\operatorname{Re}[x + i(-y - 1)] = 2$, or $x = 2$. This is the vertical line through the point $z = 2$, shown below.



- (b) Rewrite $|2z - i| = 4$ as $2\left|z - \frac{i}{2}\right| = 4$, or $\left|z - \frac{i}{2}\right| = 2$. This is the circle centered at $\frac{i}{2}$ with radius 2, shown below.



3. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned}\overline{z_1 - z_2} &= \overline{(x_1 + iy_1) - (x_2 + iy_2)} = \overline{(x_1 - x_2) + i(y_1 - y_2)} \\ &= (x_1 - x_2) - i(y_1 - y_2) = (x_1 - iy_1) - (x_2 - iy_2) = \bar{z}_1 - \bar{z}_2\end{aligned}$$

and

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)} \\ &= (x_1 x_2 - y_1 y_2) - i(y_1 x_2 + x_1 y_2) = (x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1 \bar{z}_2.\end{aligned}$$

4. (a) $\overline{z_1 z_2 z_3} = \overline{(z_1 z_2) z_3} = \overline{z_1 z_2} \overline{z_3} = (\bar{z}_1 \bar{z}_2) \bar{z}_3 = \bar{z}_1 \bar{z}_2 \bar{z}_3;$

(b) $\overline{z^4} = \overline{z^2 z^2} = \overline{z^2} \overline{z^2} = \overline{z z} \overline{z z} = (\bar{z} \bar{z})(\bar{z} \bar{z}) = \bar{z} \bar{z} \bar{z} \bar{z} = \bar{z}^4.$

6. (a) $\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\bar{z}_1}{\bar{z}_2 \bar{z}_3} = \frac{\bar{z}_1}{\bar{z}_2 \bar{z}_3};$

(b) $\left|\frac{z_1}{z_2 z_3}\right| = \frac{|z_1|}{|z_2 z_3|} = \frac{|z_1|}{|z_2| |z_3|}.$

8. In this problem, we shall use the inequalities (see Sec. 4)

$$|\operatorname{Re} z| \leq |z| \quad \text{and} \quad |z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|.$$

Specifically, when $|z| \leq 1$,

$$|\operatorname{Re}(2 + \bar{z} + z^3)| \leq |2 + \bar{z} + z^3| \leq 2 + |\bar{z}| + |z^3| = 2 + |z| + |z|^3 \leq 2 + 1 + 1 = 4.$$

10. First write $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$. Then observe that when $|z| = 2$,

$$|z^2 - 1| \geq \left| |z^2| - |1| \right| = \left| |z|^2 - 1 \right| = |4 - 1| = 3$$

and

$$|z^2 - 3| \geq \left| |z^2| - |3| \right| = \left| |z|^2 - 3 \right| = |4 - 3| = 1.$$

Thus, when $|z| = 2$,

$$|z^4 - 4z^2 + 3| = |z^2 - 1| |z^2 - 3| \geq 3 \cdot 1 = 3.$$

Consequently, when z lies on the circle $|z| = 2$,

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| = \frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}.$$

11. (a) Prove that z is real $\Leftrightarrow \bar{z} = z$.

(\Leftarrow) Suppose that $\bar{z} = z$, so that $x - iy = x + iy$. This means that $i2y = 0$, or $y = 0$. Thus $z = x + i0 = x$, or z is real.

(\Rightarrow) Suppose that z is real, so that $z = x + i0$. Then $\bar{z} = x - i0 = x + i0 = z$.

(b) Prove that z is either real or pure imaginary $\Leftrightarrow \bar{z}^2 = z^2$.

(\Leftarrow) Suppose that $\bar{z}^2 = z^2$. Then $(x - iy)^2 = (x + iy)^2$, or $i4xy = 0$. But this can be only if either $x = 0$ or $y = 0$, or possibly $x = y = 0$. Thus z is either real or pure imaginary.

(\Rightarrow) Suppose that z is either real or pure imaginary. If z is real, so that $z = x$, then $\bar{z}^2 = x^2 = z^2$. If z is pure imaginary, so that $z = iy$, then $\bar{z}^2 = (-iy)^2 = (iy)^2 = z^2$.

12. (a) We shall use mathematical induction to show that

$$\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n \quad (n = 2, 3, \dots).$$

This is known when $n = 2$ (Sec. 5). Assuming now that it is true when $n = m$, we may write

$$\begin{aligned} \overline{z_1 + z_2 + \cdots + z_m + z_{m+1}} &= \overline{(z_1 + z_2 + \cdots + z_m) + z_{m+1}} \\ &= \overline{(z_1 + z_2 + \cdots + z_m)} + \bar{z}_{m+1} \\ &= \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_m + \bar{z}_{m+1} \\ &= \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_m + \bar{z}_{m+1}. \end{aligned}$$

(b) In the same way, we can show that

$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n \quad (n = 2, 3, \dots).$$

This is true when $n = 2$ (Sec. 5). Assuming that it is true when $n = m$, we write

$$\begin{aligned} \overline{z_1 z_2 \cdots z_m z_{m+1}} &= \overline{(z_1 z_2 \cdots z_m) z_{m+1}} = \overline{(z_1 z_2 \cdots z_m)} \bar{z}_{m+1} \\ &= (\bar{z}_1 \bar{z}_2 \cdots \bar{z}_m) \bar{z}_{m+1} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_m \bar{z}_{m+1}. \end{aligned}$$

14. The identities (Sec. 5) $z\bar{z} = |z|^2$ and $\operatorname{Re} z = \frac{z + \bar{z}}{2}$ enable us to write $|z - z_0| = R$ as

$$(z - z_0)(\bar{z} - \bar{z}_0) = R^2,$$

$$z\bar{z} - (z\bar{z}_0 + \bar{z}z_0) + z_0\bar{z}_0 = R^2,$$

$$|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

15. Since $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$, the hyperbola $x^2 - y^2 = 1$ can be written in the following ways:

$$\left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 = 1,$$

$$\frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4} = 1,$$

$$\frac{2z^2 + 2\bar{z}^2}{4} = 1,$$

$$z^2 + \bar{z}^2 = 2.$$

SECTION 7

1. (a) Since

$$\arg\left(\frac{i}{-2-2i}\right) = \arg i - \arg(-2-2i),$$

one value of $\arg\left(\frac{i}{-2-2i}\right)$ is $\frac{\pi}{2} - \left(-\frac{3\pi}{4}\right)$, or $\frac{5\pi}{4}$. Consequently, the principal value is

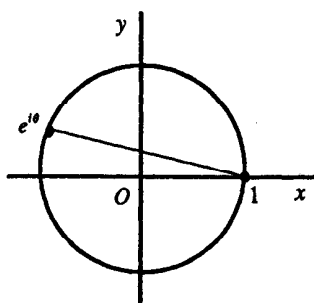
$$\frac{5\pi}{4} - 2\pi, \text{ or } -\frac{3\pi}{4}.$$

(b) Since

$$\arg(\sqrt{3} - i)^6 = 6 \arg(\sqrt{3} - i),$$

one value of $\arg(\sqrt{3} - i)^6$ is $6\left(-\frac{\pi}{6}\right)$, or $-\pi$. So the principal value is $-\pi + 2\pi$, or π .

4. The solution $\theta = \pi$ of the equation $|e^{i\theta} - 1| = 2$ in the interval $0 \leq \theta < 2\pi$ is geometrically evident if we recall that $e^{i\theta}$ lies on the circle $|z| = 1$ and that $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1. See the figure below.



5. We know from de Moivre's formula that

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta,$$

or

$$\cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta.$$

That is,

$$(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) = \cos 3\theta + i \sin 3\theta.$$

By equating real parts and then imaginary parts here, we arrive at the desired trigonometric identities:

$$(a) \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta; \quad (b) \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

8. Here $z = re^{i\theta}$ is any nonzero complex number and n a negative integer ($n = -1, -2, \dots$). Also, $m = -n = 1, 2, \dots$. By writing

$$(z^m)^{-1} = (r^m e^{im\theta})^{-1} = \frac{1}{r^m} e^{i(-m\theta)}$$

and

$$(z^{-1})^m = \left[\frac{1}{r} e^{i(-\theta)} \right]^m = \left(\frac{1}{r} \right)^m e^{i(-m\theta)} = \frac{1}{r^m} e^{i(-m\theta)},$$

we see that $(z^m)^{-1} = (z^{-1})^m$. Thus the definition $z^n = (z^{-1})^m$ can also be written as $z^n = (z^m)^{-1}$.

9. First of all, given two nonzero complex numbers z_1 and z_2 , suppose that there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \bar{c}_2$. Since

$$|z_1| = |c_1| |c_2| \quad \text{and} \quad |z_2| = |c_1| |\bar{c}_2| = |c_1| |c_2|,$$

it follows that $|z_1| = |z_2|$.

Suppose, on the other hand, that we know only that $|z_1| = |z_2|$. We may write

$$z_1 = r_1 \exp(i\theta_1) \quad \text{and} \quad z_2 = r_1 \exp(i\theta_2).$$

If we introduce the numbers

$$c_1 = r_1 \exp\left(i \frac{\theta_1 + \theta_2}{2}\right) \quad \text{and} \quad c_2 = \exp\left(i \frac{\theta_1 - \theta_2}{2}\right),$$

we find that

$$c_1 c_2 = r_1 \exp\left(i \frac{\theta_1 + \theta_2}{2}\right) \exp\left(i \frac{\theta_1 - \theta_2}{2}\right) = r_1 \exp(i\theta_1) = z_1$$

and

$$c_1 \bar{c}_2 = r_1 \exp\left(i \frac{\theta_1 + \theta_2}{2}\right) \exp\left(-i \frac{\theta_1 - \theta_2}{2}\right) = r_1 \exp(i\theta_2) = z_2.$$

That is,

$$z_1 = c_1 c_2 \quad \text{and} \quad z_2 = c_1 \bar{c}_2.$$

10. If $S = 1 + z + z^2 + \cdots + z^n$, then

$$S - zS = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}.$$

Hence $S = \frac{1 - z^{n+1}}{1 - z}$, provided $z \neq 1$. That is,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1).$$

Putting $z = e^{i\theta}$ ($0 < \theta < 2\pi$) in this identity, we have

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Now the real part of the left-hand side here is evidently

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta;$$

and, to find the real part of the right-hand side, we write that side in the form

$$\frac{1 - \exp[i(n+1)\theta]}{1 - \exp(i\theta)} \cdot \frac{\exp\left(-i\frac{\theta}{2}\right)}{\exp\left(-i\frac{\theta}{2}\right)} = \frac{\exp\left(-i\frac{\theta}{2}\right) - \exp\left[i\frac{(2n+1)\theta}{2}\right]}{\exp\left(-i\frac{\theta}{2}\right) - \exp\left(i\frac{\theta}{2}\right)},$$

which becomes

$$\frac{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} - \cos \frac{(2n+1)\theta}{2} - i \sin \frac{(2n+1)\theta}{2}}{-2i \sin \frac{\theta}{2}} \cdot \frac{i}{i},$$

or

$$\frac{\left[\sin \frac{\theta}{2} + \sin \frac{(2n+1)\theta}{2} \right] + i \left[\cos \frac{\theta}{2} - \cos \frac{(2n+1)\theta}{2} \right]}{2 \sin \frac{\theta}{2}}.$$

The real part of this is clearly

$$\frac{1}{2} + \frac{\sin \frac{(2n+1)\theta}{2}}{2 \sin \frac{\theta}{2}},$$

and we arrive at *Lagrange's trigonometric identity*:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin \frac{(2n+1)\theta}{2}}{2 \sin \frac{\theta}{2}} \quad (0 < \theta < 2\pi).$$

SECTION 9

1. (a) Since $2i = 2 \exp\left[i\left(\frac{\pi}{2} + 2k\pi\right)\right]$ ($k = 0, \pm 1, \pm 2, \dots$), the desired roots are

$$(2i)^{1/2} = \sqrt{2} \exp\left[i\left(\frac{\pi}{4} + k\pi\right)\right] \quad (k = 0, 1).$$

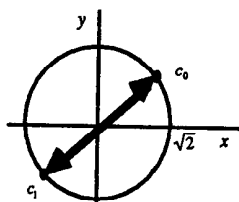
That is,

$$c_0 = \sqrt{2} e^{i\pi/4} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = 1 + i$$

and

$$c_1 = (\sqrt{2} e^{i\pi/4}) e^{i\pi} = -c_0 = -(1 + i),$$

c_0 being the principal root. These are sketched below.



- (b) Observe that $1 - \sqrt{3}i = 2 \exp\left[i\left(-\frac{\pi}{3} + 2k\pi\right)\right]$ ($k = 0, \pm 1, \pm 2, \dots$). Hence

$$(1 - \sqrt{3}i)^{1/2} = \sqrt{2} \exp\left[i\left(-\frac{\pi}{6} + k\pi\right)\right] \quad (k = 0, 1).$$

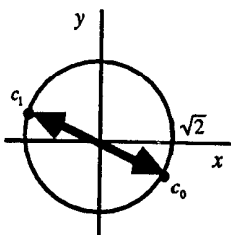
The principal root is

$$c_0 = \sqrt{2} e^{-i\pi/6} = \sqrt{2} \left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = \sqrt{2} \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) = \frac{\sqrt{3} - i}{\sqrt{2}},$$

and the other root is

$$c_1 = (\sqrt{2} e^{-i\pi/6}) e^{i\pi} = -c_0 = -\frac{\sqrt{3} - i}{\sqrt{2}}.$$

These roots are shown below.



2. (a) Since $-16 = 16 \exp[i(\pi + 2k\pi)]$ ($k = 0, \pm 1, \pm 2, \dots$), the needed roots are

$$(-16)^{1/4} = 2 \exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] \quad (k = 0, 1, 2, 3).$$

The principal root is

$$c_0 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}(1+i).$$

The other three roots are

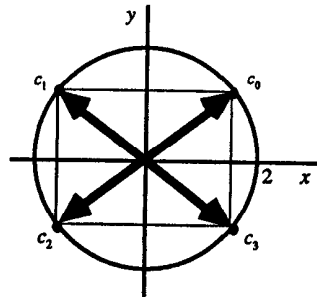
$$c_1 = (2e^{i\pi/4})e^{i\pi/2} = c_0 i = \sqrt{2}(1+i)i = -\sqrt{2}(1-i),$$

$$c_2 = (2e^{i\pi/4})e^{i\pi} = -c_0 = -\sqrt{2}(1+i),$$

and

$$c_3 = (2e^{i\pi/4})e^{i3\pi/2} = c_0(-i) = \sqrt{2}(1+i)(-i) = \sqrt{2}(1-i).$$

The four roots are shown below.



(b) First write $-8 - 8\sqrt{3}i = 16 \exp\left[i\left(-\frac{2\pi}{3} + 2k\pi\right)\right]$ ($k = 0, \pm 1, \pm 2, \dots$). Then

$$(-8 - 8\sqrt{3}i)^{1/4} = 2 \exp\left[i\left(-\frac{\pi}{6} + \frac{k\pi}{2}\right)\right] \quad (k = 0, 1, 2, 3).$$

The principal root is

$$c_0 = 2e^{-i\pi/6} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{3} - i.$$

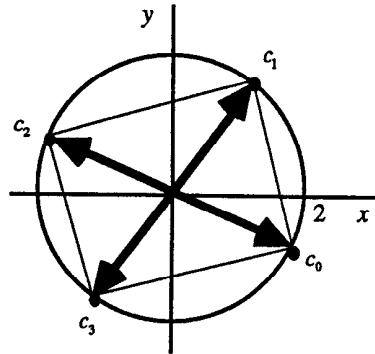
The others are

$$c_1 = (2e^{-i\pi/6})e^{i\pi/2} = c_0 i = 1 + \sqrt{3}i,$$

$$c_2 = (2e^{-i\pi/6})e^{i\pi} = -c_0 = -(\sqrt{3} - i),$$

$$c_3 = (2e^{-i\pi/6})e^{i3\pi/2} = c_0(-i) = -(1 + \sqrt{3}i).$$

These roots are all shown below.



3. (a) By writing $-1 = 1 \exp[i(\pi + 2k\pi)]$ ($k = 0, \pm 1, \pm 2, \dots$), we see that

$$(-1)^{1/3} = \exp\left[i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] \quad (k = 0, 1, 2).$$

The principal root is

$$c_0 = e^{i\pi/3} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1 + \sqrt{3}i}{2}.$$

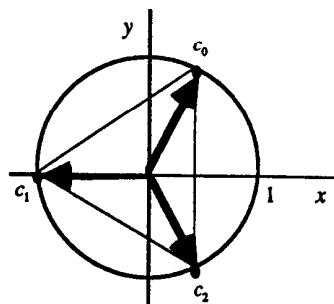
The other two roots are

$$c_1 = e^{i\pi} = -1$$

and

$$c_2 = e^{i5\pi/3} = e^{i2\pi} e^{-i\pi/3} = \cos\frac{\pi}{3} - i \sin\frac{\pi}{3} = \frac{1 - \sqrt{3}i}{2}.$$

All three roots are shown below.



(b) Since $8 = 8 \exp[i(0 + 2k\pi)]$ ($k = 0, \pm 1, \pm 2, \dots$), the desired roots of 8 are

$$8^{1/6} = \sqrt{2} \exp\left(i \frac{k\pi}{3}\right) \quad (k = 0, 1, 2, 3, 4, 5),$$

the principal one being

$$c_0 = \sqrt{2}.$$

The others are

$$c_1 = \sqrt{2} e^{i\pi/3} = \sqrt{2} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = \sqrt{2} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) = \frac{1 + \sqrt{3}i}{\sqrt{2}},$$

$$c_2 = (\sqrt{2} e^{-i\pi/3}) e^{i\pi} = \sqrt{2} \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right) (-1) = -\sqrt{2} \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) = -\frac{1 - \sqrt{3}i}{\sqrt{2}},$$

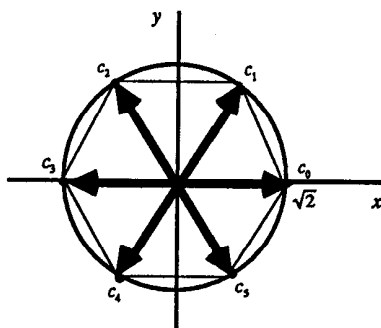
$$c_3 = \sqrt{2} e^{i\pi} = -\sqrt{2},$$

$$c_4 = (\sqrt{2} e^{i\pi/3}) e^{i\pi} = -c_1 = -\frac{1 + \sqrt{3}i}{\sqrt{2}},$$

and

$$c_5 = (\sqrt{2} e^{i2\pi/3}) e^{i\pi} = -c_2 = \frac{1 - \sqrt{3}i}{\sqrt{2}}.$$

All six roots are shown below.



4. The three cube roots of the number $z_0 = -4\sqrt{2} + 4\sqrt{2}i = 8 \exp\left(i \frac{3\pi}{4}\right)$ are evidently

$$(z_0)^{1/3} = 2 \exp\left[i \left(\frac{\pi}{4} + \frac{2k\pi}{3} \right)\right] \quad (k = 0, 1, 2).$$

In particular,

$$c_0 = 2 \exp\left(i \frac{\pi}{4}\right) = \sqrt{2}(1 + i).$$

With the aid of the number $\omega_3 = \frac{-1 + \sqrt{3}i}{2}$, we obtain the other two roots:

$$c_1 = c_0 \omega_3 = \sqrt{2}(1+i) \left(\frac{-1 + \sqrt{3}i}{2} \right) = \frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}},$$

$$c_2 = c_0 \omega_3^2 = (c_0 \omega_3) \omega_3 = \left[\frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}} \right] \left(\frac{-1 + \sqrt{3}i}{2} \right) = \frac{(\sqrt{3}-1) - (\sqrt{3}+1)i}{\sqrt{2}}.$$

5. (a) Let a denote any fixed real number. In order to find the two square roots of $a+i$ in exponential form, we write

$$A = |a+i| = \sqrt{a^2+1} \quad \text{and} \quad \alpha = \text{Arg}(a+i).$$

Since

$$a+i = A \exp[i(\alpha + 2k\pi)] \quad (k = 0, \pm 1, \pm 2, \dots),$$

we see that

$$(a+i)^{1/2} = \sqrt{A} \exp \left[i \left(\frac{\alpha}{2} + k\pi \right) \right] \quad (k = 0, 1).$$

That is, the desired square roots are

$$\sqrt{A} e^{i\alpha/2} \quad \text{and} \quad \sqrt{A} e^{i\alpha/2} e^{i\pi} = -\sqrt{A} e^{i\alpha/2}.$$

- (b) Since $a+i$ lies above the real axis, we know that $0 < \alpha < \pi$. Thus $0 < \frac{\alpha}{2} < \frac{\pi}{2}$, and this tells us that $\cos\left(\frac{\alpha}{2}\right) > 0$ and $\sin\left(\frac{\alpha}{2}\right) > 0$. Since $\cos \alpha = \frac{a}{A}$, it follows that

$$\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{a}{A}} = \frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}}$$

and

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{a}{A}} = \frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}}.$$

Consequently,

$$\begin{aligned} \pm \sqrt{A} e^{i\alpha/2} &= \pm \sqrt{A} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right) = \pm \sqrt{A} \left(\frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}} + i \frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}} \right) \\ &= \pm \frac{1}{\sqrt{2}} (\sqrt{A+a} + i\sqrt{A-a}). \end{aligned}$$

6. The four roots of the equation $z^4 + 4 = 0$ are the four fourth roots of the number -4 . To find those roots, we write $-4 = 4\exp[i(\pi + 2k\pi)]$ ($k = 0, \pm 1, \pm 2, \dots$). Then

$$(-4)^{1/4} = \sqrt{2} \exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] = \sqrt{2} e^{i\pi/4} e^{ik\pi/2} \quad (k = 0, 1, 2, 3).$$

To be specific,

$$c_0 = \sqrt{2} e^{i\pi/4} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 1 + i,$$

$$c_1 = c_0 e^{i\pi/2} = (1 + i)i = -1 + i,$$

$$c_2 = c_0 e^{i\pi} = (1 + i)(-1) = -1 - i,$$

$$c_3 = c_0 e^{i3\pi/2} = (1 + i)(-i) = 1 - i.$$

This enables us to write

$$\begin{aligned} z^4 + 4 &= (z - c_0)(z - c_1)(z - c_2)(z - c_3) \\ &= [(z - c_1)(z - c_2)] \cdot [(z - c_0)(z - c_3)] \\ &= [(z + 1) - i][(z + 1) + i] \cdot [(z - 1) - i][(z - 1) + i] \\ &= [(z + 1)^2 + 1] \cdot [(z - 1)^2 + 1] \\ &= (z^2 + 2z + 2)(z^2 - 2z + 2). \end{aligned}$$

7. Let c be any n th root of unity other than unity itself. With the aid of the identity (see Exercise 10, Sec. 7),

$$1 + z + z^2 + \dots + z^{n-1} = \frac{1 - z^n}{1 - z} \quad (z \neq 1),$$

we find that

$$1 + c + c^2 + \dots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1 - 1}{1 - c} = 0.$$

9. Observe first that

$$(z^{1/m})^{-1} = \left[\sqrt[m]{r} \exp \frac{i(\theta + 2k\pi)}{m} \right]^{-1} = \frac{1}{\sqrt[m]{r}} \exp \frac{i(-\theta - 2k\pi)}{m} = \frac{1}{\sqrt[m]{r}} \exp \frac{i(-\theta)}{m} \exp \frac{i(-2k\pi)}{m}$$

and

$$(z^{-1})^{1/m} = \sqrt[m]{\frac{1}{r}} \exp \frac{i(-\theta + 2k\pi)}{m} = \frac{1}{\sqrt[m]{r}} \exp \frac{i(-\theta)}{m} \exp \frac{i(2k\pi)}{m},$$

where $k = 0, 1, 2, \dots, m-1$. Since the set

$$\exp \frac{i(-2k\pi)}{m} \quad (k = 0, 1, 2, \dots, m-1)$$

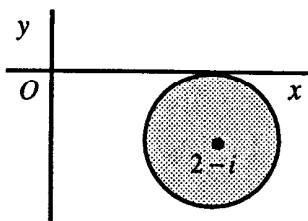
is the same as the set

$$\exp \frac{i(2k\pi)}{m} \quad (k = 0, 1, 2, \dots, m-1),$$

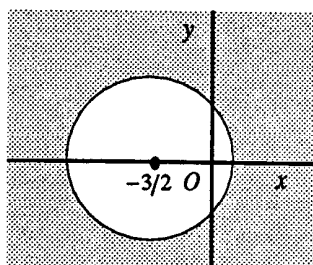
but in reverse order, we find that $(z^{1/m})^{-1} = (z^{-1})^{1/m}$.

SECTION 10

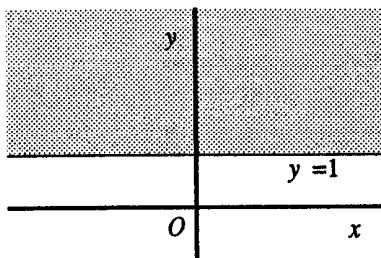
1. (a) Write $|z - 2 + i| \leq 1$ as $|z - (2 - i)| \leq 1$ to see that this is the set of points inside and on the circle centered at the point $2 - i$ with radius 1. It is *not* a domain.



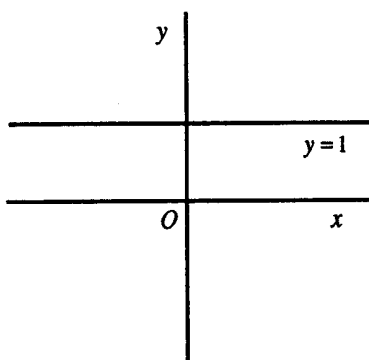
- (b) Write $|2z + 3| > 4$ as $\left|z - \left(-\frac{3}{2}\right)\right| > 2$ to see that the set in question consists of all points exterior to the circle with center at $-3/2$ and radius 2. It is a domain.



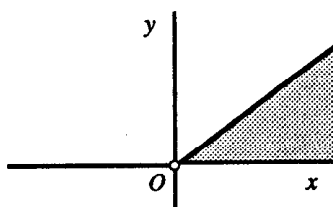
- (c) Write $\text{Im } z > 1$ as $y > 1$ to see that this is the half plane consisting of all points lying above the horizontal line $y = 1$. It is a domain.



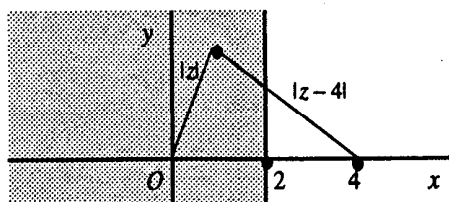
- (d) The set $\text{Im } z = 1$ is simply the horizontal line $y = 1$. It is *not* a domain.



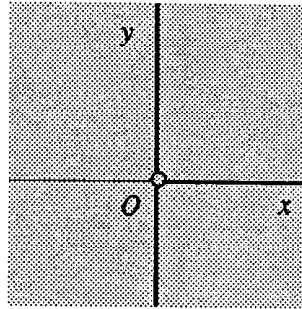
- (e) The set $0 \leq \arg z \leq \frac{\pi}{4}$ ($z \neq 0$) is indicated below. It is *not* a domain.



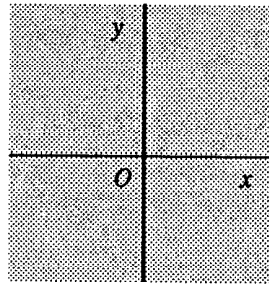
- (f) The set $|z - 4| \geq |z|$ can be written in the form $(x - 4)^2 + y^2 \geq x^2 + y^2$, which reduces to $x \leq 2$. This set, which is indicated below, is *not* a domain. The set is also geometrically evident since it consists of all points z such that the distance between z and 4 is greater than or equal to the distance between z and the origin.



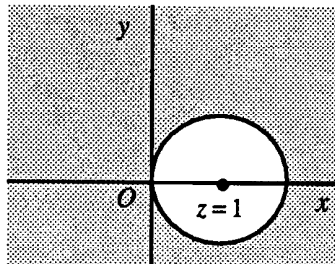
4. (a) The closure of the set $-\pi < \arg z < \pi$ ($z \neq 0$) is the entire plane.



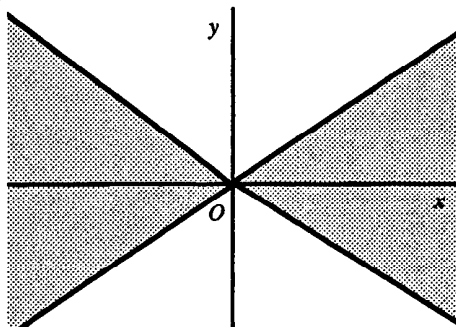
- (b) We first write the set $|\operatorname{Re} z| < |z|$ as $|x| < \sqrt{x^2 + y^2}$, or $x^2 < x^2 + y^2$. But this last inequality is the same as $y^2 > 0$, or $|y| > 0$. Hence the closure of the set $|\operatorname{Re} z| < |z|$ is the entire plane.



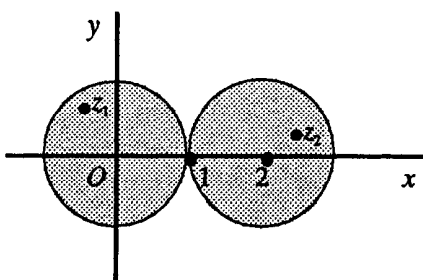
- (c) Since $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$, the set $\operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$ can be written as $\frac{x}{x^2+y^2} \leq \frac{1}{2}$, or $(x^2 - 2x) + y^2 \geq 0$. Finally, by completing the square, we arrive at the inequality $(x-1)^2 + y^2 \geq 1^2$, which describes the circle, together with its exterior, that is centered at $z=1$ with radius 1. The closure of this set is itself.



- (d) Since $z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$, the set $\operatorname{Re}(z^2) > 0$ can be written as $y^2 < x^2$, or $|y| < |x|$. The closure of this set consists of the lines $y = \pm x$ together with the shaded region shown below.



5. The set S consists of all points z such that $|z| < 1$ or $|z - 2| < 1$, as shown below.



Since every polygonal line joining z_1 and z_2 must contain at least one point that is not in S , it is clear that S is not connected.

8. We are given that a set S contains each of its accumulation points. The problem here is to show that S must be closed. We do this by contradiction. We let z_0 be a boundary point of S and suppose that it is not a point in S . The fact that z_0 is a boundary point means that every neighborhood of z_0 contains at least one point in S ; and, since z_0 is not in S , we see that every *deleted* neighborhood of S must contain at least one point in S . Thus z_0 is an accumulation point of S , and it follows that z_0 is a point in S . But this contradicts the fact that z_0 is not in S . We may conclude, then, that each boundary point z_0 must be in S . That is, S is closed.

Chapter 2

SECTION 11

1. (a) The function $f(z) = \frac{1}{z^2 + 1}$ is defined everywhere in the finite plane except at the points $z = \pm i$, where $z^2 + 1 = 0$.
- (b) The function $f(z) = \text{Arg}\left(\frac{1}{z}\right)$ is defined throughout the entire finite plane except for the point $z = 0$.
- (c) The function $f(z) = \frac{z}{z + \bar{z}}$ is defined everywhere in the finite plane except for the imaginary axis. This is because the equation $z + \bar{z} = 0$ is the same as $x = 0$.
- (d) The function $f(z) = \frac{1}{1 - |z|^2}$ is defined everywhere in the finite plane except on the circle $|z| = 1$, where $1 - |z|^2 = 0$.
3. Using $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$, write

$$\begin{aligned} f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= \frac{(z + \bar{z})^2}{4} + \frac{(z - \bar{z})^2}{4} + i(z - \bar{z}) + i(z + \bar{z}) - \frac{(z + \bar{z})(z - \bar{z})}{2} \\ &= \frac{z^2}{2} + \frac{\bar{z}^2}{2} + 2iz - \frac{z^2}{2} + \frac{\bar{z}^2}{2} = \bar{z}^2 + 2iz. \end{aligned}$$

SECTION 17

5. Consider the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x + iy}{x - iy}\right)^2 \quad (z \neq 0),$$

where $z = x + iy$. Observe that if $z = (x, 0)$, then

$$f(z) = \left(\frac{x + i0}{x - i0}\right)^2 = 1;$$

and if $z = (0, y)$,

$$f(z) = \left(\frac{0 + iy}{0 - iy}\right)^2 = 1.$$

But if $z = (x, x)$,

$$f(z) = \left(\frac{x + ix}{x - ix} \right)^2 = \left(\frac{1+i}{1-i} \right)^2 = -1.$$

This shows that $f(z)$ has value 1 at all nonzero points on the real and imaginary axes but value -1 at all nonzero points on the line $y = x$. Thus the limit of $f(z)$ as z tends to 0 cannot exist.

10. (a) To show that $\lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = 4$, we use statement (2), Sec. 16, and write

$$\lim_{z \rightarrow 0} \frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z}-1\right)^2} = \lim_{z \rightarrow 0} \frac{4}{(1-z)^2} = 4.$$

- (b) To establish the limit $\lim_{z \rightarrow 1} \frac{1}{(z-1)^3} = \infty$, we refer to statement (1), Sec. 16, and write

$$\lim_{z \rightarrow 1} \frac{1}{1/(z-1)^3} = \lim_{z \rightarrow 1} (z-1)^3 = 0.$$

- (c) To verify that $\lim_{z \rightarrow \infty} \frac{z^2+1}{z-1} = \infty$, we apply statement (3), Sec. 16, and write

$$\lim_{z \rightarrow 0} \frac{\frac{1}{z}-1}{\left(\frac{1}{z}\right)^2+1} = \lim_{z \rightarrow 0} \frac{z-z^2}{1+z^2} = 0.$$

11. In this problem, we consider the function

$$T(z) = \frac{az+b}{cz+d} \quad (ad-bc \neq 0).$$

- (a) Suppose that $c = 0$. Statement (3), Sec. 16, tells us that $\lim_{z \rightarrow \infty} T(z) = \infty$ since

$$\lim_{z \rightarrow 0} \frac{1}{T(1/z)} = \lim_{z \rightarrow 0} \frac{c+dz}{a+bz} = \frac{c}{a} = 0.$$

(b) Suppose that $c \neq 0$. Statement (2), Sec. 16, reveals that $\lim_{z \rightarrow \infty} T(z) = \frac{a}{c}$ since

$$\lim_{z \rightarrow 0} T\left(\frac{1}{z}\right) = \lim_{z \rightarrow 0} \frac{a + bz}{c + dz} = \frac{a}{c}.$$

Also, we know from statement (1), Sec. 16, that $\lim_{z \rightarrow -d/c} T(z) = \infty$ since

$$\lim_{z \rightarrow -d/c} \frac{1}{T(z)} = \lim_{z \rightarrow -d/c} \frac{cz + d}{az + b} = 0.$$

SECTION 19

1. (a) If $f(z) = 3z^2 - 2z + 4$, then

$$f'(z) = \frac{d}{dz}(3z^2 - 2z + 4) = 3\frac{d}{dz}z^2 - 2\frac{d}{dz}z + \frac{d}{dz}4 = 3(2z) - 2(1) + 0 = 6z - 2.$$

(b) If $f(z) = (1 - 4z^2)^3$, then

$$f'(z) = 3(1 - 4z^2)^2 \frac{d}{dz}(1 - 4z^2) = 3(1 - 4z^2)^2(-8z) = -24z(1 - 4z^2)^2.$$

(c) If $f(z) = \frac{z-1}{2z+1}$ ($z \neq -\frac{1}{2}$), then

$$f'(z) = \frac{(2z+1)\frac{d}{dz}(z-1) - (z-1)\frac{d}{dz}(2z+1)}{(2z+1)^2} = \frac{(2z+1)(1) - (z-1)2}{(2z+1)^2} = \frac{3}{(2z+1)^2}.$$

(d) If $f(z) = \frac{(1+z^2)^4}{z^2}$ ($z \neq 0$), then

$$\begin{aligned} f'(z) &= \frac{z^2 \frac{d}{dz}(1+z^2)^4 - (1+z^2)^4 \frac{d}{dz}z^2}{(z^2)^2} = \frac{z^2 4(1+z^2)^3(2z) - (1+z^2)^4 2z}{(z^2)^2} \\ &= \frac{2z(1+z^2)^3[4z^2 - (1+z^2)]}{z^4} = \frac{2(1+z^2)^3(3z^2 - 1)}{z^3}. \end{aligned}$$

3. If $f(z) = 1/z$ ($z \neq 0$), then

$$\Delta w = f(z + \Delta z) - f(z) = \frac{1}{z + \Delta z} - \frac{1}{z} = \frac{-\Delta z}{(z + \Delta z)z}.$$

Hence

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{-1}{(z + \Delta z)z} = -\frac{1}{z^2}.$$

4. We are given that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. According to the definition of derivative,

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{f(z)}{z - z_0}.$$

Similarly,

$$g'(z_0) = \lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{g(z)}{z - z_0}.$$

Thus

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \frac{\lim_{z \rightarrow z_0} f(z)/(z - z_0)}{\lim_{z \rightarrow z_0} g(z)/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}.$$

SECTION 22

1. (a) $f(z) = \bar{z} = x - iy$. So $u = x$, $v = -y$.

Inasmuch as $u_x = v_y \Rightarrow 1 = -1$, the Cauchy-Riemann equations are not satisfied anywhere.

(b) $f(z) = z - \bar{z} = (x + iy) - (x - iy) = 0 + i2y$. So $u = 0$, $v = 2y$.

Since $u_x = v_y \Rightarrow 0 = 2$, the Cauchy-Riemann equations are not satisfied anywhere.

(c) $f(z) = 2x + ix^2$. Here $u = 2x$, $v = xy^2$.

$$u_x = v_y \Rightarrow 2 = 2xy \Rightarrow xy = 1.$$

$$u_y = -v_x \Rightarrow 0 = -y^2 \Rightarrow y = 0.$$

Substituting $y = 0$ into $xy = 1$, we have $0 = 1$. Thus the Cauchy-Riemann equations do not hold anywhere.

(d) $f(z) = e^x e^{-iy} = e^x (\cos y - i \sin y) = e^x \cos y - ie^x \sin y$. So $u = e^x \cos y$, $v = -e^x \sin y$.

$$u_x = v_y \Rightarrow e^x \cos y = -e^x \cos y \Rightarrow 2e^x \cos y = 0 \Rightarrow \cos y = 0. \text{ Thus}$$

$$y = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

$$u_y = -v_x \Rightarrow -e^x \sin y = e^x \sin y \Rightarrow 2e^x \sin y = 0 \Rightarrow \sin y = 0. \text{ Hence}$$

$$y = n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

Since these are two different sets of values of y , the Cauchy-Riemann equations cannot be satisfied anywhere.

3. (a) $f(z) = \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}$. So

$$u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}.$$

Since

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y \quad \text{and} \quad u_y = \frac{-2xy}{(x^2 + y^2)^2} = -v_x \quad (x^2 + y^2 \neq 0),$$

$f'(z)$ exists when $z \neq 0$. Moreover, when $z \neq 0$,

$$\begin{aligned} f'(z) = u_x + iv_x &= \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \frac{2xy}{(x^2 + y^2)^2} = -\frac{x^2 - i2xy - y^2}{(x^2 + y^2)^2} \\ &= -\frac{(x - iy)^2}{(x^2 + y^2)^2} = -\frac{(\bar{z})^2}{(z\bar{z})^2} = -\frac{(\bar{z})^2}{(z)^2(\bar{z})^2} = -\frac{1}{z^2}. \end{aligned}$$

(b) $f(z) = x^2 + iy^2$. Hence $u = x^2$ and $v = y^2$. Now

$$u_x = v_y \Rightarrow 2x = 2y \Rightarrow y = x \quad \text{and} \quad u_y = -v_x \Rightarrow 0 = 0.$$

So $f'(z)$ exists only when $y = x$, and we find that

$$f'(x + ix) = u_x(x, x) + iv_x(x, x) = 2x + i0 = 2x.$$

(c) $f(z) = z \operatorname{Im} z = (x + iy)y = xy + iy^2$. Here $u = xy$ and $v = y^2$. We observe that

$$u_x = v_y \Rightarrow y = 2y \Rightarrow y = 0 \quad \text{and} \quad u_y = -v_x \Rightarrow x = 0.$$

Hence $f'(z)$ exists only when $z = 0$. In fact,

$$f'(0) = u_x(0, 0) + iv_x(0, 0) = 0 + i0 = 0.$$

4. (a) $f(z) = \frac{1}{z^4} = \left(\underbrace{\frac{1}{r^4} \cos 4\theta}_u \right) + i \left(\underbrace{-\frac{1}{r^4} \sin 4\theta}_v \right) \quad (z \neq 0)$. Since

$$ru_r = -\frac{4}{r^4} \cos 4\theta = v_\theta \quad \text{and} \quad u_\theta = -\frac{4}{r^4} \sin 4\theta = -rv_r,$$

f is analytic in its domain of definition. Furthermore,

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta} \left(-\frac{4}{r^5} \cos 4\theta + i \frac{4}{r^5} \sin 4\theta \right) \\ &= -\frac{4}{r^5} e^{-i\theta} (\cos 4\theta - i \sin 4\theta) = -\frac{4}{r^5} e^{-i\theta} e^{-i4\theta} \\ &= \frac{-4}{r^5 e^{i5\theta}} = -\frac{4}{(re^{i\theta})^5} = -\frac{4}{z^5}. \end{aligned}$$

(b) $f(z) = \sqrt{r} e^{i\theta/2} = \underbrace{\sqrt{r} \cos \frac{\theta}{2}}_u + i \underbrace{\sqrt{r} \sin \frac{\theta}{2}}_v$ ($r > 0, \alpha < \theta < \alpha + 2\pi$). Since

$$ru_r = \frac{\sqrt{r}}{2} \cos \frac{\theta}{2} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt{r}}{2} \sin \frac{\theta}{2} = -rv_r,$$

f is analytic in its domain of definition. Moreover,

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta} \left(\frac{1}{2\sqrt{r}} \cos \frac{\theta}{2} + i \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2} \right) \\ &= \frac{1}{2\sqrt{r}} e^{-i\theta} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = \frac{1}{2\sqrt{r}} e^{-i\theta} e^{i\theta/2} \\ &= \frac{1}{2\sqrt{r} e^{i\theta/2}} = \frac{1}{2f(z)}. \end{aligned}$$

(c) $f(z) = \underbrace{e^{-\theta} \cos(\ln r)}_u + i \underbrace{e^{-\theta} \sin(\ln r)}_v$ ($r > 0, 0 < \theta < 2\pi$). Since

$$ru_r = -e^{-\theta} \sin(\ln r) = v_\theta \quad \text{and} \quad u_\theta = -e^{-\theta} \cos(\ln r) = -rv_r,$$

f is analytic in its domain of definition. Also,

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta} \left[-\frac{e^{-\theta} \sin(\ln r)}{r} + i \frac{e^{-\theta} \cos(\ln r)}{r} \right] \\ &= \frac{i}{re^{i\theta}} [e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r)] = i \frac{f(z)}{z}. \end{aligned}$$

5. When $f(z) = x^3 + i(1-y)^3$, we have $u = x^3$ and $v = (1-y)^3$. Observe that

$$u_x = v_y \Rightarrow 3x^2 = -3(1-y)^2 \Rightarrow x^2 + (1-y)^2 = 0 \quad \text{and} \quad u_y = -v_x \Rightarrow 0 = 0.$$

Evidently, then, the Cauchy-Riemann equations are satisfied only when $x=0$ and $y=1$. That is, they hold only when $z=i$. Hence the expression

$$f'(z) = u_x + iv_x = 3x^2 + i0 = 3x^2$$

is valid only when $z=i$, in which case we see that $f'(i) = 0$.

6. Here u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Now

$$f(z) = \underbrace{\frac{x^3 - 3xy^2}{x^2 + y^2}}_u + i \underbrace{\frac{y^3 - 3x^2y}{x^2 + y^2}}_v$$

when $z \neq 0$, and the following calculations show that

$$u_x(0,0) = v_y(0,0) \quad \text{and} \quad u_y(0,0) = -v_x(0,0):$$

$$u_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{u(0 + \Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1,$$

$$u_y(0,0) = \lim_{\Delta y \rightarrow 0} \frac{u(0, 0 + \Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = 0,$$

$$v_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{v(0 + \Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0,$$

$$v_y(0,0) = \lim_{\Delta y \rightarrow 0} \frac{v(0, 0 + \Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta y} = 1.$$

7. Equations (2), Sec. 22, are

$$\begin{aligned} u_x \cos \theta + u_y \sin \theta &= u_r, \\ -u_x r \sin \theta + u_y r \cos \theta &= u_\theta. \end{aligned}$$

Solving these simultaneous linear equations for u_x and u_y , we find that

$$u_x = u_r \cos \theta - u_\theta \frac{\sin \theta}{r} \quad \text{and} \quad u_y = u_r \sin \theta + u_\theta \frac{\cos \theta}{r}.$$

Likewise,

$$v_x = v_r \cos \theta - v_\theta \frac{\sin \theta}{r} \quad \text{and} \quad v_y = v_r \sin \theta + v_\theta \frac{\cos \theta}{r}.$$

Assume now that the Cauchy-Riemann equations in polar form,

$$ru_r = v_\theta, \quad u_\theta = -rv_r,$$

are satisfied at z_0 . It follows that

$$\begin{aligned} u_x &= u_r \cos \theta - u_\theta \frac{\sin \theta}{r} = v_\theta \frac{\cos \theta}{r} + v_r \sin \theta = v_r \sin \theta + v_\theta \frac{\cos \theta}{r} = v_y, \\ u_y &= u_r \sin \theta + u_\theta \frac{\cos \theta}{r} = v_\theta \frac{\sin \theta}{r} - v_r \cos \theta = -\left(v_r \cos \theta - v_\theta \frac{\sin \theta}{r}\right) = -v_x. \end{aligned}$$

9. (a) Write $f(z) = u(r, \theta) + iv(r, \theta)$. Then recall the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r,$$

of the Cauchy-Riemann equations, which enables us to rewrite the expression (Sec. 22)

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

for the derivative of f at a point $z_0 = (r_0, \theta_0)$ in the following way:

$$f'(z_0) = e^{-i\theta} \left(\frac{1}{r} v_\theta - \frac{i}{r} u_\theta \right) = \frac{-i}{re^{i\theta}} (u_\theta + iv_\theta) = \frac{-i}{z_0} (u_\theta + iv_\theta).$$

(b) Consider now the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}.$$

With

$$u(r, \theta) = \frac{\cos\theta}{r} \quad \text{and} \quad v(r, \theta) = -\frac{\sin\theta}{r},$$

the final expression for $f'(z_0)$ in part (a) tells us that

$$\begin{aligned} f'(z) &= \frac{-i}{z} \left(-\frac{\sin\theta}{r} - i\frac{\cos\theta}{r} \right) = -\frac{1}{z} \left(\frac{\cos\theta - i\sin\theta}{r} \right) \\ &= -\frac{1}{z} \left(\frac{e^{-i\theta}}{r} \right) = -\frac{1}{z} \left(\frac{1}{re^{i\theta}} \right) = -\frac{1}{z^2} \end{aligned}$$

when $z \neq 0$.

10. (a) We consider a function $F(x, y)$, where

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}.$$

Formal application of the chain rule for multivariable functions yields

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{\partial F}{\partial x} \left(\frac{1}{2} \right) + \frac{\partial F}{\partial y} \left(-\frac{1}{2i} \right) = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).$$

(b) Now define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

suggested by part (a), and formally apply it to a function $f(z) = u(x, y) + iv(x, y)$:

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y} \\ &= \frac{1}{2} (u_x + iv_x) + \frac{i}{2} (u_y + iv_y) = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)]. \end{aligned}$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are satisfied, this tells us that $\partial f / \partial \bar{z} = 0$.

SECTION 24

1. (a) $f(z) = \underbrace{3x + y}_u + i\underbrace{(3y - x)}_v$ is entire since

$$u_x = 3 = v_y \quad \text{and} \quad u_y = 1 = -v_x.$$

(b) $f(z) = \underbrace{\sin x \cosh y}_u + i\underbrace{\cos x \sinh y}_v$ is entire since

$$u_x = \cos x \cosh y = v_y \quad \text{and} \quad u_y = \sin x \sinh y = -v_x.$$

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x = \underbrace{e^{-y} \sin x}_u + i\underbrace{(-e^{-y} \cos x)}_v$ is entire since

$$u_x = e^{-y} \cos x = v_y \quad \text{and} \quad u_y = -e^{-y} \sin x = -v_x.$$

(d) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire since it is the product of the entire functions

$$g(z) = z^2 - 2 \quad \text{and} \quad h(z) = e^{-x}e^{-iy} = e^{-x}(\cos y - i \sin y) = \underbrace{e^{-x} \cos y}_u + i\underbrace{(-e^{-x} \sin y)}_v.$$

The function g is entire since it is a polynomial, and h is entire since

$$u_x = -e^{-x} \cos y = v_y \quad \text{and} \quad u_y = -e^{-x} \sin y = -v_x.$$

2. (a) $f(z) = \underbrace{xy}_u + i\underbrace{y}_v$ is nowhere analytic since

$$u_x = v_y \Rightarrow y = 1 \quad \text{and} \quad u_y = -v_x \Rightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0, 1) = i$.

(c) $f(z) = e^y e^{ix} = e^y(\cos x + i \sin x) = \underbrace{e^y \cos x}_u + i\underbrace{e^y \sin x}_v$ is nowhere analytic since

$$u_x = v_y \Rightarrow -e^y \sin x = e^y \sin x \Rightarrow 2e^y \sin x = 0 \Rightarrow \sin x = 0$$

and

$$u_y = -v_x \Rightarrow e^y \cos x = -e^y \cos x \Rightarrow 2e^y \cos x = 0 \Rightarrow \cos x = 0.$$

More precisely, the roots of the equation $\sin x = 0$ are $n\pi$ ($n = 0, \pm 1, \pm 2, \dots$), and $\cos n\pi = (-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

7. (a) Suppose that a function $f(z) = u(x, y) + iv(x, y)$ is analytic and real-valued in a domain D . Since $f(z)$ is real-valued, it has the form $f(z) = u(x, y) + i0$. The Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ thus become $u_x = 0$, $u_y = 0$; and this means that $u(x, y) = a$, where a is a (real) constant. (See the proof of the theorem in Sec. 23.) Evidently, then, $f(z) = a$. That is, f is constant in D .

- (b) Suppose that a function f is analytic in a domain D and that its modulus $|f(z)|$ is constant there. Write $|f(z)| = c$, where c is a (real) constant. If $c = 0$, we see that $f(z) = 0$ throughout D . If, on the other hand, $c \neq 0$, write $f(z)\overline{f(z)} = c^2$, or

$$\overline{f(z)} = \frac{c^2}{f(z)}.$$

Since $f(z)$ is analytic and never zero in D , the conjugate $\overline{f(z)}$ must be analytic in D . Example 3 in Sec. 24 then tells us that $f(z)$ must be constant in D .

SECTION 25

(X)

1. (a) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x, y) = 2x(1 - y)$. To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = 2 - 2y$. Now

$$u_x = v_y \Rightarrow v_y = 2 - 2y \Rightarrow v(x, y) = 2y - y^2 + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow -2x = -\phi'(x) \Rightarrow \phi'(x) = 2x \Rightarrow \phi(x) = x^2 + c.$$

Consequently,

$$v(x, y) = 2y - y^2 + (x^2 + c) = x^2 - y^2 + 2y + c.$$

- (b) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x, y) = 2x - x^3 + 3xy^2$. To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \Rightarrow v_y = 2 - 3x^2 + 3y^2 \Rightarrow v(x, y) = 2y - 3x^2y + y^3 + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow 6xy = 6xy - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x, y) = 2y - 3x^2y + y^3 + c.$$

- (c) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x, y) = \sinh x \sin y$. To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = \cosh x \sin y$. Now

$$u_x = v_y \Rightarrow v_y = \cosh x \sin y \Rightarrow v(x, y) = -\cosh x \cos y + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow \sinh x \cos y = \sinh x \cos y - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x, y) = -\cosh x \cos y + c.$$

- (d) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x, y) = \frac{y}{x^2 + y^2}$. To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = -\frac{2xy}{(x^2 + y^2)^2}$. Now

$$u_x = v_y \Rightarrow v_y = -\frac{2xy}{(x^2 + y^2)^2} \Rightarrow v(x, y) = \frac{x}{x^2 + y^2} + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x, y) = \frac{x}{x^2 + y^2} + c.$$

2. Suppose that v and V are harmonic conjugates of u in a domain D . This means that

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_x = V_y, \quad u_y = -V_x.$$

If $w = v - V$, then,

$$w_x = v_x - V_x = -u_y + u_y = 0 \quad \text{and} \quad w_y = v_y - V_y = u_x - u_x = 0.$$

Hence $w(x, y) = c$, where c is a (real) constant (compare the proof of the theorem in Sec. 23). That is, $v(x, y) - V(x, y) = c$.

3. Suppose that u and v are harmonic conjugates of each other in a domain D . Then

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad v_x = u_y, \quad v_y = -u_x.$$

It follows readily from these equations that

$$u_x = 0, \quad u_y = 0 \quad \text{and} \quad v_x = 0, \quad v_y = 0.$$

Consequently, $u(x, y)$ and $v(x, y)$ must be constant throughout D (compare the proof of the theorem in Sec. 23).

5. The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

Now

$$ru_r = v_\theta \Rightarrow ru_{rr} + u_r = v_{\theta r}$$

and

$$u_{\theta} = -rv_r \Rightarrow u_{\theta\theta} = -rv_{r\theta}.$$

Thus

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = rv_{\theta r} - rv_{r\theta};$$

and, since $v_{\theta r} = v_{r\theta}$, we have

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0,$$

which is the polar form of Laplace's equation. To show that v satisfies the same equation, we observe that

$$u_{\theta} = -rv_r \Rightarrow v_r = -\frac{1}{r}u_{\theta} \Rightarrow v_{rr} = \frac{1}{r^2}u_{\theta} - \frac{1}{r}u_{\theta r}$$

and

$$ru_r = v_{\theta} \Rightarrow v_{\theta\theta} = ru_{r\theta}.$$

Since $u_{\theta r} = u_{r\theta}$, then,

$$r^2 v_{rr} + rv_r + v_{\theta\theta} = u_{\theta} - ru_{\theta r} - u_{\theta} + ru_{r\theta} = 0.$$

6. If $u(r, \theta) = \ln r$, then

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = r^2 \left(-\frac{1}{r^2} \right) + r \left(\frac{1}{r} \right) + 0 = 0.$$

This tells us that the function $u = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$. Now it follows from the Cauchy-Riemann equation $ru_r = v_{\theta}$ and the derivative $u_r = \frac{1}{r}$ that $v_{\theta} = 1$; thus $v(r, \theta) = \theta + \phi(r)$, where $\phi(r)$ is at present an arbitrary differentiable function of r . The other Cauchy-Riemann equation $u_{\theta} = -rv_r$, then becomes $0 = -r\phi'(r)$. That is, $\phi'(r) = 0$; and we see that $\phi(r) = c$, where c is an arbitrary (real) constant. Hence $v(r, \theta) = \theta + c$ is a harmonic conjugate of $u(r, \theta) = \ln r$.

Chapter 3

SECTION 28

1. (a) $\exp(2 \pm 3\pi i) = e^2 \exp(\pm 3\pi i) = -e^2$, since $\exp(\pm 3\pi i) = -1$.

(b)
$$\begin{aligned} \exp \frac{2 + \pi i}{4} &= \left(\exp \frac{1}{2} \right) \left(\exp \frac{\pi i}{4} \right) = \sqrt{e} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ &= \sqrt{e} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \sqrt{\frac{e}{2}} (1 + i). \end{aligned}$$

(c) $\exp(z + \pi i) = (\exp z)(\exp \pi i) = -\exp z$, since $\exp \pi i = -1$.

3. First write

$$\exp(\bar{z}) = \exp(x - iy) = e^x e^{-iy} = e^x \cos y - i e^x \sin y,$$

where $z = x + iy$. This tells us that $\exp(\bar{z}) = u(x, y) + iv(x, y)$, where

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = -e^x \sin y.$$

Suppose that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at some point $z = x + iy$. It is easy to see that, for the functions u and v here, these equations become $\cos y = 0$ and $\sin y = 0$. But there is no value of y satisfying this pair of equations. We may conclude that, since the Cauchy-Riemann equations fail to be satisfied anywhere, the function $\exp(\bar{z})$ is not analytic anywhere.

4. The function $\exp(z^2)$ is entire since it is a composition of the entire functions z^2 and $\exp z$; and the chain rule for derivatives tells us that

$$\frac{d}{dz} \exp(z^2) = \exp(z^2) \frac{d}{dz} z^2 = 2z \exp(z^2).$$

Alternatively, one can show that $\exp(z^2)$ is entire by writing

$$\begin{aligned} \exp(z^2) &= \exp[(x + iy)^2] = \exp(x^2 - y^2) \exp(i2xy) \\ &= \underbrace{\exp(x^2 - y^2) \cos(2xy)}_u + i \underbrace{\exp(x^2 - y^2) \sin(2xy)}_v \end{aligned}$$

and using the Cauchy-Riemann equations. To be specific,

$$u_x = 2x \exp(x^2 - y^2) \cos(2xy) - 2y \exp(x^2 - y^2) \sin(2xy) = v_y,$$

and

$$u_y = -2y \exp(x^2 - y^2) \cos(2xy) - 2x \exp(x^2 - y^2) \sin(2xy) = -v_x.$$

Furthermore,

$$\begin{aligned}\frac{d}{dz}\exp(z^2) &= u_x + iv_x = 2(x + iy)\left[\exp(x^2 - y^2)\cos(2xy) + i\exp(x^2 - y^2)\sin(2xy)\right] \\ &= 2z\exp(z^2).\end{aligned}$$

5. We first write

$$|\exp(2z + i)| = |\exp[2x + i(2y + 1)]| = e^{2x}$$

and

$$|\exp(iz^2)| = |\exp[-2xy + i(x^2 - y^2)]| = e^{-2xy}.$$

Then, since

$$|\exp(2z + i) + \exp(iz^2)| \leq |\exp(2z + i)| + |\exp(iz^2)|,$$

it follows that

$$|\exp(2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

6. First write

$$|\exp(z^2)| = |\exp[(x + iy)^2]| = |\exp(x^2 - y^2 + i2xy)| = \exp(x^2 - y^2)$$

and

$$\exp(|z|^2) = \exp(x^2 + y^2).$$

Since $x^2 - y^2 \leq x^2 + y^2$, it is clear that $\exp(x^2 - y^2) \leq \exp(x^2 + y^2)$. Hence it follows from the above that

$$|\exp(z^2)| \leq \exp(|z|^2).$$

7. To prove that $|\exp(-2z)| < 1 \Leftrightarrow \operatorname{Re} z > 0$, write

$$|\exp(-2z)| = |\exp(-2x - i2y)| = \exp(-2x).$$

It is then clear that the statement to be proved is the same as $\exp(-2x) < 1 \Leftrightarrow x > 0$, which is obvious from the graph of the exponential function in calculus.

8. (a) Write $e^z = -2$ as $e^x e^{iy} = 2e^{i\pi}$. This tells us that

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = (2n+1)\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

Hence

$$z = \ln 2 + (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

(b) Write $e^z = 1 + \sqrt{3}i$ as $e^x e^{iy} = 2e^{i(\pi/3)}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

Consequently,

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

(c) Write $\exp(2z-1) = 1$ as $e^{2x-1} e^{i2y} = 1e^{i0}$ and note how it follows that

$$e^{2x-1} = 1 \quad \text{and} \quad 2y = 0 + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

Evidently, then,

$$x = \frac{1}{2} \quad \text{and} \quad y = n\pi \quad (n = 0, \pm 1, \pm 2, \dots);$$

and this means that

$$z = \frac{1}{2} + n\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

9. This problem is actually to find all roots of the equation

$$\overline{\exp(iz)} = \exp(i\bar{z}).$$

To do this, set $z = x + iy$ and rewrite the equation as

$$e^{-y}e^{-ix} = e^ye^{ix}.$$

Now, according to the statement in italics at the beginning of Sec.8 in the text,

$$e^{-y} = e^y \quad \text{and} \quad -x = x + 2n\pi,$$

where n may have any one of the values $n = 0, \pm 1, \pm 2, \dots$. Thus

$$y = 0 \quad \text{and} \quad x = n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

The roots of the original equation are, therefore,

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

10. (a) Suppose that e^z is real. Since $e^z = e^x \cos y + ie^x \sin y$, this means that $e^x \sin y = 0$. Moreover, since e^x is never zero, $\sin y = 0$. Consequently, $y = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$); that is, $\text{Im } z = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$).

(b) On the other hand, suppose that e^z is pure imaginary. It follows that $\cos y = 0$, or that $y = \frac{\pi}{2} + n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). That is, $\text{Im } z = \frac{\pi}{2} + n\pi$ ($n = 0, \pm 1, \pm 2, \dots$).

12. We start by writing

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

Because $\text{Re}(e^z) = e^x \cos y$, it follows that

$$\text{Re}(e^{1/z}) = \exp\left(\frac{x}{x^2 + y^2}\right) \cos\left(\frac{-y}{x^2 + y^2}\right) = \exp\left(\frac{x}{x^2 + y^2}\right) \cos\left(\frac{y}{x^2 + y^2}\right).$$

Since $e^{1/z}$ is analytic in every domain that does not contain the origin, Theorem 1 in Sec. 25 ensures that $\text{Re}(e^{1/z})$ is harmonic in such a domain.

13. If $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain D , then

$$e^{f(z)} = e^{u(x, y)} \cos v(x, y) + ie^{u(x, y)} \sin v(x, y).$$

Since $e^{f(z)}$ is a composition of functions that are analytic in D , it follows from Theorem 1 in Sec. 25 that its component functions

$$U(x, y) = e^{u(x, y)} \cos v(x, y), \quad V(x, y) = e^{u(x, y)} \sin v(x, y)$$

are harmonic in D . Moreover, by Theorem 2 in Sec. 25, $V(x, y)$ is a harmonic conjugate of $U(x, y)$.

14. The problem here is to establish the identity

$$(\exp z)^n = \exp(nz) \quad (n = 0, \pm 1, \pm 2, \dots).$$

(a) To show that it is true when $n = 0, 1, 2, \dots$, we use mathematical induction. It is obviously true when $n = 0$. Suppose that it is true when $n = m$, where m is any nonnegative integer. Then

$$(\exp z)^{m+1} = (\exp z)^m (\exp z) = \exp(mz) \exp z = \exp(mz + z) = \exp[(m+1)z].$$

(b) Suppose now that n is a negative integer ($n = -1, -2, \dots$), and write $m = -n = 1, 2, \dots$. In view of part (a),

$$(\exp z)^n = \left(\frac{1}{\exp z} \right)^m = \frac{1}{(\exp z)^m} = \frac{1}{\exp(mz)} = \frac{1}{\exp(-nz)} = \exp(nz).$$

SECTION 30

1. (a) $\text{Log}(-ei) = \ln|-ei| + i\text{Arg}(-ei) = \ln e - \frac{\pi}{2}i = 1 - \frac{\pi}{2}i.$

(b) $\text{Log}(1-i) = \ln|1-i| + i\text{Arg}(1-i) = \ln\sqrt{2} - \frac{\pi}{4}i = \frac{1}{2}\ln 2 - \frac{\pi}{4}i.$

2. (a) $\log e = \ln e + i(0 + 2n\pi) = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$

(b) $\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$

3. (a) Observe that

$$\text{Log}(1+i)^2 = \text{Log}(2i) = \ln 2 + \frac{\pi}{2}i$$

and

$$2\text{Log}(1+i) = 2\left(\ln\sqrt{2} + i\frac{\pi}{4}\right) = \ln 2 + \frac{\pi}{2}i.$$

Thus

$$\text{Log}(1+i)^2 = 2\text{Log}(1+i).$$

(b) On the other hand,

$$\operatorname{Log}(-1+i)^2 = \operatorname{Log}(-2i) = \ln 2 - \frac{\pi}{2}i$$

and

$$2\operatorname{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + \frac{3\pi}{2}i.$$

Hence

$$\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i).$$

4. (a) Consider the branch

$$\log z = \ln r + i\theta \quad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

Since

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i,$$

we find that $\log(i^2) = 2\log i$ when this branch of $\log z$ is taken.

(b) Now consider the branch

$$\log z = \ln r + i\theta \quad \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

Here

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{5\pi}{2}\right) = 5\pi i.$$

Hence, for this particular branch, $\log(i^2) \neq 2\log i$.

5. (a) The two values of $i^{1/2}$ are $e^{i\pi/4}$ and $e^{i5\pi/4}$. Observe that

$$\log(e^{i\pi/4}) = \ln 1 + i\left(\frac{\pi}{4} + 2n\pi\right) = \left(2n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots)$$

and

$$\log(e^{i5\pi/4}) = \ln 1 + i\left(\frac{5\pi}{4} + 2n\pi\right) = \left[(2n+1) + \frac{1}{4}\right]\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Combining these two sets of values, we find that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

On the other hand,

$$\frac{1}{2} \log i = \frac{1}{2} \left[\ln 1 + i \left(\frac{\pi}{2} + 2n\pi \right) \right] = \left(n + \frac{1}{4} \right) \pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Thus the set of values of $\log(i^{1/2})$ is the same as the set of values of $\frac{1}{2} \log i$, and we may write

$$\log(i^{1/2}) = \frac{1}{2} \log i.$$

(b) Note that

$$\log(i^2) = \log(-1) = \ln 1 + (\pi + 2n\pi)i = (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots)$$

but that

$$2 \log i = 2 \left[\ln 1 + i \left(\frac{\pi}{2} + 2n\pi \right) \right] = (4n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Evidently, then, the set of values of $\log(i^2)$ is *not* the same as the set of values of $2 \log i$. That is,

$$\log(i^2) \neq 2 \log i.$$

7. To solve the equation $\log z = i\pi/2$, write $\exp(\log z) = \exp(i\pi/2)$, or $z = e^{i\pi/2} = i$.

10. Since $\ln(x^2 + y^2)$ is the real component of any (analytic) branch of $2 \log z$, it is harmonic in every domain that does not contain the origin. This can be verified directly by writing $u(x, y) = \ln(x^2 + y^2)$ and showing that $u_{xx}(x, y) + u_{yy}(x, y) = 0$.

SECTION 31

1. Suppose that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Then

$$z_1 = r_1 \exp i\Theta_1 \quad \text{and} \quad z_2 = r_2 \exp i\Theta_2,$$

where

$$-\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}.$$

The fact that $-\pi < \Theta_1 + \Theta_2 < \pi$ enables us to write

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \operatorname{Log}[(r_1 r_2) \exp i(\Theta_1 + \Theta_2)] = \ln(r_1 r_2) + i(\Theta_1 + \Theta_2) \\ &= (\ln r_1 + i\Theta_1) + (\ln r_2 + i\Theta_2) = \operatorname{Log}(r_1 \exp i\Theta_1) + \operatorname{Log}(r_2 \exp i\Theta_2) \\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2.\end{aligned}$$

3. We are asked to show in two different ways that

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \quad (z_1 \neq 0, z_2 \neq 0).$$

(a) One way is to refer to the relation $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ in Sec. 7 and write

$$\log\left(\frac{z_1}{z_2}\right) = \ln\left|\frac{z_1}{z_2}\right| + i \arg\left(\frac{z_1}{z_2}\right) = (\ln|z_1| + i \arg z_1) - (\ln|z_2| + i \arg z_2) = \log z_1 - \log z_2.$$

(b) Another way is to first show that $\log\left(\frac{1}{z}\right) = -\log z$ ($z \neq 0$). To do this, we write $z = re^{i\theta}$

and then

$$\log\left(\frac{1}{z}\right) = \log\left(\frac{1}{r} e^{-i\theta}\right) = \ln\left(\frac{1}{r}\right) + i(-\theta + 2n\pi) = -[\ln r + i(\theta - 2n\pi)] = -\log z,$$

where $n = 0, \pm 1, \pm 2, \dots$. This enables us to use the relation

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

and write

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(z_1 \frac{1}{z_2}\right) = \log z_1 + \log\left(\frac{1}{z_2}\right) = \log z_1 - \log z_2.$$

5. The problem here is to verify that

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \quad (n = -1, -2, \dots),$$

given that it is valid when $n = 1, 2, \dots$. To do this, we put $m = -n$, where n is a negative integer. Then, since m is a positive integer, we may use the relations $z^{-1} = 1/z$ and $1/e^z = e^{-z}$ to write

$$\begin{aligned} z^{1/n} &= (z^{1/m})^{-1} = \left[\exp\left(\frac{1}{m} \log z\right) \right]^{-1} \\ &= 1 / \left[\exp\left(\frac{1}{m} \log z\right) \right] = \exp\left(-\frac{1}{m} \log z\right) = \exp\left(\frac{1}{n} \log z\right). \end{aligned}$$

SECTION 32

1. In each part below, $n = 0, \pm 1, \pm 2, \dots$.

$$\begin{aligned} (a) \quad (1+i)^i &= \exp[i \log(1+i)] = \exp\left\{i \left[\ln \sqrt{2} + i \left(\frac{\pi}{4} + 2n\pi \right) \right] \right\} \\ &= \exp\left[\frac{i}{2} \ln 2 - \left(\frac{\pi}{4} + 2n\pi \right) \right] = \exp\left(-\frac{\pi}{4} - 2n\pi\right) \exp\left(\frac{i}{2} \ln 2\right). \end{aligned}$$

Since n takes on all integral values, the term $-2n\pi$ here can be replaced by $+2n\pi$. Thus

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(\frac{i}{2} \ln 2\right).$$

$$(b) \quad (-1)^{1/\pi} = \exp\left[\frac{1}{\pi} \log(-1)\right] = \exp\left\{\frac{1}{\pi} [\ln 1 + i(\pi + 2n\pi)]\right\} = \exp[(2n+1)i].$$

$$2. \quad (a) \quad \text{P.V. } i^i = \exp(i \text{Log} i) = \exp\left[i \left(\ln 1 + i \frac{\pi}{2} \right)\right] = \exp\left(-\frac{\pi}{2}\right).$$

$$\begin{aligned} (b) \quad \text{P.V. } \left[\frac{e}{2} (-1 - \sqrt{3}i) \right]^{3\pi i} &= \exp\left\{3\pi i \text{Log} \left[\frac{e}{2} (-1 - \sqrt{3}i) \right]\right\} = \exp\left[3\pi i \left(\ln e - i \frac{2\pi}{3} \right)\right] \\ &= \exp(2\pi^2) \exp(i3\pi) = -\exp(2\pi^2). \end{aligned}$$

$$\begin{aligned}
 (c) \text{ P.V. } (1-i)^{4i} &= \exp[4i \operatorname{Log}(1-i)] = \exp\left[4i\left(\ln\sqrt{2} - i\frac{\pi}{4}\right)\right] = e^\pi e^{i4\ln\sqrt{2}} \\
 &= e^\pi [\cos(4\ln\sqrt{2}) + i\sin(4\ln\sqrt{2})] = e^\pi [\cos(2\ln 2) + i\sin(2\ln 2)].
 \end{aligned}$$

3. Since $-1 + \sqrt{3}i = 2e^{2\pi i/3}$, we may write

$$\begin{aligned}
 (-1 + \sqrt{3}i)^{3/2} &= \exp\left[\frac{3}{2}\log(-1 + \sqrt{3}i)\right] = \exp\left\{\frac{3}{2}\left[\ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right)\right]\right\} \\
 &= \exp[\ln(2^{3/2}) + (3n+1)\pi i] = 2\sqrt{2} \exp[(3n+1)\pi i],
 \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$. Observe that if n is even, then $3n+1$ is odd; and so $\exp[(3n+1)\pi i] = -1$. On the other hand, if n is odd, $3n+1$ is even; and this means that $\exp[(3n+1)\pi i] = 1$. So only two distinct values of $(-1 + \sqrt{3}i)^{3/2}$ arise. Specifically,

$$(-1 + \sqrt{3}i)^{3/2} = \pm 2\sqrt{2}.$$

5. We consider here any nonzero complex number z_0 in the exponential form $z_0 = r_0 \exp i\Theta_0$, where $-\pi < \Theta_0 \leq \pi$. According to Sec. 8, the principal value of $z^{1/n}$ is $\sqrt[n]{r_0} \exp\left(i\frac{\Theta_0}{n}\right)$; and, according to Sec. 32, that value is

$$\exp\left(\frac{1}{n}\operatorname{Log}z\right) = \exp\left[\frac{1}{n}(\ln r_0 + i\Theta_0)\right] = \exp(\ln \sqrt[n]{r_0}) \exp\left(i\frac{\Theta_0}{n}\right) = \sqrt[n]{r_0} \exp\left(i\frac{\Theta_0}{n}\right).$$

These two expressions are evidently the same.

7. Observe that when $c = a + bi$ is any fixed complex number, where $c \neq 0, \pm 1, \pm 2, \dots$, the power i^c can be written as

$$\begin{aligned}
 i^c &= \exp(c \log i) = \exp\left\{(a + bi)\left[\ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right\} \\
 &= \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right) + ia\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad (n = 0, \pm 1, \pm 2, \dots).
 \end{aligned}$$

Thus

$$|i^c| = \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad (n = 0, \pm 1, \pm 2, \dots),$$

and it is clear that $|i^c|$ is multiple-valued unless $b = 0$, or c is real. Note that the restriction $c \neq 0, \pm 1, \pm 2, \dots$ ensures that i^c is multiple-valued even when $b = 0$.

SECTION 33

1. The desired derivatives can be found by writing

$$\begin{aligned}\frac{d}{dz} \sin z &= \frac{d}{dz} \left(\frac{e^{iz} - e^{-iz}}{2i} \right) = \frac{1}{2i} \left(\frac{d}{dz} e^{iz} - \frac{d}{dz} e^{-iz} \right) \\ &= \frac{1}{2i} (ie^{iz} + ie^{-iz}) = \frac{e^{iz} + e^{-iz}}{2} = \cos z\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dz} \cos z &= \frac{d}{dz} \left(\frac{e^{iz} + e^{-iz}}{2} \right) = \frac{1}{2} \left(\frac{d}{dz} e^{iz} + \frac{d}{dz} e^{-iz} \right) \\ &= \frac{1}{2} (ie^{iz} - ie^{-iz}) \cdot \frac{i}{i} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.\end{aligned}$$

2. From the expressions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

we see that

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} = e^{iz}.$$

3. Equation (4), Sec. 33 is

$$2 \sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2).$$

Interchanging z_1 and z_2 here and using the fact that $\sin z$ is an odd function, we have

$$2 \cos z_1 \sin z_2 = \sin(z_1 + z_2) - \sin(z_1 - z_2).$$

Addition of corresponding sides of these two equations now yields

$$2(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) = 2 \sin(z_1 + z_2),$$

or

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

4. Differentiating each side of equation (5), Sec. 33, with respect to z_1 , we have

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

7. (a) From the identity $\sin^2 z + \cos^2 z = 1$, we have

$$\frac{\sin^2 z}{\cos^2 z} + \frac{\cos^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}, \quad \text{or} \quad 1 + \tan^2 z = \sec^2 z.$$

(b) Also,

$$\frac{\sin^2 z}{\sin^2 z} + \frac{\cos^2 z}{\sin^2 z} = \frac{1}{\sin^2 z}, \quad \text{or} \quad 1 + \cot^2 z = \csc^2 z.$$

9. From the expression

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

we find that

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x) \sinh^2 y \\ &= \sin^2 x + \sinh^2 y. \end{aligned}$$

The expression

$$\cos z = \cos x \cosh y + i \sin x \sinh y,$$

on the other hand, tells us that

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x(1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y \\ &= \cos^2 x + \sinh^2 y. \end{aligned}$$

10. Since $\sinh^2 y$ is never negative, it follows from Exercise 9 that

$$(a) \quad |\sin z|^2 \geq \sin^2 x, \quad \text{or} \quad |\sin z| \geq |\sin x|$$

and that

$$(b) \quad |\cos z|^2 \geq \cos^2 x, \quad \text{or} \quad |\cos z| \geq |\cos x|.$$

11. In this problem we shall use the identities

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

(a) Observe that

$$\sinh^2 y = |\sin z|^2 - \sin^2 x \leq |\sin z|^2$$

and

$$\begin{aligned} |\sin z|^2 &= \sin^2 x + (\cosh^2 y - 1) = \cosh^2 y - (1 - \sin^2 x) \\ &= \cosh^2 y - \cos^2 x \leq \cosh^2 y. \end{aligned}$$

Thus

$$\sinh^2 y \leq |\sin z|^2 \leq \cosh^2 y, \quad \text{or} \quad |\sinh y| \leq |\sin z| \leq \cosh y.$$

(b) On the other hand,

$$\sinh^2 y = |\cos z|^2 - \cos^2 x \leq |\cos z|^2$$

and

$$\begin{aligned} |\cos z|^2 &= \cos^2 x + (\cosh^2 y - 1) = \cosh^2 y - (1 - \cos^2 x) \\ &= \cosh^2 y - \sin^2 x \leq \cosh^2 y. \end{aligned}$$

Hence

$$\sinh^2 y \leq |\cos z|^2 \leq \cosh^2 y, \quad \text{or} \quad |\sinh y| \leq |\cos z| \leq \cosh y.$$

13. By writing $f(z) = \sin \bar{z} = \sin(x - iy) = \sin x \cosh y - i \cos x \sinh y$, we have

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = \sin x \cosh y \quad \text{and} \quad v(x, y) = -\cos x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are to hold, it is easy to see that

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0.$$

Since $\cosh y$ is never zero, it follows from the first of these equations that $\cos x = 0$; that is, $x = \frac{\pi}{2} + n\pi$ ($n = 0 \pm 1, \pm 2, \dots$). Furthermore, since $\sin x$ is nonzero for each of these values of x , the second equation tells us that $\sinh y = 0$, or $y = 0$. Thus the Cauchy-Riemann equations hold only at the points

$$z = \frac{\pi}{2} + n\pi \quad (n = 0 \pm 1, \pm 2, \dots).$$

Evidently, then, there is no neighborhood of any point throughout which f is analytic, and we may conclude that $\sin \bar{z}$ is not analytic anywhere.

The function $f(z) = \cos \bar{z} = \cos(x - iy) = \cos x \cosh y + i \sin x \sinh y$ can be written as

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = \cos x \cosh y \quad \text{and} \quad v(x, y) = \sin x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

So there is no neighborhood throughout which f is analytic, and this means that $\cos \bar{z}$ is nowhere analytic.

16. (a) Use expression (12), Sec. 33, to write

$$\overline{\cos(iz)} = \overline{\cos(-y + ix)} = \cos y \cosh x - i \sin y \sinh x$$

and

$$\cos(i\bar{z}) = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x.$$

This shows that $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z .

(b) Use expression (11), Sec. 33, to write

$$\overline{\sin(iz)} = \overline{\sin(-y + ix)} = -\sin y \cosh x - i \cos y \sinh x$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Evidently, then, the equation $\overline{\sin(iz)} = \sin(i\bar{z})$ is equivalent to the pair of equations

$$\sin y \cosh x = 0, \quad \cos y \sinh x = 0.$$

Since $\cosh x$ is never zero, the first of these equations tells us that $\sin y = 0$. Consequently, $y = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Since $\cos n\pi = (-1)^n \neq 0$, the second equation tells us that $\sinh x = 0$, or that $x = 0$. So we may conclude that $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = 0 + in\pi = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$).

17. Rewriting the equation $\sin z = \cosh 4$ as $\sin x \cosh y + i \cos x \sinh y = \cosh 4$, we see that we need to solve the pair of equations

$$\sin x \cosh y = \cosh 4, \quad \cos x \sinh y = 0$$

for x and y . If $y = 0$, the first equation becomes $\sin x = \cosh 4$, which cannot be satisfied by any x since $\sin x \leq 1$ and $\cosh 4 > 1$. So $y \neq 0$, and the second equation requires that $\cos x = 0$. Thus

$$x = \frac{\pi}{2} + n\pi \quad (n = 0 \pm 1, \pm 2, \dots).$$

Since

$$\sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n,$$

the first equation then becomes $(-1)^n \cosh y = \cosh 4$, which cannot hold when n is odd. If n is even, it follows that $y = \pm 4$. Finally, then, the roots of $\sin z = \cosh 4$ are

$$z = \left(\frac{\pi}{2} + 2n\pi\right) \pm 4i \quad (n = 0 \pm 1, \pm 2, \dots).$$

18. The problem here is to find all roots of the equation $\cos z = 2$. We start by writing that equation as $\cos x \cosh y - i \sin x \sinh y = 2$. Thus we need to solve the pair of equations

$$\cos x \cosh y = 2, \quad \sin x \sinh y = 0$$

for x and y . We note that $y \neq 0$ since $\cos x = 2$ if $y = 0$, and that is impossible. So the second in the pair of equations to be solved tells us that $\sin x = 0$, or that $x = n\pi$ ($n = 0 \pm 1, \pm 2, \dots$). The first equation then tells us that $(-1)^n \cosh y = 2$; and, since $\cosh y$ is always positive, n must be even. That is, $x = 2n\pi$ ($n = 0 \pm 1, \pm 2, \dots$). But this means that $\cosh y = 2$, or $y = \cosh^{-1} 2$. Consequently, the roots of the given equation are

$$z = 2n\pi + i \cosh^{-1} 2 \quad (n = 0 \pm 1, \pm 2, \dots).$$

To express $\cosh^{-1} 2$, which has two values, in a different way, we begin with $y = \cosh^{-1} 2$, or $\cosh y = 2$. This tells us that $e^y + e^{-y} = 4$; and, rewriting this as

$$(e^y)^2 - 4(e^y) + 1 = 0,$$

we may apply the quadratic formula to obtain $e^y = 2 \pm \sqrt{3}$, or $y = \ln(2 \pm \sqrt{3})$. Finally, with the observation that

$$\ln(2 - \sqrt{3}) = \ln\left[\frac{(2 - \sqrt{3})(2 + \sqrt{3})}{2 + \sqrt{3}}\right] = \ln\left(\frac{1}{2 + \sqrt{3}}\right) = -\ln(2 + \sqrt{3}),$$

we arrive at this alternative form of the roots:

$$z = 2n\pi \pm i \ln(2 + \sqrt{3}) \quad (n = 0 \pm 1, \pm 2, \dots).$$

SECTION 34

1. To find the derivatives of $\sinh z$ and $\cosh z$, we write

$$\frac{d}{dz} \sinh z = \frac{d}{dz} \left(\frac{e^z - e^{-z}}{2} \right) = \frac{1}{2} \frac{d}{dz} (e^z - e^{-z}) = \frac{e^z + e^{-z}}{2} = \cosh z$$

and

$$\frac{d}{dz} \cosh z = \frac{d}{dz} \left(\frac{e^z + e^{-z}}{2} \right) = \frac{1}{2} \frac{d}{dz} (e^z + e^{-z}) = \frac{e^z - e^{-z}}{2} = \sinh z.$$

3. Identity (7), Sec. 33, is $\sin^2 z + \cos^2 z = 1$. Replacing z by iz here and using the identities

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z,$$

we find that $i^2 \sinh^2 z + \cosh^2 z = 1$, or

$$\cosh^2 z - \sinh^2 z = 1.$$

Identity (6), Sec. 33, is $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Replacing z_1 by iz_1 and z_2 by iz_2 here, we have $\cos[i(z_1 + z_2)] = \cos(iz_1) \cos(iz_2) - \sin(iz_1) \sin(iz_2)$. The same identities that were used just above then lead to

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

6. We wish to show that

$$|\sinh x| \leq |\cosh z| \leq \cosh x$$

in two different ways.

(a) Identity (12), Sec. 34, is $|\cosh z|^2 = \sinh^2 x + \cos^2 y$. Thus $|\cosh z|^2 - \sinh^2 x \geq 0$; and this tells us that $\sinh^2 x \leq |\cosh z|^2$, or $|\sinh x| \leq |\cosh z|$. On the other hand, since $|\cosh z|^2 = (\cosh^2 x - 1) + \cos^2 y = \cosh^2 x - (1 - \cos^2 y) = \cosh^2 x - \sin^2 y$, we know that $|\cosh z|^2 - \cosh^2 x \leq 0$. Consequently, $|\cosh z|^2 \leq \cosh^2 x$, or $|\cosh z| \leq \cosh x$.

(b) Exercise 11(b), Sec. 33, tells us that $|\sinh y| \leq |\cos z| \leq \cosh y$. Replacing z by iz here and recalling that $\cos iz = \cosh z$ and $iz = -y + ix$, we obtain the desired inequalities.

7. (a) Observe that

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} - e^{-z} e^{-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z.$$

(b) Also,

$$\cosh(z + \pi i) = \frac{e^{z+\pi i} + e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} + e^{-z} e^{-\pi i}}{2} = \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z.$$

(c) From parts (a) and (b), we find that

$$\tanh(z + \pi i) = \frac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z.$$

9. The zeros of the hyperbolic tangent function

$$\tanh z = \frac{\sinh z}{\cosh z}$$

are the same as the zeros of $\sinh z$, which are $z = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$). The singularities of $\tanh z$ are the zeros of $\cosh z$, or $z = \left(\frac{\pi}{2} + n\pi\right)i$ ($n = 0, \pm 1, \pm 2, \dots$).

15. (a) Observe that, since $\sinh z = i$ can be written as $\sinh x \cos y + i \cosh x \sin y = i$, we need to solve the pair of equations

$$\sinh x \cos y = 0, \quad \cosh x \sin y = 1.$$

If $x = 0$, the second of these equations becomes $\sin y = 1$; and so $y = \frac{\pi}{2} + 2n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Hence

$$z = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

If $x \neq 0$, the first equation requires that $\cos y = 0$, or $y = \frac{\pi}{2} + n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). The second then becomes $(-1)^n \cosh x = 1$. But there is no nonzero value of x satisfying this equation, and we have no additional roots of $\sinh z = i$.

(b) Rewriting $\cosh z = \frac{1}{2}$ as $\cosh x \cos y + i \sinh x \sin y = \frac{1}{2}$, we see that x and y must satisfy the pair of equations

$$\cosh x \cos y = \frac{1}{2}, \quad \sinh x \sin y = 0.$$

If $x = 0$, the second equation is satisfied and the first equation becomes $\cos y = \frac{1}{2}$. Thus $y = \cos^{-1} \frac{1}{2} = \pm \frac{\pi}{3} + 2n\pi$ ($n = 0, \pm 1, \pm 2, \dots$), and this means that

$$z = \left(2n \pm \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

If $x \neq 0$, the second equation tells us that $y = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). The first then becomes $(-1)^n \cosh x = \frac{1}{2}$. But this equation in x has no solution since $\cosh x \geq 1$ for all x . Thus no additional roots of $\cosh z = \frac{1}{2}$ are obtained.

16. Let us rewrite $\cosh z = -2$ as $\cosh x \cos y + i \sinh x \sin y = -2$. The problem is evidently to solve the pair of equations

$$\cosh x \cos y = -2, \quad \sinh x \sin y = 0.$$

If $x = 0$, the second equation is satisfied and the first reduces to $\cos y = -2$. Since there is no y satisfying this equation, no roots of $\cosh z = -2$ arise.

If $x \neq 0$, we find from the second equation that $\sin y = 0$, or $y = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Since $\cos n\pi = (-1)^n$, it follows from the first equation that $(-1)^n \cosh x = -2$. But this equation can hold only when n is odd, in which case $x = \cosh^{-1} 2$. Consequently,

$$z = \cosh^{-1} 2 + (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Recalling from the solution of Exercise 18, Sec 33, that $\cosh^{-1} 2 = \pm \ln(2 + \sqrt{3})$, we note that these roots can also be written as

$$z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Chapter 4

SECTION 37

$$2. (a) \int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i \int_1^2 \frac{dt}{t} = -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4;$$

$$(b) \int_0^{\pi/6} e^{i2t} dt = \left[\frac{e^{i2t}}{2i} \right]_0^{\pi/6} = \frac{1}{2i} \left[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} - 1 \right] = \frac{\sqrt{3}}{4} + \frac{i}{4};$$

(c) Since $|e^{-bz}| = e^{-bx}$, we find that

$$\int_0^{\infty} e^{-zt} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-zt} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{-zt}}{-z} \right]_{t=0}^{t=b} = \frac{1}{z} \lim_{b \rightarrow \infty} (1 - e^{-bz}) = \frac{1}{z} \text{ when } \operatorname{Re} z > 0.$$

3. The problem here is to verify that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

To do this, we write

$$I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

and observe that when $m \neq n$,

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

When $m = n$, I becomes

$$I = \int_0^{2\pi} d\theta = 2\pi;$$

and the verification is complete.

4. First of all,

$$\int_0^{\pi} e^{(1+i)x} dx = \int_0^{\pi} e^x \cos x dx + i \int_0^{\pi} e^x \sin x dx.$$

But also,

$$\int_0^{\pi} e^{(1+i)x} dx = \left[\frac{e^{(1+i)x}}{1+i} \right]_0^{\pi} = \frac{e^{\pi} e^{i\pi} - 1}{1+i} = \frac{-e^{\pi} - 1}{1+i} \cdot \frac{1-i}{1-i} = -\frac{1+e^{\pi}}{2} + i \frac{1+e^{\pi}}{2}.$$

Equating the real parts and then the imaginary parts of these two expressions, we find that

$$\int_0^{\pi} e^x \cos x \, dx = -\frac{1+e^{\pi}}{2} \quad \text{and} \quad \int_0^{\pi} e^x \sin x \, dx = \frac{1+e^{\pi}}{2}.$$

5. Consider the function $w(t) = e^{it}$ and observe that

$$\int_0^{2\pi} w(t) \, dt = \int_0^{2\pi} e^{it} \, dt = \left[\frac{e^{it}}{i} \right]_0^{2\pi} = \frac{1}{i} - \frac{1}{i} = 0.$$

Since $|w(c)(2\pi - 0)| = |e^{ic}|2\pi = 2\pi$ for every real number c , it is clear that there is no number c in the interval $0 < t < 2\pi$ such that

$$\int_0^{2\pi} w(t) \, dt = w(c)(2\pi - 0).$$

6. (a) Suppose that $w(t)$ is even. It is straightforward to show that $u(t)$ and $v(t)$ must be even. Thus

$$\begin{aligned} \int_{-a}^a w(t) \, dt &= \int_{-a}^a u(t) \, dt + i \int_{-a}^a v(t) \, dt = 2 \int_0^a u(t) \, dt + 2i \int_0^a v(t) \, dt \\ &= 2 \left[\int_0^a u(t) \, dt + i \int_0^a v(t) \, dt \right] = 2 \int_0^a w(t) \, dt. \end{aligned}$$

(b) Suppose, on the other hand, that $w(t)$ is odd. It follows that $u(t)$ and $v(t)$ are odd, and so

$$\int_{-a}^a w(t) \, dt = \int_{-a}^a u(t) \, dt + i \int_{-a}^a v(t) \, dt = 0 + i0 = 0.$$

7. Consider the functions

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x + i\sqrt{1-x^2} \cos \theta)^n \, d\theta \quad (n = 0, 1, 2, \dots),$$

where $-1 \leq x \leq 1$. Since

$$\left| x + i\sqrt{1-x^2} \cos \theta \right| = \sqrt{x^2 + (1-x^2)\cos^2 \theta} \leq \sqrt{x^2 + (1-x^2)} = 1,$$

it follows that

$$|P_n(x)| \leq \frac{1}{\pi} \int_0^{\pi} \left| x + i\sqrt{1-x^2} \cos \theta \right|^n \, d\theta \leq \frac{1}{\pi} \int_0^{\pi} d\theta = 1.$$

SECTION 38

1. (a) Start by writing

$$I = \int_{-b}^{-a} w(-t) dt = \int_{-b}^{-a} u(-t) dt + i \int_{-b}^{-a} v(-t) dt.$$

The substitution $\tau = -t$ in each of these two integrals on the right then yields

$$I = -\int_b^a u(\tau) d\tau - i \int_b^a v(\tau) d\tau = \int_a^b u(\tau) d\tau + i \int_a^b v(\tau) d\tau = \int_a^b w(\tau) d\tau.$$

That is,

$$\int_{-b}^{-a} w(-t) dt = \int_a^b w(\tau) d\tau.$$

(b) Start with

$$I = \int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

and then make the substitution $t = \phi(\tau)$ in each of the integrals on the right. The result is

$$I = \int_{\alpha}^{\beta} u[\phi(\tau)]\phi'(\tau) d\tau + i \int_{\alpha}^{\beta} v[\phi(\tau)]\phi'(\tau) d\tau = \int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau) d\tau.$$

That is,

$$\int_a^b w(t) dt = \int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau) d\tau.$$

3. The slope of the line through the points (α, a) and (β, b) in the τt plane is

$$m = \frac{b-a}{\beta-\alpha}.$$

So the equation of that line is

$$t - a = \frac{b-a}{\beta-\alpha}(\tau - \alpha).$$

Solving this equation for t , one can rewrite it as

$$t = \frac{b-a}{\beta-\alpha} \tau + \frac{a\beta - b\alpha}{\beta-\alpha}.$$

Since $t = \phi(\tau)$, then,

$$\phi(\tau) = \frac{b-a}{\beta-\alpha} \tau + \frac{a\beta - b\alpha}{\beta-\alpha}.$$

4. If $Z(\tau) = z[\phi(\tau)]$, where $z(t) = x(t) + iy(t)$ and $t = \phi(\tau)$, then

$$Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)].$$

Hence

$$\begin{aligned} Z'(\tau) &= \frac{d}{d\tau} x[\phi(\tau)] + i \frac{d}{d\tau} y[\phi(\tau)] = x'[\phi(\tau)]\phi'(\tau) + iy'[\phi(\tau)]\phi'(\tau) \\ &= \{x'[\phi(\tau)] + iy'[\phi(\tau)]\}\phi'(\tau) = z'[\phi(\tau)]\phi'(\tau). \end{aligned}$$

5. If $w(t) = f[z(t)]$ and $f(z) = u(x, y) + iv(x, y)$, $z(t) = x(t) + iy(t)$, we have

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

The chain rule tells us that

$$\frac{du}{dt} = u_x x' + u_y y' \quad \text{and} \quad \frac{dv}{dt} = v_x x' + v_y y',$$

and so

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y').$$

In view of the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, then,

$$w'(t) = (u_x x' - v_x y') + i(v_x x' + u_x y') = (u_x + iv_x)(x' + iy').$$

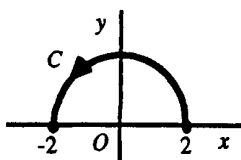
That is,

$$w'(t) = \{u_x[x(t), y(t)] + iv_x[x(t), y(t)]\}[x'(t) + iy'(t)] = f'[z(t)]z'(t)$$

when $t = t_0$.

SECTION 40

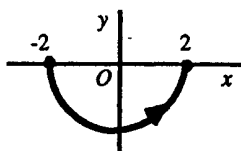
1. (a) Let C be the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$), shown below.



Then

$$\begin{aligned} \int_C \frac{z+2}{z} dz &= \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^\pi \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta} d\theta = 2i \int_0^\pi (e^{i\theta} + 1) d\theta \\ &= 2i \left[\frac{e^{i\theta}}{i} + \theta \right]_0^\pi = 2i(i + \pi + i) = -4 + 2\pi i. \end{aligned}$$

- (b) Now let C be the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$) just below.



This is the same as part (a), except for the limits of integration. Thus

$$\int_C \frac{z+2}{z} dz = 2i \left[\frac{e^{i\theta}}{i} + \theta \right]_\pi^{2\pi} = 2i(-i + 2\pi - i - \pi) = 4 + 2\pi i.$$

- (c) Finally, let C denote the entire circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$). In this case,

$$\int_C \frac{z+2}{z} dz = 4\pi i,$$

the value here being the sum of the values of the integrals in parts (a) and (b).

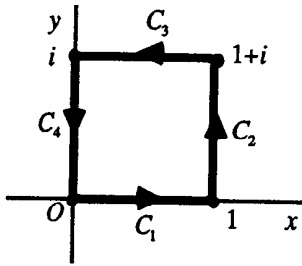
2. (a) The arc is $C: z = 1 + e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$). Then

$$\begin{aligned} \int_C (z-1) dz &= \int_\pi^{2\pi} (1 + e^{i\theta} - 1) i e^{i\theta} d\theta = i \int_\pi^{2\pi} e^{i2\theta} d\theta = i \left[\frac{e^{i2\theta}}{2i} \right]_\pi^{2\pi} \\ &= \frac{1}{2} (e^{i4\pi} - e^{i2\pi}) = \frac{1}{2} (1 - 1) = 0. \end{aligned}$$

(b) Here $C: z = x$ ($0 \leq x \leq 2$). Then

$$\int_C (z-1) dz = \int_0^2 (x-1) dx = \left[\frac{x^2}{2} - x \right]_0^2 = 0.$$

3. In this problem, the path C is the sum of the paths C_1 , C_2 , C_3 , and C_4 that are shown below.



The function to be integrated around the closed path C is $f(z) = \pi e^{\pi \bar{z}}$. We observe that $C = C_1 + C_2 + C_3 + C_4$ and find the values of the integrals along the individual legs of the square C .

(i) Since C_1 is $z = x$ ($0 \leq x \leq 1$),

$$\int_{C_1} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi x} dx = e^\pi - 1.$$

(ii) Since C_2 is $z = 1 + iy$ ($0 \leq y \leq 1$),

$$\int_{C_2} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi(1-iy)} i dy = e^\pi \pi i \int_0^1 e^{-i\pi y} dy = 2e^\pi.$$

(iii) Since C_3 is $z = (1-x) + i$ ($0 \leq x \leq 1$),

$$\int_{C_3} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi(1-x)-i} (-1) dx = \pi e^\pi \int_0^1 e^{-\pi x} dx = e^\pi - 1.$$

(iv) Since C_4 is $z = i(1-y)$ ($0 \leq y \leq 1$),

$$\int_{C_4} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{-\pi(1-y)i} (-i) dy = \pi i \int_0^1 e^{i\pi y} dy = -2.$$

Finally, then, since

$$\int_C \pi e^{\pi \bar{z}} dz = \int_{C_1} \pi e^{\pi \bar{z}} dz + \int_{C_2} \pi e^{\pi \bar{z}} dz + \int_{C_3} \pi e^{\pi \bar{z}} dz + \int_{C_4} \pi e^{\pi \bar{z}} dz,$$

we find that

$$\int_C \pi e^{\pi \bar{z}} dz = 4(e^\pi - 1).$$

4. The path C is the sum of the paths

$$C_1: z = x + ix^3 \quad (-1 \leq x \leq 0) \quad \text{and} \quad C_2: z = x + ix^3 \quad (0 \leq x \leq 1).$$

Using

$$f(z) = 1 \text{ on } C_1 \quad \text{and} \quad f(z) = 4y = 4x^3 \text{ on } C_2,$$

we have

$$\begin{aligned} \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz = \int_{-1}^0 1(1 + i3x^2) dx + \int_0^1 4x^3(1 + i3x^2) dx \\ &= \int_{-1}^0 dx + 3i \int_{-1}^0 x^2 dx + 4 \int_0^1 x^3 dx + 12i \int_0^1 x^5 dx \\ &= [x]_{-1}^0 + i[x^3]_{-1}^0 + [x^4]_0^1 + 2i[x^6]_0^1 = 1 + i + 1 + 2i = 2 + 3i. \end{aligned}$$

5. The contour C has some parametric representation $z = z(t)$ ($a \leq t \leq b$), where $z(a) = z_1$ and $z(b) = z_2$. Then

$$\int_C dz = \int_a^b z'(t) dt = [z(t)]_a^b = z(b) - z(a) = z_2 - z_1.$$

6. To integrate the branch

$$z^{-1+i} = e^{(-1+i)\log z} \quad (|z| > 0, 0 < \arg z < 2\pi)$$

around the circle $C: z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), write

$$\int_C z^{-1+i} dz = \int_C e^{(-1+i)\log z} dz = \int_0^{2\pi} e^{(-1+i)(\ln 1 + i\theta)} i e^{i\theta} d\theta = i \int_0^{2\pi} e^{-i\theta - \theta} e^{i\theta} d\theta = i \int_0^{2\pi} e^{-\theta} d\theta = i(1 - e^{-2\pi}).$$

7. Let C be the positively oriented circle $|z|=1$, with parametric representation $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), and let m and n be integers. Then

$$\int_C z^m \bar{z}^n dz = \int_0^{2\pi} (e^{i\theta})^m (e^{-i\theta})^n i e^{i\theta} d\theta = i \int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta.$$

But we know from Exercise 3, Sec. 37, that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Consequently,

$$\int_C z^m \bar{z}^n dz = \begin{cases} 0 & \text{when } m+1 \neq n, \\ 2\pi i & \text{when } m+1 = n. \end{cases}$$

8. Note that C is the right-hand half of the circle $x^2 + y^2 = 4$. So, on C , $x = \sqrt{4 - y^2}$. This suggests the parametric representation $C: z = \sqrt{4 - y^2} + iy$ ($-2 \leq y \leq 2$), to be used here. With that representation, we have

$$\begin{aligned} \int_C \bar{z} dz &= \int_{-2}^2 (\sqrt{4 - y^2} - iy) \left(\frac{-y}{\sqrt{4 - y^2}} + i \right) dy \\ &= \int_{-2}^2 (-y + y) dy + i \int_{-2}^2 \left(\frac{y^2}{\sqrt{4 - y^2}} + \sqrt{4 - y^2} \right) dy \\ &= i \int_{-2}^2 \frac{y^2 + 4 - y^2}{\sqrt{4 - y^2}} dy = 4i \int_{-2}^2 \frac{dy}{\sqrt{4 - y^2}} = 4i \left[\sin^{-1} \left(\frac{y}{2} \right) \right]_{-2}^2 \\ &= 4i [\sin^{-1}(1) - \sin^{-1}(-1)] = 4i \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = 4\pi i. \end{aligned}$$

10. Let C_0 be the circle $z = z_0 + Re^{i\theta}$ ($-\pi \leq \theta \leq \pi$).

$$(a) \int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} d\theta = i \int_{-\pi}^{\pi} d\theta = 2\pi i.$$

(b) When $n = \pm 1, \pm 2, \dots$,

$$\begin{aligned} \int_{C_0} (z - z_0)^{n-1} dz &= \int_{-\pi}^{\pi} (Re^{i\theta})^{n-1} Rie^{i\theta} d\theta = iR^n \int_{-\pi}^{\pi} e^{in\theta} d\theta \\ &= \frac{R^n}{n} (e^{in\pi} - e^{-in\pi}) = i \frac{2R^n}{n} \sin n\pi = 0. \end{aligned}$$

11. In this case, where a is any real number other than zero, the same steps as in Exercise 10(b), with a instead of n , yield the result

$$\int_{C_0} (z - z_0)^{a-1} dz = i \frac{2R^a}{a} \sin(a\pi).$$

12. (a) The function $f(z)$ is continuous on a smooth arc C , which has a parametric representation $z = z(t)$ ($a \leq t \leq b$). Exercise 1(b), Sec. 38, enables us to write

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]z'[\phi(\tau)]\phi'(\tau)d\tau,$$

where

$$Z(\tau) = z[\phi(\tau)] \quad (\alpha \leq \tau \leq \beta).$$

But expression (14), Sec 38, tells us that

$$z'[\phi(\tau)]\phi'(\tau) = Z'(\tau);$$

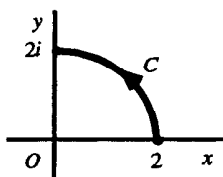
and so

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)d\tau.$$

- (b) Suppose that C is any contour and that $f(z)$ is piecewise continuous on C . Since C can be broken up into a finite chain of smooth arcs on which $f(z)$ is continuous, the identity obtained in part (a) remains valid.

SECTION 41

1. Let C be the arc of the circle $|z|=2$ shown below.



Without evaluating the integral, let us find an upper bound for $\left| \int_C \frac{dz}{z^2 - 1} \right|$. To do this, we note that if z is a point on C ,

$$|z^2 - 1| \geq ||z^2| - 1| = ||z|^2 - 1| = |4 - 1| = 3.$$

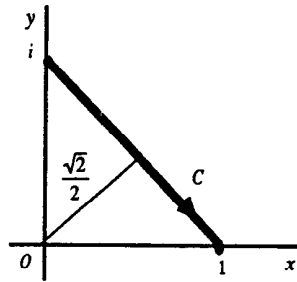
Thus

$$\left| \frac{1}{z^2 - 1} \right| = \frac{1}{|z^2 - 1|} \leq \frac{1}{3}.$$

Also, the length of C is $\frac{1}{4}(4\pi) = \pi$. So, taking $M = \frac{1}{3}$ and $L = \pi$, we find that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq ML = \frac{\pi}{3}.$$

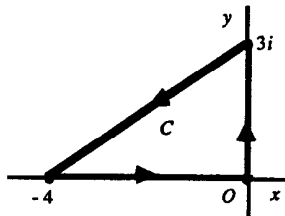
2. The path C is as shown in the figure below. The midpoint of C is clearly the closest point on C to the origin. The distance of that midpoint from the origin is clearly $\frac{\sqrt{2}}{2}$, the length of C being $\sqrt{2}$.



Hence if z is any point on C , $|z| \geq \frac{\sqrt{2}}{2}$. This means that, for such a point $\left| \frac{1}{z^4} \right| = \frac{1}{|z|^4} \leq 4$. Consequently, by taking $M = 4$ and $L = \sqrt{2}$, we have

$$\left| \int_C \frac{dz}{z^4} \right| \leq ML = 4\sqrt{2}.$$

3. The contour C is the closed triangular path shown below.



To find an upper bound for $\left| \int_C (e^z - \bar{z}) dz \right|$, we let z be a point on C and observe that

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = e^x + \sqrt{x^2 + y^2}.$$

But $e^x \leq 1$ since $x \leq 0$, and the distance $\sqrt{x^2 + y^2}$ of the point z from the origin is always less than or equal to 4. Thus $|e^z - \bar{z}| \leq 5$ when z is on C . The length of C is evidently 12. Hence, by writing $M = 5$ and $L = 12$, we have

$$\left| \int_C (e^z - \bar{z}) dz \right| \leq ML = 60.$$

4. Note that if $|z| = R$ ($R > 2$), then

$$|2z^2 - 1| \leq 2|z|^2 + 1 = 2R^2 + 1$$

and

$$|z^4 + 5z^2 + 4| = |z^2 + 1||z^2 + 4| \geq ||z|^2 - 1| ||z|^2 - 4| = (R^2 - 1)(R^2 - 4).$$

Thus

$$\left| \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \right| = \frac{|2z^2 - 1|}{|z^4 + 5z^2 + 4|} \leq \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$$

when $|z| = R$ ($R > 2$). Since the length of C_R is πR , then,

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} = \frac{\frac{\pi}{R} \left(2 + \frac{1}{R^2} \right)}{\left(1 - \frac{1}{R^2} \right) \left(1 - \frac{4}{R^2} \right)};$$

and it is clear that the value of the integral tends to zero as R tends to infinity.

5. Here C_R is the positively oriented circle $|z| = R$ ($R > 1$). If z is a point on C_R , then

$$\left| \frac{\text{Log } z}{z^2} \right| = \frac{|\ln R + i\Theta|}{R^2} \leq \frac{\ln R + |\Theta|}{R^2} \leq \frac{\pi + \ln R}{R^2},$$

since $-\pi < \Theta \leq \pi$. The length of C_R is, of course, $2\pi R$. Consequently, by taking

$$M = \frac{\pi + \ln R}{R^2} \quad \text{and} \quad L = 2\pi R,$$

we see that

$$\left| \int_{C_R} \frac{\text{Log } z}{z^2} dz \right| \leq ML = 2\pi \left(\frac{\pi + \ln R}{R} \right).$$

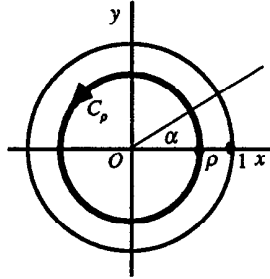
Since

$$\lim_{R \rightarrow \infty} \frac{\pi + \ln R}{R} = \lim_{R \rightarrow \infty} \frac{1/R}{1} = 0,$$

it follows that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{\text{Log } z}{z^2} dz = 0.$$

6. Let C_ρ be the positively oriented circle $|z| = \rho$ ($0 < \rho < 1$), shown in the figure below, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$.



We let $z^{-1/2}$ represent any particular branch

$$z^{-1/2} = \exp\left(-\frac{1}{2} \log z\right) = \exp\left[-\frac{1}{2}(\ln r + i\theta)\right] = \frac{1}{\sqrt{r}} \exp\left(-i\frac{\theta}{2}\right) \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the power function here; and we note that, since $f(z)$ is continuous on the *closed bounded* disk $|z| \leq 1$, there is a nonnegative constant M such that $|f(z)| \leq M$ for each point z in that disk. We are asked to find an upper bound for $\left| \int_{C_\rho} z^{-1/2} f(z) dz \right|$. To do this, we observe that if z is a point on C_ρ ,

$$|z^{-1/2} f(z)| = |z^{-1/2}| |f(z)| \leq \frac{M}{\sqrt{\rho}}.$$

Since the length of the path C_ρ is $2\pi\rho$, we may conclude that

$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq \frac{M}{\sqrt{\rho}} 2\pi\rho = 2\pi M \sqrt{\rho}.$$

Note that, inasmuch as M is independent of ρ , it follows that

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} z^{-1/2} f(z) dz = 0.$$

SECTION 43

1. The function z^n ($n = 0, 1, 2, \dots$) has the antiderivative $z^{n+1} / (n+1)$ everywhere in the finite plane. Consequently, for any contour C from a point z_1 to a point z_2 ,

$$\int_C z^n dz = \int_{z_1}^{z_2} z^n dz = \left. \frac{z^{n+1}}{n+1} \right]_{z_1}^{z_2} = \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1} = \frac{1}{n+1} (z_2^{n+1} - z_1^{n+1}).$$

2. (a)
$$\int_i^{i/2} e^{\pi z} dz = \left. \frac{e^{\pi z}}{\pi} \right]_i^{i/2} = \frac{e^{i\pi/2} - e^{i\pi}}{\pi} = \frac{i+1}{\pi} = \frac{1+i}{\pi}.$$

(b)
$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2 \sin\left(\frac{z}{2}\right) \Big|_0^{\pi+2i} = 2 \sin\left(\frac{\pi}{2} + i\right) = 2 \frac{e^{i(\frac{\pi}{2}+i)} - e^{-i(\frac{\pi}{2}+i)}}{2i} = -i(e^{i\pi/2} e^{-1} - e^{-i\pi/2} e)$$

$$= -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$$

(c)
$$\int_1^3 (z-2)^3 dz = \left. \frac{(z-2)^4}{4} \right]_1^3 = \frac{1}{4} - \frac{1}{4} = 0.$$

3. Note the function $(z - z_0)^{n-1}$ ($n = \pm 1, \pm 2, \dots$) always has an antiderivative in any domain that does not contain the point $z = z_0$. So, by the theorem in Sec. 42,

$$\int_{C_0} (z - z_0)^{n-1} dz = 0$$

for any closed contour C_0 that does not pass through z_0 .

5. Let C denote any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. This exercise asks us to evaluate the integral

$$I = \int_{-1}^1 z^i dz,$$

where z^i denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi).$$

An antiderivative of this branch *cannot* be used since the branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

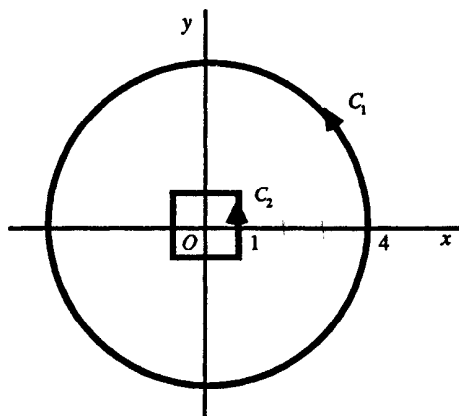
$$z^i = \exp(i \log z) \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right)$$

since it agrees with the integrand along C . Using an antiderivative of this new branch, we can now write

$$\begin{aligned} I &= \left. \frac{z^{i+1}}{i+1} \right|_{-1}^1 = \frac{1}{i+1} [(1)^{i+1} - (-1)^{i+1}] = \frac{1}{i+1} [e^{(i+1)\log 1} - e^{(i+1)\log(-1)}] \\ &= \frac{1}{i+1} [e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)}] = \frac{1}{i+1} (1 - e^{-\pi} e^{i\pi}) = \frac{1 + e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} \\ &= \frac{1 + e^{-\pi}}{2} (1-i). \end{aligned}$$

SECTION 46

2. The contours C_1 and C_2 are as shown in the figure below.



In each of the cases below, the singularities of the integrand lie outside C_1 or inside C_2 ; and so the integrand is analytic on the contours and between them. Consequently,

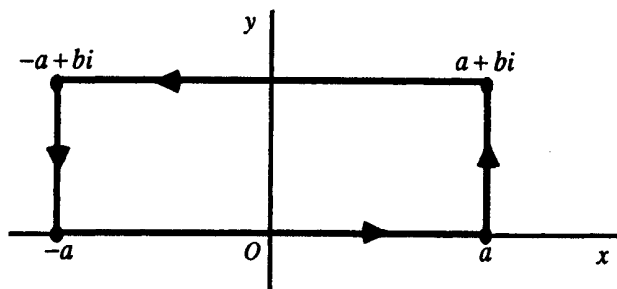
$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

(a) When $f(z) = \frac{1}{3z^2 + 1}$, the singularities are the points $z = \pm \frac{1}{\sqrt{3}}i$.

(b) When $f(z) = \frac{z+2}{\sin(z/2)}$, the singularities are at $z = 2n\pi$ ($n = 0, \pm 1, \pm 2, \dots$).

(c) When $f(z) = \frac{z}{1-e^z}$, the singularities are at $z = 2n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$).

4. (a) In order to derive the integration formula in question, we integrate the function e^{-z^2} around the closed rectangular path shown below.



Since the lower horizontal leg is represented by $z = x$ ($-a \leq x \leq a$), the integral of e^{-z^2} along that leg is

$$\int_{-a}^a e^{-x^2} dx = 2 \int_0^a e^{-x^2} dx.$$

Since the opposite direction of the upper horizontal leg has parametric representation $z = x + bi$ ($-a \leq x \leq a$), the integral of e^{-z^2} along the upper leg is

$$-\int_{-a}^a e^{-(x+bi)^2} dx = -e^{b^2} \int_{-a}^a e^{-x^2} e^{-i2bx} dx = -e^{b^2} \int_{-a}^a e^{-x^2} \cos 2bx dx + ie^{b^2} \int_{-a}^a e^{-x^2} \sin 2bx dx,$$

or simply

$$-2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx.$$

Since the right-hand vertical leg is represented by $z = a + iy$ ($0 \leq y \leq b$), the integral of e^{-z^2} along it is

$$\int_0^b e^{-(a+iy)^2} i dy = ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy.$$

Finally, since the opposite direction of the left-hand vertical leg has the representation $z = -a + iy$ ($0 \leq y \leq b$), the integral of e^{-z^2} along that vertical leg is

$$-\int_0^b e^{-(-a+iy)^2} i dy = -ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy.$$

According to the Cauchy-Goursat theorem, then,

$$2 \int_0^a e^{-x^2} dx - 2e^{-b^2} \int_0^a e^{-x^2} \cos 2bx dx + ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy = 0;$$

and this reduces to

$$\int_0^a e^{-x^2} \cos 2bx dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy.$$

- (b) We now let $a \rightarrow \infty$ in the final equation in part (a), keeping in mind the known integration formula

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

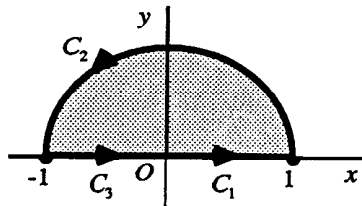
and the fact that

$$\left| e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy \right| \leq e^{-(a^2+b^2)} \int_0^b e^{y^2} dy \rightarrow 0 \text{ as } a \rightarrow \infty.$$

The result is

$$\int_0^{\infty} e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

6. We let C denote the entire boundary of the semicircular region appearing below. It is made up of the leg C_1 from the origin to the point $z = 1$, the semicircular arc C_2 that is shown, and the leg C_3 from $z = -1$ to the origin. Thus $C = C_1 + C_2 + C_3$.



We also let $f(z)$ be a continuous function that is defined on this closed semicircular region by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. The problem here is to evaluate the integral of $f(z)$ around C by evaluating the integrals along the individual paths C_1 , C_2 , and C_3 and then adding the results. In each case, we write a parametric representation for the path (or a related one) and then use it to evaluate the integral along the particular path.

(i) $C_1: z = re^{i0}$ ($0 \leq r \leq 1$). Then

$$\int_{C_1} f(z) dz = \int_0^1 \sqrt{r} \cdot 1 dr = \left[\frac{2}{3} r^{3/2} \right]_0^1 = \frac{2}{3}.$$

(ii) $C_2: z = 1 \cdot e^{i\theta}$ ($0 \leq \theta \leq \pi$). Then

$$\int_{C_2} f(z) dz = \int_0^\pi e^{i\theta/2} \cdot ie^{i\theta} d\theta = i \int_0^\pi e^{i3\theta/2} d\theta = i \left[\frac{2}{3i} e^{i3\theta/2} \right]_0^\pi = \frac{2}{3}(-i - 1) = -\frac{2}{3}(1 + i).$$

(iii) $-C_3: z = re^{i\pi}$ ($0 \leq r \leq 1$). Then

$$\int_{-C_3} f(z) dz = -\int_{C_3} f(z) dz = -\int_0^1 \sqrt{r}e^{i\pi/2}(-1)dr = i \int_0^1 \sqrt{r} dr = i \left[\frac{2}{3} r^{3/2} \right]_0^1 = \frac{2}{3}i.$$

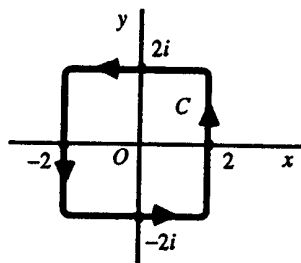
The desired result is

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \int_{C_3} f(z) dz = \frac{2}{3} - \frac{2}{3}(1 + i) + \frac{2}{3}i = 0.$$

The Cauchy-Goursat theorem does not apply since $f(z)$ is not analytic at the origin, or even defined on the negative imaginary axis.

SECTION 48

1. In this problem, we let C denote the square contour shown in the figure below.



$$(a) \int_C \frac{e^{-z} dz}{z - (\pi i / 2)} = 2\pi i [e^{-z}]_{z=\pi i/2} = 2\pi i(-i) = 2\pi.$$

$$(b) \int_C \frac{\cos z}{z(z^2 + 8)} dz = \int_C \frac{(\cos z) / (z^2 + 8)}{z - 0} dz = 2\pi i \left[\frac{\cos z}{z^2 + 8} \right]_{z=0} = 2\pi i \left(\frac{1}{8} \right) = \frac{\pi i}{4}.$$

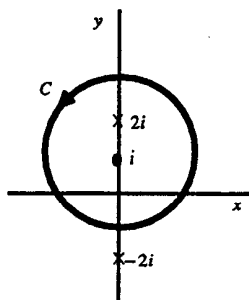
$$(c) \int_C \frac{z dz}{2z + 1} = \int_C \frac{z/2}{z - (-1/2)} dz = 2\pi i \left[\frac{z}{2} \right]_{z=-1/2} = 2\pi i \left(-\frac{1}{4} \right) = -\frac{\pi i}{2}.$$

$$(d) \int_C \frac{\cosh z}{z^4} dz = \int_C \frac{\cosh z}{(z-0)^{3+1}} dz = \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} \cosh z \right]_{z=0} = \frac{\pi i}{3} (0) = 0.$$

$$(e) \int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = \int_C \frac{\tan(z/2)}{(z-x_0)^{1+1}} dz = \frac{2\pi i}{1!} \left[\frac{d}{dz} \tan\left(\frac{z}{2}\right) \right]_{z=x_0}$$

$$= 2\pi i \left(\frac{1}{2} \sec^2 \frac{x_0}{2} \right) = i\pi \sec^2 \left(\frac{x_0}{2} \right) \text{ when } -2 < x_0 < 2.$$

2. Let C denote the positively oriented circle $|z - i| = 2$, shown below.



(a) The Cauchy integral formula enables us to write

$$\int_C \frac{dz}{z^2 + 4} = \int_C \frac{dz}{(z-2i)(z+2i)} = \int_C \frac{1/(z+2i)}{z-2i} dz = 2\pi i \left(\frac{1}{z+2i} \right)_{z=2i} = 2\pi i \left(\frac{1}{4i} \right) = \frac{\pi}{2}.$$

(b) Applying the extended form of the Cauchy integral formula, we have

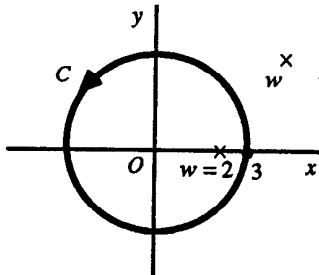
$$\int_C \frac{dz}{(z^2 + 4)^2} = \int_C \frac{dz}{(z-2i)^2(z+2i)^2} = \int_C \frac{1/(z+2i)^2}{(z-2i)^{1+1}} dz = \frac{2\pi i}{1!} \left[\frac{d}{dz} \frac{1}{(z+2i)^2} \right]_{z=2i}$$

$$= 2\pi i \left[\frac{-2}{(z+2i)^3} \right]_{z=2i} = \frac{-4\pi i}{(4i)^3} = \frac{-4\pi i}{-(16)(4)i} = \frac{\pi}{16}.$$

3. Let C be the positively oriented circle $|z|=3$, and consider the function

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w} dz \quad (|w| \neq 3).$$

We wish to find $g(w)$ when $w = 2$ and when $|w| > 3$ (see the figure below).



We observe that

$$g(2) = \int_C \frac{2z^2 - z - 2}{z - 2} dz = 2\pi i [2z^2 - z - 2]_{z=2} = 2\pi i(4) = 8\pi i.$$

On the other hand, when $|w| > 3$, the Cauchy-Goursat theorem tells us that $g(w) = 0$.

5. Suppose that a function f is analytic inside and on a simple closed contour C and that z_0 is not on C . If z_0 is inside C , then

$$\int_C \frac{f'(z) dz}{z - z_0} = 2\pi i f'(z_0) \quad \text{and} \quad \int_C \frac{f(z) dz}{(z - z_0)^2} = \int_C \frac{f(z) dz}{(z - z_0)^{1+1}} = \frac{2\pi i}{1!} f'(z_0).$$

Thus

$$\int_C \frac{f'(z) dz}{z - z_0} = \int_C \frac{f(z) dz}{(z - z_0)^2}.$$

The Cauchy-Goursat theorem tells us that this last equation is also valid when z_0 is exterior to C , each side of the equation being 0.

7. Let C be the unit circle $z = e^{i\theta}$ ($-\pi \leq \theta \leq \pi$), and let a denote any real constant. The Cauchy integral formula reveals that

$$\int_C \frac{e^{az}}{z} dz = \int_C \frac{e^{az}}{z - 0} dz = 2\pi i [e^{az}]_{z=0} = 2\pi i.$$

On the other hand, the stated parametric representation for C gives us

$$\begin{aligned}\int_C \frac{e^{az}}{z} dz &= \int_{-\pi}^{\pi} \frac{\exp(ae^{i\theta})}{e^{i\theta}} ie^{i\theta} d\theta = i \int_{-\pi}^{\pi} \exp[a(\cos \theta + i \sin \theta)] d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} e^{ia \sin \theta} d\theta = i \int_{-\pi}^{\pi} e^{a \cos \theta} [\cos(a \sin \theta) + i \sin(a \sin \theta)] d\theta \\ &= - \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta + i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta.\end{aligned}$$

Equating these two different expressions for the integral $\int_C \frac{e^{az}}{z} dz$, we have

$$- \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta + i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi i.$$

Then, by equating the imaginary parts on each side of this last equation, we see that

$$\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi;$$

and, since the integrand here is even,

$$\int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi.$$

8. (a) The binomial formula enables us to write

$$P_n(z) = \frac{1}{n! 2^n} \frac{d^n}{dz^n} (z^2 - 1)^n = \frac{1}{n! 2^n} \frac{d^n}{dz^n} \sum_{k=0}^n \binom{n}{k} z^{2n-2k} (-1)^k.$$

We note that the highest power of z appearing under the derivative is z^{2n} , and differentiating it n times brings it down to z^n . So $P_n(z)$ is a polynomial of degree n .

(b) We let C denote any positively oriented simple closed contour surrounding a fixed point z . The Cauchy integral formula for derivatives tells us that

$$\frac{d^n}{dz^n} (z^2 - 1)^n = \frac{n!}{2\pi i} \int_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} ds \quad (n = 0, 1, 2, \dots).$$

Hence the polynomials $P_n(z)$ in part (a) can be written

$$P_n(z) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} ds \quad (n = 0, 1, 2, \dots).$$

(c) Note that

$$\frac{(s^2 - 1)^n}{(s - 1)^{n+1}} = \frac{(s - 1)^n (s + 1)^n}{(s - 1)^{n+1}} = \frac{(s + 1)^n}{s - 1}.$$

Referring to the final result in part (b), then, we have

$$P_n(1) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s - 1)^{n+1}} ds = \frac{1}{2^n} \frac{1}{2 \pi i} \int_C \frac{(s + 1)^n}{s - 1} ds = \frac{1}{2^n} 2^n = 1 \quad (n = 0, 1, 2, \dots).$$

Also, since

$$\frac{(s^2 - 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n (s + 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n}{s + 1},$$

we have

$$P_n(-1) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s + 1)^{n+1}} ds = \frac{1}{2^n} \frac{1}{2 \pi i} \int_C \frac{(s - 1)^n}{s + 1} ds = \frac{1}{2^n} (-2)^n = (-1)^n \quad (n = 0, 1, 2, \dots).$$

9. We are asked to show that

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s) ds}{(s - z)^3}.$$

(a) In view of the expression for $f'(z)$ in the lemma,

$$\begin{aligned} \frac{f'(z + \Delta z) - f'(z)}{\Delta z} &= \frac{1}{2 \pi i} \int_C \left[\frac{1}{(s - z - \Delta z)^2} - \frac{1}{(s - z)^2} \right] \frac{f(s) ds}{\Delta z} \\ &= \frac{1}{2 \pi i} \int_C \frac{2(s - z) - \Delta z}{(s - z - \Delta z)^2 (s - z)^2} f(s) ds. \end{aligned}$$

Then

$$\begin{aligned} \frac{f'(z + \Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i} \int_C \frac{f(s) ds}{(s - z)^3} &= \frac{1}{2 \pi i} \int_C \left[\frac{2(s - z) - \Delta z}{(s - z - \Delta z)^2 (s - z)^2} - \frac{2}{(s - z)^3} \right] f(s) ds \\ &= \frac{1}{2 \pi i} \int_C \frac{3(s - z) \Delta z - 2(\Delta z)^2}{(s - z - \Delta z)^2 (s - z)^3} f(s) ds. \end{aligned}$$

(b) We must show that

$$\left| \int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s) ds \right| \leq \frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d-|\Delta z|)^2 d^3} L.$$

Now D , d , M , and L are as in the statement of the exercise in the text. The triangle inequality tells us that

$$|3(s-z)\Delta z - 2(\Delta z)^2| \leq 3|s-z||\Delta z| + 2|\Delta z|^2 \leq 3D|\Delta z| + 2|\Delta z|^2.$$

Also, we know from the verification of the expression for $f'(z)$ in the lemma that $|s-z-\Delta z| \geq d-|\Delta z| > 0$; and this means that

$$|(s-z-\Delta z)^2(s-z)^3| \geq (d-|\Delta z|)^2 d^3 > 0.$$

This gives the desired inequality.

(c) If we let Δz tend to 0 in the inequality obtained in part (b) we find that

$$\lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s) ds = 0.$$

This, together with the result in part (a), yields the desired expression for $f''(z)$.

Chapter 5

SECTION 52

1. We are asked to show in two ways that the sequence

$$z_n = -2 + i \frac{(-1)^n}{n^2} \quad (n = 1, 2, \dots)$$

converges to -2 . One way is to note that the two sequences

$$x_n = -2 \quad \text{and} \quad y_n = \frac{(-1)^n}{n^2} \quad (n = 1, 2, \dots)$$

of real numbers converge to -2 and 0 , respectively, and then to apply the theorem in Sec.

51. Another way is to observe that $|z_n - (-2)| = \frac{1}{n^2}$. Thus for each $\varepsilon > 0$,

$$|z_n - (-2)| < \varepsilon \quad \text{whenever} \quad n > n_0,$$

where n_0 is any positive integer such that $n_0 \geq \frac{1}{\sqrt{\varepsilon}}$.

2. Observe that if $z_n = -2 + i \frac{(-1)^n}{n^2}$ ($n = 1, 2, \dots$), then

$$r_n = |z_n| = \sqrt{4 + \frac{1}{n^4}} \rightarrow 2.$$

But, since

$$\Theta_{2n} = \text{Arg } z_{2n} \rightarrow \pi \quad \text{and} \quad \Theta_{2n-1} = \text{Arg } z_{2n-1} \rightarrow -\pi \quad (n = 1, 2, \dots),$$

the sequence Θ_n ($n = 1, 2, \dots$) does not converge.

3. Suppose that $\lim_{n \rightarrow \infty} z_n = z$. That is, for each $\varepsilon > 0$, there is a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$. In view of the inequality (see Sec. 4)

$$|z_n - z| \geq ||z_n| - |z||,$$

it follows that $||z_n| - |z|| < \varepsilon$ whenever $n > n_0$. That is, $\lim_{n \rightarrow \infty} |z_n| = |z|$.

4. The summation formula found in the example in Sec. 52 can be written

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when } |z| < 1.$$

If we put $z = re^{i\theta}$, where $0 < r < 1$, the left-hand side becomes

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta;$$

and the right-hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta} - r^2}{1-r(e^{i\theta} + e^{-i\theta}) + r^2} = \frac{r \cos \theta - r^2 + ir \sin \theta}{1-2r \cos \theta + r^2}.$$

Thus

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \cos \theta - r^2}{1-2r \cos \theta + r^2} + i \frac{r \sin \theta}{1-2r \cos \theta + r^2}.$$

Equating the real parts on each side here and then the imaginary parts, we arrive at the summation formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r \cos \theta - r^2}{1-2r \cos \theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \sin \theta}{1-2r \cos \theta + r^2},$$

where $0 < r < 1$. These formulas clearly hold when $r = 0$ too.

6. Suppose that $\sum_{n=1}^{\infty} z_n = S$. To show that $\sum_{n=1}^{\infty} \bar{z}_n = \bar{S}$, we write $z_n = x_n + iy_n$, $S = X + iY$ and appeal to the theorem in Sec. 52. First of all, we note that

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Then, since $\sum_{n=1}^{\infty} (-y_n) = -Y$, it follows that

$$\sum_{n=1}^{\infty} \bar{z}_n = \sum_{n=1}^{\infty} (x_n - iy_n) = \sum_{n=1}^{\infty} [x_n + i(-y_n)] = X - iY = \bar{S}.$$

8. Suppose that $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$. In order to use the theorem in Sec. 52, we write

$$z_n = x_n + iy_n, \quad S = X + iY \quad \text{and} \quad w_n = u_n + iv_n, \quad T = U + iV.$$

Now

$$\sum_{n=1}^{\infty} x_n = X, \quad \sum_{n=1}^{\infty} y_n = Y \quad \text{and} \quad \sum_{n=1}^{\infty} u_n = U, \quad \sum_{n=1}^{\infty} v_n = V.$$

Since

$$\sum_{n=1}^{\infty} (x_n + u_n) = X + U \quad \text{and} \quad \sum_{n=1}^{\infty} (y_n + v_n) = Y + V,$$

it follows that

$$\sum_{n=1}^{\infty} [(x_n + u_n) + i(y_n + v_n)] = X + U + i(Y + V).$$

That is,

$$\sum_{n=1}^{\infty} [(x_n + iy_n) + (u_n + iv_n)] = X + iY + (U + iV),$$

or

$$\sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

SECTION 54

1. Replace z by z^2 in the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (|z| < \infty)$$

to get

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \quad (|z| < \infty).$$

Then, multiplying through this last equation by z , we have the desired result:

$$z \cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \quad (|z| < \infty).$$

2. (b) Replacing z by $z - 1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty),$$

we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z| < \infty).$$

So

$$e^z = e^{z-1} e = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z| < \infty).$$

3. We want to find the Maclaurin series for the function

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 + (z^4/9)}.$$

To do this, we first replace z by $-(z^4/9)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1),$$

as well as its condition of validity, to get

$$\frac{1}{1+(z^4/9)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} \quad (|z| < \sqrt{3}).$$

Then, if we multiply through this last equation by $\frac{z}{9}$, we have the desired expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{3}).$$

6. Replacing z by z^2 in the representation

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty),$$

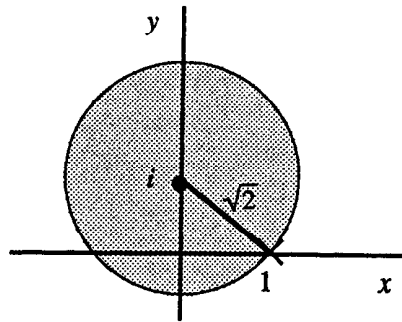
we have

$$\sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \quad (|z| < \infty).$$

Since the coefficient of z^n in the Maclaurin series for a function $f(z)$ is $f^{(n)}(0)/n!$, this shows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, \dots).$$

7. The function $\frac{1}{1-z}$ has a singularity at $z = 1$. So the Taylor series about $z = i$ is valid when $|z - i| < \sqrt{2}$, as indicated in the figure below.



To find the series, we start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i) - (z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - (z-i)/(1-i)}.$$

This suggests that we replace z by $(z-i)/(1-i)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

and then multiply through by $\frac{1}{1-i}$. The desired Taylor series is then obtained:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad (|z-i| < \sqrt{2}).$$

- 9 The identity $\sinh(z + \pi i) = -\sinh z$ and the periodicity of $\sinh z$, with period $2\pi i$, tell us that

$$\sinh z = -\sinh(z + \pi i) = -\sinh(z - \pi i).$$

So, if we replace z by $z - \pi i$ in the known representation

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

and then multiply through by -1 , we find that

$$\sinh z = -\sum_{n=0}^{\infty} \frac{(z - \pi i)^{2n+1}}{(2n+1)!} \quad (|z - \pi i| < \infty).$$

13. Suppose that $0 < |z| < 4$. Then $0 < |z/4| < 1$, and we can use the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1).$$

To be specific, when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}} = \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

SECTION 56

1. We may use the expansion

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

to see that when $0 < |z| < \infty$,

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}.$$

3. Suppose that $1 < |z| < \infty$ and recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1).$$

This enables us to write

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1 + \frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \quad (1 < |z| < \infty).$$

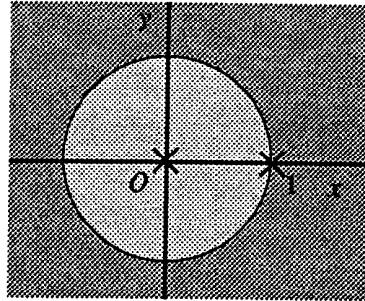
Replacing n by $n-1$ in this last series and then noting that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1},$$

we arrive at the desired expansion:

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \quad (1 < |z| < \infty).$$

4. The singularities of the function $f(z) = \frac{1}{z^2(1-z)}$ are at the points $z=0$ and $z=1$. Hence there are Laurent series in powers of z for the domains $0 < |z| < 1$ and $1 < |z| < \infty$ (see the figure below).



To find the series when $0 < |z| < 1$, recall that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ ($|z| < 1$) and write

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1-z} = \frac{1}{z^2} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}.$$

As for the domain $1 < |z| < \infty$, note that $|1/z| < 1$ and write

$$f(z) = -\frac{1}{z^3} \cdot \frac{1}{1-(1/z)} = -\frac{1}{z^3} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty} \frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty} \frac{1}{z^n}.$$

5. (a) The Maclaurin series for the function $\frac{z+1}{z-1}$ is valid when $|z| < 1$. To find it, we recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

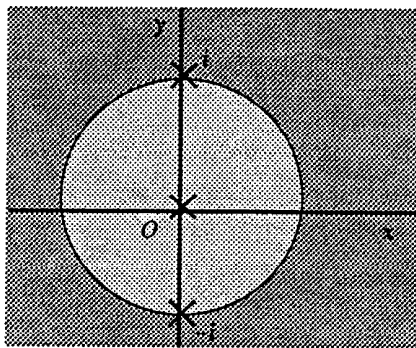
for $\frac{1}{1-z}$ and write

$$\begin{aligned} \frac{z+1}{z-1} &= -(z+1) \frac{1}{1-z} = (-z-1) \sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \\ &= -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -1 - 2 \sum_{n=1}^{\infty} z^n \end{aligned} \quad (|z| < 1).$$

(b) To find the Laurent series for the same function when $1 < |z| < \infty$, we recall the Maclaurin series for $\frac{1}{1-z}$ that was used in part (a). Since $\left|\frac{1}{z}\right| < 1$ here, we may write

$$\begin{aligned} \frac{z+1}{z-1} &= \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right) \frac{1}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right) \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{z^n} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{z^n} \end{aligned} \quad (1 < |z| < \infty).$$

7. The function $f(z) = \frac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0 < |z| < 1$ and also one for the domain $1 < |z| < \infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1).$$

For the domain $0 < |z| < 1$, we have

$$f(z) = \frac{1}{z} \cdot \frac{1}{1+z^2} = \frac{1}{z} \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z}.$$

On the other hand, when $1 < |z| < \infty$,

$$f(z) = \frac{1}{z^3} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^3} \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}.$$

In this second expansion, we have used the fact that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.

8. (a) Let a denote a real number, where $-1 < a < 1$. Recalling that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

enables us to write

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-(a/z)} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{z^{n+1}},$$

or

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty).$$

- (b) Putting $z = e^{i\theta}$ on each side of the final result in part (a), we have

$$\frac{a}{e^{i\theta} - a} = \sum_{n=1}^{\infty} a^n e^{-in\theta}.$$

But

$$\frac{a}{e^{i\theta} - a} = \frac{a}{(\cos \theta - a) + i \sin \theta} \cdot \frac{(\cos \theta - a) - i \sin \theta}{(\cos \theta - a) - i \sin \theta} = \frac{a \cos \theta - a^2 - i a \sin \theta}{1 - 2a \cos \theta + a^2}$$

and

$$\sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos n\theta - i \sum_{n=1}^{\infty} a^n \sin n\theta.$$

Consequently,

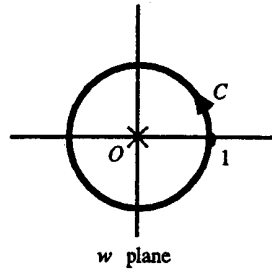
$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a \cos \theta - a^2}{1 - 2a \cos \theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a \sin \theta}{1 - 2a \cos \theta + a^2}$$

when $-1 < a < 1$.

10. (a) Let z be any fixed complex number and C the unit circle $w = e^{i\phi}$ ($-\pi \leq \phi \leq \pi$) in the w plane. The function

$$f(w) = \exp \left[\frac{z}{2} \left(w - \frac{1}{w} \right) \right]$$

has the one singularity $w = 0$ in the w plane. That singularity is, of course, interior to C , as shown in the figure below.



Now the function $f(w)$ has a Laurent series representation in the domain $0 < |w| < \infty$. According to expression (5), Sec. 55, then,

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \quad (0 < |w| < \infty),$$

where the coefficients $J_n(z)$ are

$$J_n(z) = \frac{1}{2\pi i} \int_C \frac{\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right]}{w^{n+1}} dw \quad (n = 0, \pm 1, \pm 2, \dots).$$

Using the parametric representation $w = e^{i\phi}$ ($-\pi \leq \phi \leq \pi$) for C , let us rewrite this expression for $J_n(z)$ as follows:

$$J_n(z) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\exp\left[\frac{z}{2}(e^{i\phi} - e^{-i\phi})\right]}{e^{i(n+1)\phi}} ie^{i\phi} d\phi = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp[iz \sin \phi] e^{-in\phi} d\phi.$$

That is,

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(n\phi - z \sin \phi)] d\phi \quad (n = 0, \pm 1, \pm 2, \dots).$$

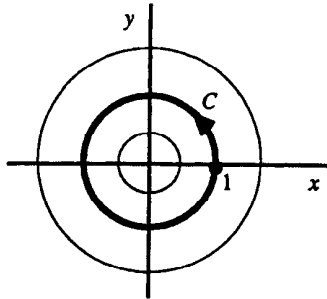
(b) The last expression for $J_n(z)$ in part (a) can be written as

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(n\phi - z \sin \phi) - i \sin(n\phi - z \sin \phi)] d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\phi - z \sin \phi) d\phi - \frac{i}{2\pi} \int_{-\pi}^{\pi} \sin(n\phi - z \sin \phi) d\phi \\ &= \frac{1}{2\pi} 2 \int_0^{\pi} \cos(n\phi - z \sin \phi) d\phi - \frac{i}{2\pi} 0 \quad (n = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

That is,

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\phi - z \sin \phi) d\phi \quad (n = 0, \pm 1, \pm 2, \dots).$$

11. (a) The function $f(z)$ is analytic in some annular domain centered at the origin; and the unit circle $C: z = e^{i\phi}$ ($-\pi \leq \phi \leq \pi$) is contained in that domain, as shown below.



For each point z in the annular domain, there is a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(n+1)}} i e^{i\phi} d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{-in\phi} d\phi \quad (n = 0, 1, 2, \dots)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z^{-n+1}} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(-n+1)}} i e^{i\phi} d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \quad (n = 1, 2, \dots).$$

Substituting these values of a_n and b_n into the series, we then have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{-in\phi} d\phi z^n + \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \frac{1}{z^n},$$

or

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[\left(\frac{z}{e^{i\phi}} \right)^n + \left(\frac{e^{i\phi}}{z} \right)^n \right] d\phi.$$

(b) Put $z = e^{i\theta}$ in the final result in part (a) to get

$$f(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) [e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}] d\phi,$$

or

$$f(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \cos[n(\theta - \phi)] d\phi.$$

If $u(\theta) = \operatorname{Re} f(e^{i\theta})$, then, equating the real parts on each side of this last equation yields

$$u(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} u(\phi) \cos[n(\theta - \phi)] d\phi.$$

SECTION 60

1. Differentiating each side of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1),$$

we find that

$$\frac{1}{(1-z)^2} = \frac{d}{dz} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{d}{dz} z^n = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n \quad (|z| < 1).$$

Another differentiation gives

$$\frac{2}{(1-z)^3} = \frac{d}{dz} \sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=0}^{\infty} (n+1) \frac{d}{dz} z^n = \sum_{n=1}^{\infty} n(n+1) z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \quad (|z| < 1).$$

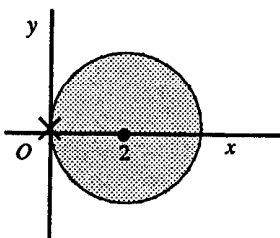
2. Replace z by $1/(1-z)$ on each side of the Maclaurin series representation (Exercise 1)

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \quad (|z| < 1),$$

as well as in its condition of validity. This yields the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} \quad (1 < |z-1| < \infty).$$

3. Since the function $f(z) = 1/z$ has a singular point at $z = 0$, its Taylor series about $z_0 = 2$ is valid in the open disk $|z - 2| < 2$, as indicated in the figure below.



To find that series, write

$$\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2}$$

to see that it can be obtained by replacing z by $-(z-2)/2$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1).$$

Specifically,

$$\frac{1}{z} = \frac{1}{2} \sum_{n=0}^{\infty} \left[-\frac{(z-2)}{2} \right]^n \quad (|z-2| < 2),$$

or

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^n \quad (|z-2| < 2).$$

Differentiating this series term by term, we have

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} n (z-2)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}} (n+1) (z-2)^n \quad (|z-2| < 2).$$

Thus

$$\frac{1}{z^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2} \right)^n \quad (|z-2| < 2).$$

4. Consider the function defined by the equations

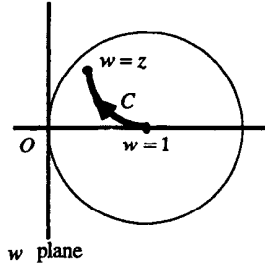
$$f(z) = \begin{cases} \frac{e^z - 1}{z} & \text{when } z \neq 0, \\ 1 & \text{when } z = 0. \end{cases}$$

When $z \neq 0$, $f(z)$ has the power series representation

$$f(z) = \frac{1}{z} \left[\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) - 1 \right] = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$

Since this representation clearly holds when $z = 0$ too, it is actually valid for all z . Hence f is entire.

6. Let C be a contour lying in the open disk $|w - 1| < 1$ in the w plane that extends from the point $w = 1$ to a point $w = z$, as shown in the figure below.



According to Theorem 1 in Sec. 59, we can integrate the Taylor series representation

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \quad (|w-1| < 1)$$

term by term along the contour C . Thus

$$\int_C \frac{dw}{w} = \int_C \sum_{n=0}^{\infty} (-1)^n (w-1)^n dw = \sum_{n=0}^{\infty} (-1)^n \int_C (w-1)^n dw.$$

But

$$\int_C \frac{dw}{w} = \int_1^z \frac{dw}{w} = [\text{Log } w]_1^z = \text{Log } z - \text{Log } 1 = \text{Log } z$$

and

$$\int_C (w-1)^n = \int_1^z (w-1)^n dw = \left[\frac{(w-1)^{n+1}}{n+1} \right]_1^z = \frac{(z-1)^{n+1}}{n+1}.$$

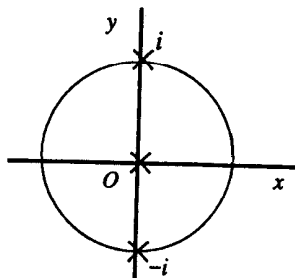
Hence

$$\text{Log } z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n \quad (|z-1| < 1);$$

and, since $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$, this result becomes

$$\text{Log } z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n \quad (|z-1| < 1).$$

1. The singularities of the function $f(z) = \frac{e^z}{z(z^2+1)}$ are at $z = 0, \pm i$. The problem here is to find the Laurent series for f that is valid in the punctured disk $0 < |z| < 1$, shown below.



We begin by recalling the Maclaurin series representations

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad (|z| < \infty)$$

and

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \quad (|z| < 1),$$

which enable us to write

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots \quad (|z| < \infty)$$

and

$$\frac{1}{z^2+1} = 1 - z^2 + z^4 - z^6 + \dots \quad (|z| < 1).$$

Multiplying these last two series term by term, we have the Maclaurin series representation

$$\begin{aligned} \frac{e^z}{z^2+1} &= 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots \\ &\quad - z^2 \quad - z^3 \dots \\ &\quad \quad \quad z^4 + \dots \\ &\quad \quad \quad \vdots \\ &= 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \dots, \end{aligned}$$

which is valid when $|z| < 1$. The desired Laurent series is then obtained by multiplying each side of the above representation by $\frac{1}{z}$:

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots \quad (0 < |z| < 1).$$

4. We know the Laurent series representation

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360} z + \dots \quad (0 < |z| < \pi)$$

from Example 2, Sec. 61. Expression (3), Sec. 55, for the coefficients b_n in a Laurent series tells us that the coefficient b_1 of $\frac{1}{z}$ in this series can be written

$$b_1 = \frac{1}{2\pi i} \int_C \frac{dz}{z^2 \sinh z},$$

where C is the circle $|z|=1$, taken counterclockwise. Since $b_1 = -\frac{1}{6}$, then,

$$\int_C \frac{dz}{z^2 \sinh z} = 2\pi i \left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

6. The problem here is to use mathematical induction to verify the differentiation formula

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \quad (n = 1, 2, \dots).$$

The formula is clearly true when $n = 1$ since in that case it becomes

$$[f(z)g(z)]' = f(z)g'(z) + f'(z)g(z).$$

We now assume that the formula is true when $n = m$ and show how, as a consequence, it is true when $n = m + 1$. We start by writing

$$\begin{aligned} [f(z)g(z)]^{(m+1)} &= \{[f(z)g(z)]'\}^{(m)} = [f(z)g'(z) + f'(z)g(z)]^{(m)} \\ &= [f(z)g'(z)]^{(m)} + [f'(z)g(z)]^{(m)} \\ &= \sum_{k=0}^m \binom{m}{k} f^{(k)}(z)g^{(m-k+1)}(z) + \sum_{k=0}^m \binom{m}{k} f^{(k+1)}(z)g^{(m-k)}(z) \\ &= \sum_{k=0}^m \binom{m}{k} f^{(k)}(z)g^{(m-k+1)}(z) + \sum_{k=1}^{m+1} \binom{m}{k-1} f^{(k)}(z)g^{(m-k+1)}(z) \\ &= f(z)g^{(m+1)}(z) + \sum_{k=1}^m \left[\binom{m}{k} + \binom{m}{k-1} \right] f^{(k)}(z)g^{(m+1-k)}(z) + f^{(m+1)}(z)g(z). \end{aligned}$$

But

$$\binom{m}{k} + \binom{m}{k-1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!} = \frac{(m+1)!}{k!(m+1-k)!} = \binom{m+1}{k};$$

and so

$$[f(z)g(z)]^{(m+1)} = f(z)g^{(m+1)}(z) + \sum_{k=1}^m \binom{m+1}{k} f^{(k)}(z)g^{(m+1-k)}(z) + f^{(m+1)}(z)g(z),$$

or

$$[f(z)g(z)]^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)}(z)g^{(m+1-k)}(z).$$

The desired verification is now complete.

7. We are given that $f(z)$ is an entire function represented by a series of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \quad (|z| < \infty).$$

(a) Write $g(z) = f[f(z)]$ and observe that

$$f[f(z)] = g(0) + \frac{g'(0)}{1!}z + \frac{g''(0)}{2!}z^2 + \frac{g'''(0)}{3!}z^3 + \dots \quad (|z| < \infty).$$

It is straightforward to show that

$$g'(z) = f'[f(z)]f'(z),$$

$$g''(z) = f''[f(z)][f'(z)]^2 + f'[f(z)]f''(z),$$

and

$$g'''(z) = f'''[f(z)][f'(z)]^3 + 2f'(z)f''(z)f''[f(z)] + f''[f(z)]f'(z)f''(z) + f'[f(z)]f'''(z).$$

Thus

$$g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 4a_2, \quad \text{and} \quad g'''(0) = 12(a_2^2 + a_3),$$

and so

$$f[f(z)] = z + 2a_2z^2 + 2(a_2^2 + a_3)z^3 + \dots \quad (|z| < \infty).$$

(b) Proceeding formally, we have

$$\begin{aligned}
 f[f(z)] &= f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots \\
 &= (z + a_2z^2 + a_3z^3 + \cdots) + a_2(z + a_2z^2 + a_3z^3 + \cdots)^2 + a_3(z + a_2z^2 + a_3z^3 + \cdots)^3 + \cdots \\
 &= (z + a_2z^2 + a_3z^3 + \cdots) + (a_2z^2 + 2a_2^2z^3 + \cdots) + (a_3z^3 + \cdots) \\
 &= z + 2a_2z^2 + 2(a_2^2 + a_3)z^3 + \cdots.
 \end{aligned}$$

(c) Since

$$\sin z = z - \frac{z^3}{3!} + \cdots = z + 0z^2 + \left(-\frac{1}{6}\right)z^3 + \cdots \quad (|z| < \infty),$$

the result in part (a), with $a_2 = 0$ and $a_3 = -\frac{1}{6}$, tells us that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \quad (|z| < \infty).$$

8. We need to find the first four nonzero coefficients in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \quad \left(|z| < \frac{\pi}{2}\right).$$

This representation is valid in the stated disk since the zeros of $\cosh z$ are the numbers $z = \left(\frac{\pi}{2} + n\pi\right)i$ ($n = 0, \pm 1, \pm 2, \dots$), the ones nearest to the origin being $z = \pm \frac{\pi}{2}i$. The series contains only even powers of z since $\cosh z$ is an even function; that is, $E_{2n+1} = 0$ ($n = 0, 1, 2, \dots$). To find the series, we divide the series

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots = 1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots \quad (|z| < \infty)$$

into 1. The result is

$$\frac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \quad \left(|z| < \frac{\pi}{2}\right),$$

or

$$\frac{1}{\cosh z} = 1 - \frac{1}{2!}z^2 + \frac{5}{4!}z^4 - \frac{61}{6!}z^6 + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

Since

$$\frac{1}{\cosh z} = E_0 + \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 + \frac{E_6}{6!}z^6 + \dots \quad \left(|z| < \frac{\pi}{2}\right),$$

this tells us that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$

Chapter 6

SECTION 64

1. (a) Let us write

$$\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z} (1 - z + z^2 - z^3 + \dots) = \frac{1}{z} - 1 + z - z^2 + \dots \quad (0 < |z| < 1).$$

The residue at $z = 0$, which is the coefficient of $\frac{1}{z}$, is clearly 1.

(b) We may use the expansion

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad (|z| < \infty)$$

to write

$$z \cos\left(\frac{1}{z}\right) = z \left(1 - \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{4!} \cdot \frac{1}{z^4} - \frac{1}{6!} \cdot \frac{1}{z^6} + \dots \right) = z - \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^3} - \frac{1}{6!} \cdot \frac{1}{z^5} + \dots \quad (0 < |z| < \infty).$$

The residue at $z = 0$, or coefficient of $\frac{1}{z}$, is now seen to be $-\frac{1}{2}$.

(c) Observe that

$$\frac{z - \sin z}{z} = \frac{1}{z} (z - \sin z) = \frac{1}{z} \left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \right] = \frac{z^2}{3!} - \frac{z^4}{5!} + \dots \quad (0 < |z| < \infty).$$

Since the coefficient of $\frac{1}{z}$ in this Laurent series is 0, the residue at $z = 0$ is 0.

(d) Write

$$\frac{\cot z}{z^4} = \frac{1}{z^4} \cdot \frac{\cos z}{\sin z}$$

and recall that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots \quad (|z| < \infty)$$

and

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots \quad (|z| < \infty).$$

Dividing the series for $\sin z$ into the one for $\cos z$, we find that

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \dots \quad (0 < |z| < \pi).$$

Thus

$$\frac{\cot z}{z^4} = \frac{1}{z^4} \left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \dots \right) = \frac{1}{z^5} - \frac{1}{3} \cdot \frac{1}{z^3} - \frac{1}{45} \cdot \frac{1}{z} + \dots \quad (0 < |z| < \pi).$$

Note that the condition of validity for this series is due to the fact that $\sin z = 0$ when $z = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). It is now evident that $\frac{\cot z}{z^4}$ has residue $-\frac{1}{45}$ at $z = 0$.

(e) Recall that

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \quad (|z| < \infty)$$

and

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad (|z| < \infty).$$

There is a Laurent series for the function

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4} \cdot (\sinh z) \left(\frac{1}{1-z^2} \right)$$

that is valid for $0 < |z| < 1$. To find it, we first multiply the Maclaurin series for $\sinh z$ and $\frac{1}{1-z^2}$:

$$\begin{aligned} (\sinh z) \left(\frac{1}{1-z^2} \right) &= \left(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \dots \right) (1 + z^2 + z^4 + \dots) \\ &= z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \dots \\ &\quad z^3 + \frac{1}{6}z^5 + \dots \\ &\quad z^5 + \dots \\ &= z + \frac{7}{6}z^3 + \dots \quad (0 < |z| < 1). \end{aligned}$$

We then see that

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3} + \frac{7}{6} \cdot \frac{1}{z} + \dots \quad (0 < |z| < 1).$$

This shows that the residue of $\frac{\sinh z}{z^4(1-z^2)}$ at $z=0$ is $\frac{7}{6}$.

2. In each part, C denotes the positively oriented circle $|z|=3$.

(a) To evaluate $\int_C \frac{\exp(-z)}{z^2} dz$, we need the residue of the integrand at $z=0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2} \left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots \right) = \frac{1}{z^2} - \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \dots \quad (0 < |z| < \infty),$$

we see that the required residue is -1 . Thus

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i(-1) = -2\pi i.$$

(c) Likewise, to evaluate the integral $\int_C z^2 \exp\left(\frac{1}{z}\right) dz$, we must find the residue of the integrand at $z=0$. The Laurent series

$$\begin{aligned} z^2 \exp\left(\frac{1}{z}\right) &= z^2 \left(1 + \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{4!} \cdot \frac{1}{z^4} + \dots \right) \\ &= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^2} + \dots, \end{aligned}$$

which is valid for $0 < |z| < \infty$, tells us that the needed residue is $\frac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i \left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

(d) As for the integral $\int_C \frac{z+1}{z^2-2z} dz$, we need the two residues of

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)},$$

one at $z=0$ and one at $z=2$. The residue at $z=0$ can be found by writing

$$\begin{aligned} \frac{z+1}{z(z-2)} &= \left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right) = \left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right) \cdot \frac{1}{1-(z/2)} \\ &= \left(-\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{z}\right)\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \dots\right), \end{aligned}$$

which is valid when $0 < |z| < 2$, and observing that the coefficient of $\frac{1}{z}$ in this last product is $-\frac{1}{2}$. To obtain the residue at $z=2$, we write

$$\begin{aligned} \frac{z+1}{z(z-2)} &= \frac{(z-2)+3}{z-2} \cdot \frac{1}{2+(z-2)} = \frac{1}{2} \left(1 + \frac{3}{z-2}\right) \cdot \frac{1}{1+(z-2)/2} \\ &= \frac{1}{2} \left(1 + \frac{3}{z-2}\right) \left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \dots\right], \end{aligned}$$

which is valid when $0 < |z-2| < 2$, and note that the coefficient of $\frac{1}{z-2}$ in this product is $\frac{3}{2}$. Finally, then, by the residue theorem,

$$\int_C \frac{z+1}{z^2-2z} dz = 2\pi i \left(-\frac{1}{2} + \frac{3}{2}\right) = 2\pi i.$$

3. In each part of this problem, C is the positively oriented circle $|z|=2$.

(a) If $f(z) = \frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^7-z^4} = -\frac{1}{z^4} \cdot \frac{1}{1-z^3} = -\frac{1}{z^4} (1+z^3+z^6+\dots) = -\frac{1}{z^4} - \frac{1}{z} - z^2 - \dots$$

when $0 < |z| < 1$. This tells us that

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(-1) = -2\pi i.$$

(b) When $f(z) = \frac{1}{1+z^2}$, we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - \dots \quad (0 < |z| < 1).$$

Thus

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(0) = 0.$$

(c) If $f(z) = \frac{1}{z}$, it follows that $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z}$. Evidently, then,

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(1) = 2\pi i.$$

4. Let C denote the circle $|z|=1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ ($|z| < \infty$) enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z} dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!} dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \quad (n = 0, 1, 2, \dots).$$

Now the $\frac{1}{z}$ in this series occurs when $n-k = -1$, or $k = n+1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \quad (n = 0, 1, 2, \dots).$$

The final result in part (a) thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

5. We are given two polynomials

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \quad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1z + b_2z^2 + \cdots + b_mz^m \quad (b_m \neq 0),$$

where $m \geq n + 2$.

It is straightforward to show that

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = \frac{a_0z^{m-2} + a_1z^{m-3} + a_2z^{m-4} + \cdots + a_nz^{m-n-2}}{b_0z^m + b_1z^{m-1} + b_2z^{m-2} + \cdots + b_m} \quad (z \neq 0).$$

Observe that the numerator here is, in fact, a polynomial since $m - n - 2 \geq 0$. Also, since $b_m \neq 0$, the quotient of these polynomials is represented by a series of the form $d_0 + d_1z + d_2z^2 + \cdots$. That is,

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = d_0 + d_1z + d_2z^2 + \cdots \quad (0 < |z| < R_2);$$

and we see that $\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)}$ has residue 0 $z = 0$.

Suppose now that all of the zeros of $Q(z)$ lie inside a simple closed contour C , and assume that C is positively oriented. Since $P(z)/Q(z)$ is analytic everywhere in the finite plane except at the zeros of $Q(z)$, it follows from the theorem in Sec. 64 and the residue just obtained that

$$\int_C \frac{P(z)}{Q(z)} dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} \right] = 2\pi i \cdot 0 = 0.$$

If C is negatively oriented, this result is still true since then

$$\int_C \frac{P(z)}{Q(z)} dz = - \int_{-C} \frac{P(z)}{Q(z)} dz = 0.$$

SECTION 65

1. (a) From the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z| < \infty),$$

we see that

$$z \exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \cdots \quad (0 < |z| < \infty).$$

The principal part of $z \exp\left(\frac{1}{z}\right)$ at the isolated singular point $z = 0$ is, then,

$$\frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \dots;$$

and $z = 0$ is an essential singular point of that function.

- (b) The isolated singular point of $\frac{z^2}{1+z}$ is at $z = -1$. Since the principal part at $z = -1$ involves powers of $z + 1$, we begin by observing that

$$z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1.$$

This enables us to write

$$\frac{z^2}{1+z} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}.$$

Since the principal part is $\frac{1}{z+1}$, the point $z = -1$ is a (simple) pole.

- (c) The point $z = 0$ is the isolated singular point of $\frac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z} = \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \quad (0 < |z| < \infty).$$

The principal part here is evidently 0, and so $z = 0$ is a removable singular point of the function $\frac{\sin z}{z}$.

- (d) The isolated singular point of $\frac{\cos z}{z}$ is $z = 0$. Since

$$\frac{\cos z}{z} = \frac{1}{z} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \dots \quad (0 < |z| < \infty),$$

the principal part is $\frac{1}{z}$. This means that $z = 0$ is a (simple) pole of $\frac{\cos z}{z}$.

- (e) Upon writing $\frac{1}{(2-z)^3} = \frac{-1}{(z-2)^3}$, we find that the principal part of $\frac{1}{(2-z)^3}$ at its isolated singular point $z = 2$ is simply the function itself. That point is evidently a pole (of order 3).

2. (a) The singular point is $z = 0$. Since

$$\frac{1 - \cosh z}{z^3} = \frac{1}{z^3} \left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots \right) \right] = -\frac{1}{2!} \cdot \frac{1}{z} - \frac{z}{4!} - \frac{z^3}{6!} - \dots$$

when $0 < |z| < \infty$, we have $m = 1$ and $B = -\frac{1}{2!} = -\frac{1}{2}$.

(b) Here the singular point is also $z = 0$. Since

$$\begin{aligned} \frac{1 - \exp(2z)}{z^4} &= \frac{1}{z^4} \left[1 - \left(1 + \frac{2z}{1!} + \frac{2^2 z^2}{2!} + \frac{2^3 z^3}{3!} + \frac{2^4 z^4}{4!} + \frac{2^5 z^5}{5!} + \dots \right) \right] \\ &= -\frac{2}{1!} \cdot \frac{1}{z^3} - \frac{2^2}{2!} \cdot \frac{1}{z^2} - \frac{2^3}{3!} \cdot \frac{1}{z} - \frac{2^4}{4!} - \frac{2^5}{5!} z - \dots \end{aligned}$$

when $0 < |z| < \infty$, we have $m = 3$ and $B = -\frac{2^3}{3!} = -\frac{4}{3}$.

(c) The singular point of $\frac{\exp(2z)}{(z-1)^2}$ is $z = 1$. The Taylor series

$$\exp(2z) = e^{2(z-1)} e^2 = e^2 \left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \dots \right] \quad (|z| < \infty)$$

enables us to write the Laurent series

$$\frac{\exp(2z)}{(z-1)^2} = e^2 \left[\frac{1}{(z-1)^2} + \frac{2}{1!} \cdot \frac{1}{z-1} + \frac{2^2}{2!} + \frac{2^2}{3!} (z-1) + \dots \right] \quad (0 < |z-1| < \infty).$$

Thus $m = 2$ and $B = e^2 \frac{2}{1!} = 2e^2$.

3. Since f is analytic at z_0 , it has a Taylor series representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!} (z - z_0) + \frac{f''(z_0)}{2!} (z - z_0)^2 + \dots \quad (|z - z_0| < R_0).$$

Let g be defined by means of the equation

$$g(z) = \frac{f(z)}{z - z_0}.$$

(a) Suppose that $f(z_0) \neq 0$. Then

$$\begin{aligned} g(z) &= \frac{1}{z-z_0} \left[f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots \right] \\ &= \frac{f(z_0)}{z-z_0} + \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z-z_0) + \dots \end{aligned} \quad (0 < |z-z_0| < R_0).$$

This shows that g has a simple pole at z_0 , with residue $f(z_0)$.

(b) Suppose, on the other hand, that $f(z_0) = 0$. Then

$$\begin{aligned} g(z) &= \frac{1}{z-z_0} \left[\frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots \right] \\ &= \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z-z_0) + \dots \end{aligned} \quad (0 < |z-z_0| < R_0).$$

Since the principal part of g at z_0 is just 0, the point $z = z_0$ is a removable singular point of g .

4. Write the function

$$f(z) = \frac{8a^3 z^2}{(z^2 + a^2)^3} \quad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3 z^2}{(z+ai)^3}.$$

Since the only singularity of $\phi(z)$ is at $z = -ai$, $\phi(z)$ has a Taylor series representation

$$\phi(z) = \phi(ai) + \frac{\phi'(ai)}{1!}(z-ai) + \frac{\phi''(ai)}{2!}(z-ai)^2 + \dots \quad (|z-ai| < 2a)$$

about $z = ai$. Thus

$$f(z) = \frac{1}{(z-ai)^3} \left[\phi(ai) + \frac{\phi'(ai)}{1!}(z-ai) + \frac{\phi''(ai)}{2!}(z-ai)^2 + \dots \right] \quad (0 < |z-ai| < 2a).$$

Now straightforward differentiation reveals that

$$\phi'(z) = \frac{16a^4 iz - 8a^3 z^2}{(z+ai)^4} \quad \text{and} \quad \phi''(z) = \frac{16a^3(z^2 - 4aiz - a^2)}{(z+ai)^5}.$$

Consequently,

$$\phi(ai) = -a^2i, \quad \phi'(ai) = -\frac{a}{2}, \quad \text{and} \quad \phi''(ai) = -i.$$

This enables us to write

$$f(z) = \frac{1}{(z-ai)^3} \left[-a^2i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2 + \dots \right] \quad (0 < |z-ai| < 2a).$$

The principal part of f at the point $z = ai$ is, then,

$$-\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2i}{(z-ai)^3}.$$

SECTION 67

1. (a) The function $f(z) = \frac{z^2+2}{z-1}$ has an isolated singular point at $z = 1$. Writing $f(z) = \frac{\phi(z)}{z-1}$, where $\phi(z) = z^2+2$, and observing that $\phi(z)$ is analytic and nonzero at $z = 1$, we see that $z = 1$ is a pole of order $m = 1$ and that the residue there is $B = \phi(1) = 3$.

(b) If we write

$$f(z) = \left(\frac{z}{2z+1} \right)^3 = \frac{\phi(z)}{\left[z - \left(-\frac{1}{2} \right) \right]^3}, \quad \text{where} \quad \phi(z) = \frac{z^3}{8},$$

we see that $z = -\frac{1}{2}$ is a singular point of f . Since $\phi(z)$ is analytic and nonzero at that point, f has a pole of order $m = 3$ there. The residue is

$$B = \frac{\phi''(-1/2)}{2!} = -\frac{3}{16}.$$

(c) The function

$$\frac{\exp z}{z^2 + \pi^2} = \frac{\exp z}{(z - \pi i)(z + \pi i)}$$

has poles of order $m = 1$ at the two points $z = \pm \pi i$. The residue at $z = \pi i$ is

$$B_1 = \frac{\exp \pi i}{2\pi i} = \frac{-1}{2\pi i} = \frac{i}{2\pi},$$

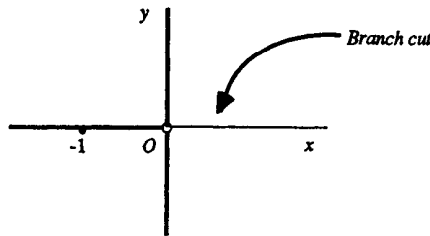
and the one at $z = -\pi i$ is

$$B_2 = \frac{\exp(-\pi i)}{-2\pi i} = \frac{-1}{-2\pi i} = -\frac{i}{2\pi}.$$

2. (a) Write the function $f(z) = \frac{z^{1/4}}{z+1}$ ($|z| > 0, 0 < \arg z < 2\pi$) as

$$f(z) = \frac{\phi(z)}{z+1}, \quad \text{where } \phi(z) = z^{1/4} = e^{\frac{1}{4} \log z} \quad (|z| > 0, 0 < \arg z < 2\pi).$$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the figure below.



Also,

$$\phi(-1) = (-1)^{1/4} = e^{\frac{1}{4} \log(-1)} = e^{\frac{1}{4} (\ln 1 + i\pi)} = e^{i\pi/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1+i}{\sqrt{2}} \neq 0.$$

This shows that the function f has a pole of order $m=1$ at $z=-1$, the residue there being

$$B = \phi(-1) = \frac{1+i}{\sqrt{2}}.$$

- (b) Write the function $f(z) = \frac{\text{Log } z}{(z^2+1)^2}$ as

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \text{where } \phi(z) = \frac{\text{Log } z}{(z+i)^2}.$$

From this, it is clear that $f(z)$ has a pole of order $m=2$ at $z=i$. Straightforward differentiation then reveals that

$$\text{Res}_{z=i} \frac{\text{Log } z}{(z^2+1)^2} = \phi'(i) = \frac{\pi+2i}{8}.$$

(c) Write the function

$$f(z) = \frac{z^{1/2}}{(z^2 + 1)^2} \quad (|z| > 0, 0 < \arg z < 2\pi)$$

as

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{z^{1/2}}{(z+i)^2}.$$

Since

$$\phi'(z) = \frac{(z+i)z^{-1/2} - 4z^{1/2}}{2(z+i)^3}$$

and

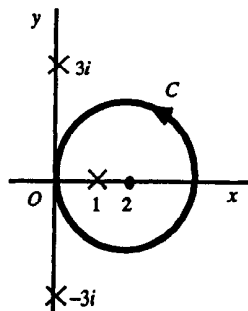
$$i^{-1/2} = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \quad i^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},$$

$$\text{Res}_{z=i} \frac{z^{1/2}}{(z^2 + 1)^2} = \phi'(i) = \frac{1-i}{8\sqrt{2}}.$$

3. (a) We wish to evaluate the integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2 + 9)} dz,$$

where C is the circle $|z-2|=2$, taken in the counterclockwise direction. That circle and the singularities $z=1, \pm 3i$ of the integrand are shown in the figure just below.



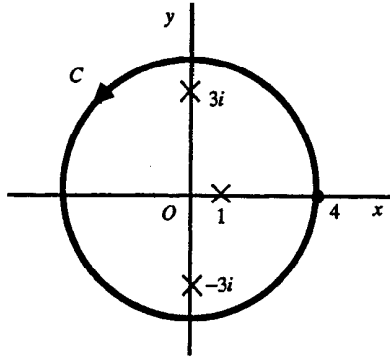
Observe that the point $z=1$, which is the only singularity inside C , is a simple pole of the integrand and that

$$\text{Res}_{z=1} \frac{3z^3 + 2}{(z-1)(z^2 + 9)} = \left. \frac{3z^3 + 2}{z^2 + 9} \right|_{z=1} = \frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2 + 9)} dz = 2\pi i \left(\frac{1}{2} \right) = \pi i.$$

- (b) Let us redo part (a) when C is changed to be the positively oriented circle $|z| = 4$, shown in the figure below.



In this case, all three singularities $z = 1, \pm 3i$ of the integrand are interior to C . We already know from part (a) that

$$\operatorname{Res}_{z=1} \frac{3z^3 + 2}{(z-1)(z^2 + 9)} = \frac{1}{2}.$$

It is, moreover, straightforward to show that

$$\operatorname{Res}_{z=3i} \frac{3z^3 + 2}{(z-1)(z^2 + 9)} = \frac{3z^3 + 2}{(z-1)(z+3i)} \Big|_{z=3i} = \frac{15 + 49i}{12}$$

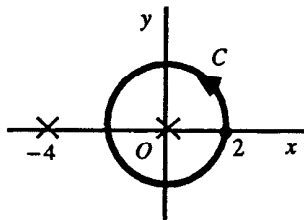
and

$$\operatorname{Res}_{z=-3i} \frac{3z^3 + 2}{(z-1)(z^2 + 9)} = \frac{3z^3 + 2}{(z-1)(z-3i)} \Big|_{z=-3i} = \frac{15 - 49i}{12}.$$

The residue theorem now tells us that

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2 + 9)} dz = 2\pi i \left(\frac{1}{2} + \frac{15 + 49i}{12} + \frac{15 - 49i}{12} \right) = 6\pi i.$$

4. (a) Let C denote the positively oriented circle $|z| = 2$, and note that the integrand of the integral $\int_C \frac{dz}{z^3(z+4)}$ has singularities at $z = 0$ and $z = -4$. (See the figure below.)



To find the residue of the integrand at $z = 0$, we recall the expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

and write

$$\frac{1}{z^3(z+4)} = \frac{1}{4z^3} \left[\frac{1}{1+(z/4)} \right] = \frac{1}{4z^3} \sum_{n=0}^{\infty} \left(-\frac{z}{4} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} z^{n-3} \quad (0 < |z| < 4).$$

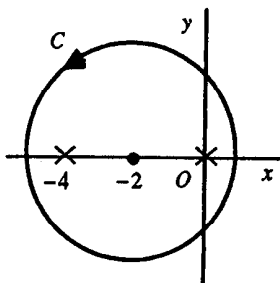
Now the coefficient of $\frac{1}{z}$ here occurs when $n = 2$, and we see that

$$\operatorname{Res}_{z=0} \frac{1}{z^3(z+4)} = \frac{1}{64}.$$

Consequently,

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\frac{1}{64} \right) = \frac{\pi i}{32}.$$

- (b) Let us replace the path C in part (a) by the positively oriented circle $|z+2| = 3$, centered at -2 and with radius 3. It is shown below.



We already know from part (a) that

$$\operatorname{Res}_{z=0} \frac{1}{z^3(z+4)} = \frac{1}{64}.$$

To find the residue at -4 , we write

$$\frac{1}{z^3(z+4)} = \frac{\phi(z)}{z - (-4)}, \quad \text{where } \phi(z) = \frac{1}{z^3}.$$

This tells us that $z = -4$ is a simple pole of the integrand and that the residue there is $\phi(-4) = -1/64$. Consequently,

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\frac{1}{64} - \frac{1}{64} \right) = 0.$$

5. Let us evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2+1)} dz$, where C is the positively oriented circle $|z|=2$.

All three isolated singularities $z=0, \pm i$ of the integrand are interior to C . The desired residues are

$$\operatorname{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \left. \frac{\cosh \pi z}{z^2+1} \right]_{z=0} = 1,$$

$$\operatorname{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} = \left. \frac{\cosh \pi z}{z(z+i)} \right]_{z=i} = \frac{1}{2},$$

and

$$\operatorname{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \left. \frac{\cosh \pi z}{z(z-i)} \right]_{z=-i} = \frac{1}{2}.$$

Consequently,

$$\int_C \frac{\cosh \pi z}{z(z^2+1)} dz = 2\pi i \left(1 + \frac{1}{2} + \frac{1}{2} \right) = 4\pi i.$$

6. In each part of this problem, C denotes the positively oriented circle $|z|=3$.

(a) It is straightforward to show that

$$\text{if } f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}, \text{ then } \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

This function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z=0$, and

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \left(\frac{9}{2} \right) = 9\pi i.$$

(b) Likewise,

$$\text{if } f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}, \text{ then } \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{z-3}{z(z+1)(z^4+2)}.$$

The function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z=0$, and we find here that

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \left(-\frac{3}{2} \right) = -3\pi i.$$

(c) Finally,

$$\text{if } f(z) = \frac{z^3 e^{1/z}}{1+z^3}, \text{ then } \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{e^z}{z^2(1+z^3)}.$$

The point $z = 0$ is a pole of order 2 of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$. The residue is $\phi'(0)$, where

$$\phi(z) = \frac{e^z}{1+z^3}.$$

Since

$$\phi'(z) = \frac{(1+z^3)e^z - e^z 3z^2}{(1+z^3)^2},$$

the value of $\phi'(0)$ is 1. So

$$\int_C \frac{z^3 e^{1/z}}{1+z^3} dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i(1) = 2\pi i.$$

SECTION 69

1. (a) Write

$$\csc z = \frac{1}{\sin z} = \frac{p(z)}{q(z)}, \text{ where } p(z) = 1 \text{ and } q(z) = \sin z.$$

Since

$$p(0) = 1 \neq 0, \quad q(0) = \sin 0 = 0, \quad \text{and} \quad q'(0) = \cos 0 = 1 \neq 0,$$

$z = 0$ must be a simple pole of $\csc z$, with residue

$$\frac{p(0)}{q'(0)} = \frac{1}{1} = 1.$$

(b) From Exercise 2, Sec. 61, we know that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \dots \quad (0 < |z| < \pi).$$

Since the coefficient of $\frac{1}{z}$ here is 1, it follows that $z = 0$ is a simple pole of $\csc z$, the residue being 1.

2. (a) Write

$$\frac{z - \sinh z}{z^2 \sinh z} = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = z - \sinh z \text{ and } q(z) = z^2 \sinh z.$$

Since

$$p(\pi i) = \pi i \neq 0, \quad q(\pi i) = 0, \quad \text{and} \quad q'(\pi i) = \pi^2 \neq 0,$$

it follows that

$$\operatorname{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}.$$

(b) Write

$$\frac{\exp(zt)}{\sinh z} = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = \exp(zt) \text{ and } q(z) = \sinh z.$$

It is easy to see that

$$\operatorname{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} = \frac{p(\pi i)}{q'(\pi i)} = -\exp(i\pi t) \quad \text{and} \quad \operatorname{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = \frac{p(-\pi i)}{q'(-\pi i)} = -\exp(-i\pi t).$$

Evidently, then,

$$\operatorname{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2 \frac{\exp(i\pi t) + \exp(-i\pi t)}{2} = -2 \cos \pi t.$$

3. (a) Write

$$f(z) = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = z \text{ and } q(z) = \cos z.$$

Observe that

$$q\left(\frac{\pi}{2} + n\pi\right) = 0 \quad (n = 0, \pm 1, \pm 2, \dots).$$

Also, for the stated values of n ,

$$p\left(\frac{\pi}{2} + n\pi\right) = \frac{\pi}{2} + n\pi \neq 0 \quad \text{and} \quad q'\left(\frac{\pi}{2} + n\pi\right) = -\sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^{n+1} \neq 0.$$

So the function $f(z) = \frac{z}{\cos z}$ has poles of order $m = 1$ at each of the points

$$z_n = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

The corresponding residues are

$$B = \frac{p(z_n)}{q'(z_n)} = (-1)^{n+1} z_n.$$

(b) Write

$$\tanh z = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = \sinh z \text{ and } q(z) = \cosh z.$$

Both p and q are entire, and the zeros of q are (Sec. 34)

$$z = \left(\frac{\pi}{2} + n\pi\right)i \quad (n = 0, \pm 1, \pm 2, \dots)$$

In addition to the fact that $q\left(\left(\frac{\pi}{2} + n\pi\right)i\right) = 0$, we see that

$$p\left(\left(\frac{\pi}{2} + n\pi\right)i\right) = \sinh\left(\frac{\pi}{2}i + n\pi i\right) = i \cos n\pi = i(-1)^n \neq 0$$

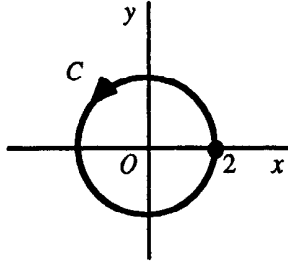
and

$$q'\left(\left(\frac{\pi}{2} + n\pi\right)i\right) = \sinh\left(\frac{\pi}{2}i + n\pi i\right) = i(-1)^n \neq 0.$$

So the points $z = \left(\frac{\pi}{2} + n\pi\right)i$ ($n = 0, \pm 1, \pm 2, \dots$) are poles of order $m = 1$ of $\tanh z$, the residue in each case being

$$B = \frac{p\left(\left(\frac{\pi}{2} + n\pi\right)i\right)}{q'\left(\left(\frac{\pi}{2} + n\pi\right)i\right)} = \frac{i(-1)^n}{i(-1)^n} = 1.$$

4. Let C be the positively oriented circle $|z|=2$, shown just below.



- (a) To evaluate the integral $\int_C \tan z \, dz$, we write the integrand as

$$\tan z = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = \sin z \text{ and } q(z) = \cos z,$$

and recall that the zeros of $\cos z$ are $z = \frac{\pi}{2} + n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Only two of those zeros, namely $z = \pm \pi/2$, are interior to C , and they are the isolated singularities of $\tan z$ interior to C . Observe that

$$\operatorname{Res}_{z=\pi/2} \tan z = \frac{p(\pi/2)}{q'(\pi/2)} = -1 \quad \text{and} \quad \operatorname{Res}_{z=-\pi/2} \tan z = \frac{p(-\pi/2)}{q'(-\pi/2)} = -1.$$

Hence

$$\int_C \tan z \, dz = 2\pi i(-1-1) = -4\pi i.$$

- (b) The problem here is to evaluate the integral $\int_C \frac{dz}{\sinh 2z}$. To do this, we write the integrand as

$$\frac{1}{\sinh 2z} = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = 1 \text{ and } q(z) = \sinh 2z.$$

Now $\sinh 2z = 0$ when $2z = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$), or when

$$z = \frac{n\pi i}{2} \quad (n = 0, \pm 1, \pm 2, \dots).$$

Three of these zeros of $\sinh 2z$, namely 0 and $\pm \frac{\pi i}{2}$, are inside C and are the isolated singularities of the integrand that need to be considered here. It is straightforward to show that

$$\operatorname{Res}_{z=0} \frac{1}{\sinh 2z} = \frac{p(0)}{q'(0)} = \frac{1}{2 \cosh 0} = \frac{1}{2},$$

$$\operatorname{Res}_{z=\pi i/2} \frac{1}{\sinh 2z} = \frac{p(\pi i/2)}{q'(\pi i/2)} = \frac{1}{2 \cosh(\pi i)} = \frac{1}{2 \cos \pi} = -\frac{1}{2},$$

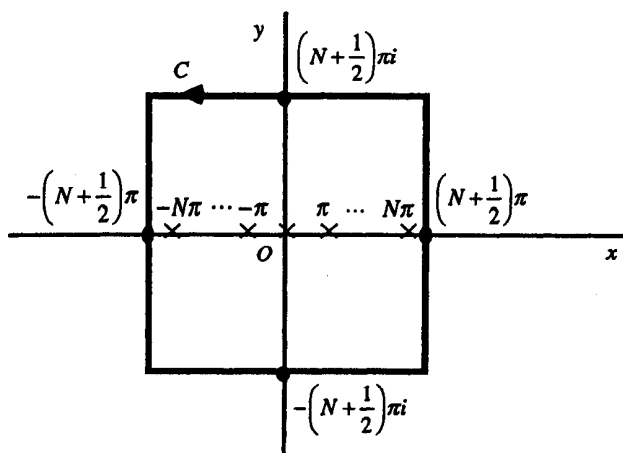
and

$$\operatorname{Res}_{z=-\pi i/2} \frac{1}{\sinh 2z} = \frac{p(-\pi i/2)}{q'(-\pi i/2)} = \frac{1}{2 \cosh(-\pi i)} = \frac{1}{2 \cos(-\pi)} = -\frac{1}{2}.$$

Thus

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right) = -\pi i.$$

5. The simple closed contour C_N is as shown in the figure below.



Within C_N , the function $\frac{1}{z^2 \sin z}$ has isolated singularities at

$$z = 0 \quad \text{and} \quad z = \pm n\pi \quad (n = 1, 2, \dots, N).$$

To find the residue at $z = 0$, we recall the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 61, and write

$$\begin{aligned} \frac{1}{z^2 \sin z} &= \frac{1}{z^2} \csc z = \frac{1}{z^2} \left\{ \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \dots \right\} \\ &= \frac{1}{z^3} + \frac{1}{6} \cdot \frac{1}{z} + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z + \dots \end{aligned} \quad (0 < |z| < \pi).$$

This tells us that $\frac{1}{z^2 \sin z}$ has a pole of order 3 at $z = 0$ and that

$$\operatorname{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

As for the points $z = \pm n\pi$ ($n = 1, 2, \dots, N$), write

$$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = 1 \text{ and } q(z) = z^2 \sin z.$$

Since

$$p(\pm n\pi) = 1 \neq 0, \quad q(\pm n\pi) = 0, \quad \text{and } q'(\pm n\pi) = n^2 \pi^2 \cos n\pi = (-1)^n n^2 \pi^2 \neq 0,$$

it follows that

$$\operatorname{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2 \pi^2} \cdot \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2 \pi^2}.$$

So, by the residue theorem,

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^N \frac{(-1)^n}{n^2 \pi^2} \right].$$

Rewriting this equation in the form

$$\sum_{n=1}^N \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

and recalling from Exercise 7, Sec. 41, that the value of the integral here tends to zero as N tends to infinity, we arrive at the desired summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. The path C here is the positively oriented boundary of the rectangle with vertices at the points ± 2 and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3}.$$

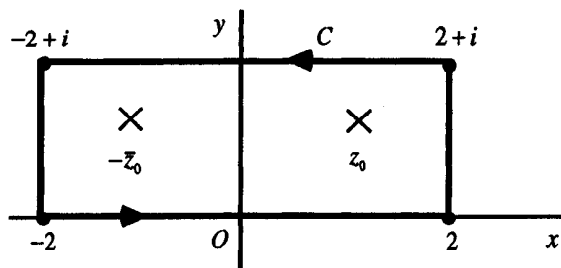
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2 - 1)^2 + 3.$$

Setting this polynomial equal to zero and solving for z^2 , we find that any zero z of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1 + \sqrt{3}i$ and also the two square roots of $1 - \sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3}+i}{\sqrt{2}},$$

lie inside C . They are shown in the figure below.



To find the residues at z_0 and $-\bar{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2 - 1)^2 + 3} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \quad \text{and} \quad q(z) = (z^2 - 1)^2 + 3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0) = 0$. Note, too, that p and q are analytic at z_0 and that $p(z_0) \neq 0$. Finally, it is straightforward to show that $q'(z) = 4z(z^2 - 1)$ and hence that

$$q'(z_0) = 4z_0(z_0^2 - 1) = -2\sqrt{6} + 6\sqrt{2}i \neq 0.$$

We may conclude, then, that z_0 is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)} = \frac{1}{-2\sqrt{6} + 6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\bar{z}_0$. To be specific, it is easy to see that

$$q'(-\bar{z}_0) = -q'(\bar{z}_0) = -\overline{q'(z_0)} = 2\sqrt{6} + 6\sqrt{2}i \neq 0,$$

the residue of the integrand at $-\bar{z}_0$ being

$$\frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)} = \frac{1}{2\sqrt{6} + 6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3} = 2\pi i \left(\frac{1}{-2\sqrt{6} + 6\sqrt{2}i} + \frac{1}{2\sqrt{6} + 6\sqrt{2}i} \right) = \frac{\pi}{2\sqrt{2}}.$$

7. We are given that $f(z) = 1/[q(z)]^2$, where q is analytic at z_0 , $q(z_0) = 0$, and $q'(z_0) \neq 0$. These conditions on q tell us that q has a zero of order $m = 1$ at z_0 . Hence $q(z) = (z - z_0)g(z)$, where g is a function that is analytic and nonzero at z_0 ; and this enables us to write

$$f(z) = \frac{\phi(z)}{(z - z_0)^2}, \quad \text{where } \phi(z) = \frac{1}{[g(z)]^2}.$$

So f has a pole of order 2 at z_0 , and

$$\operatorname{Res}_{z=z_0} f(z) = \phi'(z_0) = -\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z) = (z - z_0)g(z)$, we know that

$$q'(z) = (z - z_0)g'(z) + g(z) \quad \text{and} \quad q''(z) = (z - z_0)g''(z) + 2g'(z).$$

Then, by setting $z = z_0$ in these last two equations, we find that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname{Res}_{z=z_0} f(z) = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

8. (a) To find the residue of the function $\csc^2 z$ at $z = 0$, we write

$$\csc^2 z = \frac{1}{[q(z)]^2}, \quad \text{where } q(z) = \sin z.$$

Since q is entire, $q(0) = 0$, and $q'(0) = 1 \neq 0$, the result in Exercise 7 tells us that

$$\operatorname{Res}_{z=0} \csc^2 z = -\frac{q''(0)}{[q'(0)]^3} = 0.$$

(b) The residue of the function $\frac{1}{(z+z^2)^2}$ at $z=0$ can be obtained by writing

$$\frac{1}{(z+z^2)^2} = \frac{1}{[q(z)]^2}, \quad \text{where } q(z) = z+z^2.$$

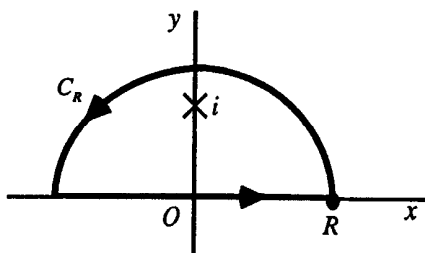
Inasmuch as q is entire, $q(0)=0$, and $q'(0) = 1 \neq 0$, we know from Exercise 7 that

$$\operatorname{Res}_{z=0} \frac{1}{(z+z^2)^2} = -\frac{q''(0)}{[q'(0)]^3} = -2.$$

Chapter 7

SECTION 72

1. To evaluate the integral $\int_0^{\infty} \frac{dx}{x^2+1}$, we integrate the function $f(z) = \frac{1}{z^2+1}$ around the simple closed contour shown below, where $R > 1$.



We see that

$$\int_{-R}^R \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1} = 2\pi i B,$$

where

$$B = \operatorname{Res}_{z=i} \frac{1}{z^2+1} = \operatorname{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \frac{1}{z+i} \Big|_{z=i} = \frac{1}{2i}.$$

Thus

$$\int_{-R}^R \frac{dx}{x^2+1} = \pi - \int_{C_R} \frac{dz}{z^2+1}.$$

Now if z is a point on C_R ,

$$|z^2+1| \geq ||z|^2-1| = R^2-1;$$

and so

$$\left| \int_{C_R} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{R^2-1} = \frac{\frac{\pi}{R}}{1-\frac{1}{R^2}} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Finally, then

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi, \quad \text{or} \quad \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

2. The integral $\int_0^{\infty} \frac{dx}{(x^2+1)^2}$ can be evaluated using the function $f(z) = \frac{1}{(z^2+1)^2}$ and the same simple closed contour as in Exercise 1. Here

$$\int_{-R}^R \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i B,$$

where $B = \operatorname{Res}_{z=i} \frac{1}{(z^2+1)^2}$. Since

$$\frac{1}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2}, \quad \text{where } \phi(z) = \frac{1}{(z+i)^2},$$

we readily find that $B = \phi'(i) = \frac{1}{4i}$, and so

$$\int_{-R}^R \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.$$

If z is a point on C_R , we know from Exercise 1 that

$$|z^2+1| \geq R^2-1;$$

thus

$$\left| \int_{C_R} \frac{dz}{(z^2+1)^2} \right| \leq \frac{\pi R}{(R^2-1)^2} = \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

The desired result is, then,

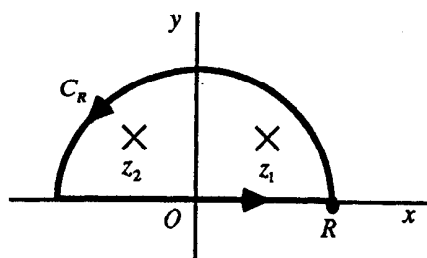
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}, \quad \text{or} \quad \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

3. We begin the evaluation of $\int_0^{\infty} \frac{dx}{x^4+1}$ by finding the zeros of the polynomial z^4+1 , which are the fourth roots of -1 , and noting that two of them are below the real axis. In fact, if we consider the simple closed contour shown below, where $R > 1$, that contour encloses only the two roots

$$z_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

and

$$z_2 = e^{i3\pi/4} = e^{i\pi/4} e^{i\pi/2} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)i = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$



Now

$$\int_{-R}^R \frac{dx}{x^4+1} + \int_{C_R} \frac{dz}{z^4+1} = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname{Res}_{z=z_1} \frac{1}{z^4+1} \quad \text{and} \quad B_2 = \operatorname{Res}_{z=z_2} \frac{1}{z^4+1}.$$

The method of Theorem 2 in Sec. 69 tells us that z_1 and z_2 are simple poles of $\frac{1}{z^4+1}$ and that

$$B_1 = \frac{1}{4z_1^3} \cdot \frac{z_1}{z_1} = -\frac{z_1}{4} \quad \text{and} \quad B_2 = \frac{1}{4z_2^3} \cdot \frac{z_2}{z_2} = -\frac{z_2}{4},$$

since $z_1^4 = -1$ and $z_2^4 = -1$. Furthermore,

$$B_1 + B_2 = -\frac{1}{4}(z_1 + z_2) = -\frac{1}{4} \left[\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) + \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \right] = -\frac{i}{2\sqrt{2}}.$$

Hence

$$\int_{-R}^R \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}} - \int_{C_R} \frac{dz}{z^4+1}.$$

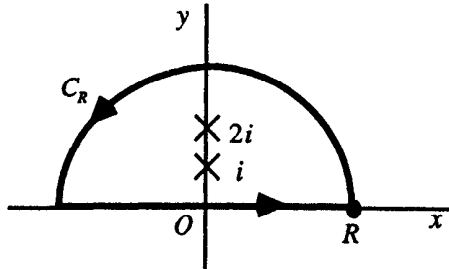
Since

$$\left| \int_{C_R} \frac{dz}{z^4+1} \right| \leq \frac{\pi R}{R^4-1} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}, \quad \text{or} \quad \int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

4. We wish to evaluate the integral $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$. We use the simple closed contour shown below, where $R > 2$.



We must find the residues of the function $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ at its simple poles $z = i$ and $z = 2i$. They are

$$B_1 = \text{Res}_{z=i} f(z) = \left. \frac{z^2}{(z+i)(z^2+4)} \right]_{z=i} = -\frac{1}{6i}$$

and

$$B_2 = \text{Res}_{z=2i} f(z) = \left. \frac{z^2}{(z^2+1)(z+2i)} \right]_{z=2i} = \frac{1}{3i}.$$

Thus

$$\int_{-R}^R \frac{x^2 dx}{(x^2+1)(x^2+4)} + \int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)} = 2\pi i(B_1 + B_2),$$

or

$$\int_{-R}^R \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3} - \int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)}.$$

If z is a point on C_R , then

$$|z^2+1| \geq ||z|^2-1| = R^2-1 \quad \text{and} \quad |z^2+4| \geq ||z|^2-4| = R^2-4.$$

Consequently,

$$\left| \int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)} \right| \leq \frac{\pi R^3}{(R^2-1)(R^2-4)} = \frac{\frac{\pi}{R}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \rightarrow 0 \text{ as } R \rightarrow \infty;$$

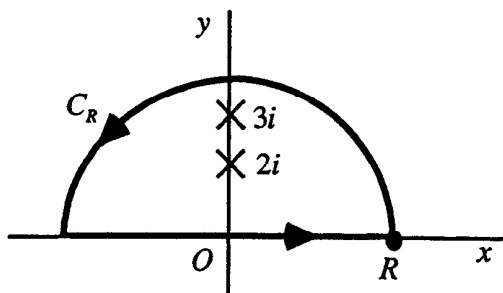
and we may conclude that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}, \quad \text{or} \quad \int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$$

5. The integral $\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$ can be evaluated with the aid of the function

$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$$

and the simple closed contour shown below, where $R > 3$.



We start by writing

$$\int_{-R}^R \frac{x^2 dx}{(x^2+9)(x^2+4)^2} + \int_{C_R} \frac{z^2 dz}{(z^2+9)(z^2+4)^2} = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname{Res}_{z=3i} \frac{z^2}{(z^2+9)(z^2+4)^2} \quad \text{and} \quad B_2 = \operatorname{Res}_{z=2i} \frac{z^2}{(z^2+9)(z^2+4)^2}.$$

Now

$$B_1 = \left. \frac{z^2}{(z+3i)(z^2+4)^2} \right]_{z=3i} = -\frac{3}{50i}.$$

To find B_2 , we write

$$\frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{\phi(z)}{(z-2i)^2}, \quad \text{where} \quad \phi(z) = \frac{z^2}{(z^2+9)(z+2i)^2}.$$

Then

$$B_2 = \phi'(2i) = \frac{13}{200i}.$$

This tells us that

$$\int_{-R}^R \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{100} - \int_{C_R} \frac{z^2 dz}{(z^2+9)(z^2+4)^2}.$$

Finally, since

$$\left| \int_{C_R} \frac{z^2 dz}{(z^2+9)(z^2+4)^2} \right| \leq \frac{\pi R^3}{(R^2-9)(R^2-4)^2} \rightarrow 0 \text{ as } R \rightarrow \infty,$$

we find that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{100}, \quad \text{or} \quad \int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$$

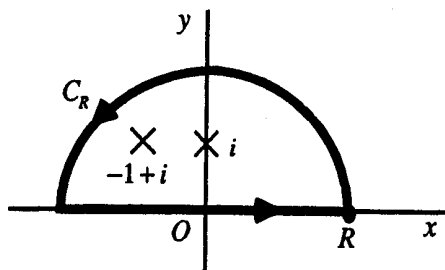
7. In order to show that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x dx}{(x^2 + 1)(x^2 + 2x + 2)} = -\frac{\pi}{5},$$

we introduce the function

$$f(z) = \frac{z}{(z^2 + 1)(z^2 + 2z + 2)}$$

and the simple closed contour shown below.



Observe that the singularities of $f(z)$ are at i , $z_0 = -1+i$ and their conjugates $-i$, $\bar{z}_0 = -1-i$ in the lower half plane. Also, if $R > \sqrt{2}$, we see that

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i(B_0 + B_1),$$

where

$$B_0 = \text{Res}_{z=z_0} f(z) = \left. \frac{z}{(z^2 + 1)(z - \bar{z}_0)} \right]_{z=z_0} = -\frac{1}{10} + \frac{3}{10}i$$

and

$$B_1 = \text{Res}_{z=i} f(z) = \left. \frac{z}{(z+i)(z^2 + 2z + 2)} \right]_{z=i} = \frac{1}{10} - \frac{1}{5}i.$$

Evidently, then,

$$\int_{-R}^R \frac{x dx}{(x^2 + 1)(x^2 + 2x + 2)} = -\frac{\pi}{5} - \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)}.$$

Since

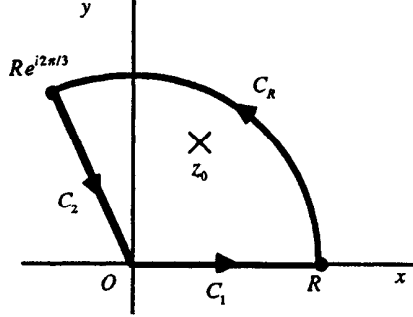
$$\left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} \right| = \left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z - z_0)(z - \bar{z}_0)} \right| \leq \frac{\pi R^2}{(R^2 - 1)(R - \sqrt{2})^2} \rightarrow 0$$

as $R \rightarrow \infty$, this means that

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{x dx}{(x^2 + 1)(x^2 + 2x + 2)} = -\frac{\pi}{5}.$$

This is the desired result.

8. The problem here is to establish the integration formula $\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$ using the simple closed contour shown below, where $R > 1$.



There is only one singularity of the function $f(z) = \frac{1}{z^3+1}$, namely $z_0 = e^{i\pi/3}$, that is interior to the closed contour when $R > 1$. According to the residue theorem,

$$\int_{C_1} \frac{dz}{z^3+1} + \int_{C_2} \frac{dz}{z^3+1} + \int_{C_3} \frac{dz}{z^3+1} + \int_{C_4} \frac{dz}{z^3+1} = 2\pi i \operatorname{Res}_{z=z_0} \frac{1}{z^3+1},$$

where the legs of the closed contour are as indicated in the figure. Since C_1 has parametric representation $z = r$ ($0 \leq r \leq R$),

$$\int_{C_1} \frac{dz}{z^3+1} = \int_0^R \frac{dr}{r^3+1};$$

and, since $-C_2$ can be represented by $z = re^{i2\pi/3}$ ($0 \leq r \leq R$),

$$\int_{C_2} \frac{dz}{z^3+1} = - \int_{-C_2} \frac{dz}{z^3+1} = - \int_0^R \frac{e^{i2\pi/3} dr}{(re^{i2\pi/3})^3+1} = -e^{i2\pi/3} \int_0^R \frac{dr}{r^3+1}.$$

Furthermore,

$$\operatorname{Res}_{z=z_0} \frac{1}{z^3+1} = \frac{1}{3z_0^2} = \frac{1}{3e^{i2\pi/3}}.$$

Consequently,

$$(1 - e^{i2\pi/3}) \int_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3e^{i2\pi/3}} - \int_{C_3} \frac{dz}{z^3+1}.$$

But

$$\left| \int_{C_3} \frac{dz}{z^3+1} \right| \leq \frac{1}{R^3-1} \cdot \frac{2\pi R}{3} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

This gives us the desired result, with the variable of integration r instead of x :

$$\int_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{i4\pi/3} \cdot e^{-i6\pi/3})} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{-i2\pi/3})} = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$

9. Let m and n be integers, where $0 \leq m < n$. The problem here is to derive the integration formula

$$\int_0^{\infty} \frac{x^{2m}}{x^{2n} + 1} dx = \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n} \pi\right).$$

- (a) The zeros of the polynomial $z^{2n} + 1$ occur when $z^{2n} = -1$. Since

$$(-1)^{1/(2n)} = \exp\left[i \frac{(2k+1)\pi}{2n}\right] \quad (k = 0, 1, 2, \dots, 2n-1),$$

it is clear that the zeros of $z^{2n} + 1$ in the upper half plane are

$$c_k = \exp\left[i \frac{(2k+1)\pi}{2n}\right] \quad (k = 0, 1, 2, \dots, n-1)$$

and that there are none on the real axis.

- (b) With the aid of Theorem 2 in Sec. 69, we find that

$$\operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = \frac{c_k^{2m}}{2n c_k^{2n-1}} = \frac{1}{2n} c_k^{2(m-n)+1} \quad (k = 0, 1, 2, \dots, n-1).$$

Putting $\alpha = \frac{2m+1}{2n} \pi$, we can write

$$\begin{aligned} c_k^{2(m-n)+1} &= \exp\left[i \frac{(2k+1)\pi(2m-2n+1)}{2n}\right] \\ &= \exp\left[i \frac{(2k+1)(2m+1)\pi}{2n}\right] \exp[-i(2k+1)\pi] = -e^{i(2k+1)\alpha}. \end{aligned}$$

Thus

$$\operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = -\frac{1}{2n} e^{i(2k+1)\alpha} \quad (k = 0, 1, 2, \dots, n-1).$$

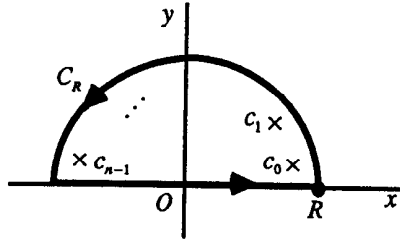
In view of the identity (see Exercise 10, Sec. 7)

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \quad (z \neq 1),$$

then,

$$\begin{aligned} 2\pi i \sum_{k=0}^{n-1} \operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} &= -\frac{\pi i}{n} e^{i\alpha} \sum_{k=0}^{n-1} (e^{i2\alpha})^k = -\frac{\pi i}{n} e^{i\alpha} \frac{1-e^{i2\alpha n}}{1-e^{i2\alpha}} \cdot \frac{e^{-i\alpha}}{e^{-i\alpha}} = -\frac{\pi i}{n} \cdot \frac{e^{i2\alpha n}-1}{e^{i\alpha}-e^{-i\alpha}} \\ &= -\frac{\pi i}{n} \cdot \frac{e^{i(2m+1)\pi}-1}{e^{i\alpha}-e^{-i\alpha}} = \frac{\pi}{n} \cdot \frac{2i}{e^{i\alpha}-e^{-i\alpha}} = \frac{\pi}{n \sin \alpha}. \end{aligned}$$

(c) Consider the path shown below, where $R > 1$.



The residue theorem tells us that

$$\int_{-R}^R \frac{x^{2m}}{x^{2n}+1} dx + \int_{C_R} \frac{z^{2m}}{z^{2n}+1} dz = 2\pi i \sum_{k=0}^{n-1} \operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1},$$

or

$$\int_{-R}^R \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{n \sin \alpha} - \int_{C_R} \frac{z^{2m}}{z^{2n}+1} dz.$$

Observe that if z is a point on C_R , then

$$|z^{2m}| = R^{2m} \quad \text{and} \quad |z^{2n}+1| \geq R^{2n}-1.$$

Consequently,

$$\left| \int_{C_R} \frac{z^{2m}}{z^{2n}+1} dz \right| \leq \frac{R^{2m}}{R^{2n}-1} \pi R \cdot \frac{R^{-2n}}{R^{-2n}} = \pi \frac{R^{2(m-n)+1}}{1-\frac{1}{R^{2n}}} \rightarrow 0;$$

and the desired integration formula follows.

10. The problem here is to evaluate the integral

$$\int_0^{\infty} \frac{dx}{[(x^2-a)^2+1]^2},$$

where a is any real number. We do this by following the steps below.

(a) Let us first find the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1.$$

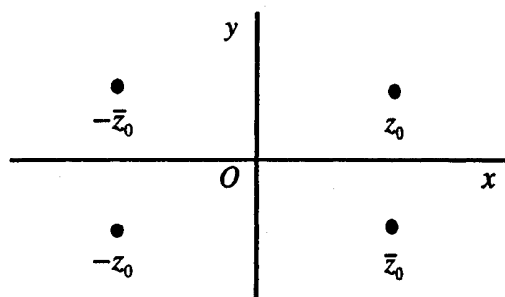
Solving the equation $q(z) = 0$ for z^2 , we obtain $z^2 = a \pm i$. Thus two of the zeros are the square roots of $a + i$, and the other two are the square roots of $a - i$. By Exercise 5, Sec. 9, the two square roots of $a + i$ are the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a}) \quad \text{and} \quad -z_0,$$

where $A = \sqrt{a^2 + 1}$. Since $(\pm \bar{z}_0)^2 = \bar{z}_0^2 = \overline{a+i} = a-i$, the two square roots of $a-i$, are evidently

$$\bar{z}_0 \quad \text{and} \quad -\bar{z}_0.$$

The four zeros of $q(z)$ just obtained are located in the plane in the figure below, which tells us that z_0 and $-\bar{z}_0$ lie above the real axis and that the other two zeros lie below it.



(b) Let $q(z)$ denote the polynomial in part (a); and define the function

$$f(z) = \frac{1}{[q(z)]^2},$$

which becomes the integrand in the integral to be evaluated when $z = x$. The method developed in Exercise 7, Sec. 69, reveals that z_0 is a pole of order 2 of f . To be specific, we note that q is entire and recall from part (a) that $q(z_0) = 0$. Furthermore, $q'(z) = 4z(z^2 - a)$ and $z_0^2 = a + i$, as pointed out above in part (a). Consequently, $q'(z_0) = 4z_0(z_0^2 - a) = 4iz_0 \neq 0$. The exercise just mentioned, together with the relations $z_0^2 = a + i$ and $1 + a^2 = A^2$, also enables us to write the residue B_1 of f at z_0 :

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = -\frac{12z_0^2 - 4a}{(4iz_0)^3} = \frac{3z_0^2 - a}{16iz_0^2 z_0} = \frac{3(a+i) - a}{16i(a+i)z_0} \cdot \frac{a-i}{a-i} = \frac{a-i(2a^2+3)}{16A^2 z_0}.$$

As for the point $-\bar{z}_0$, we observe that

$$q'(-\bar{z}) = -\overline{q'(z)} \quad \text{and} \quad q''(-\bar{z}) = \overline{q''(z)}.$$

Since $q(-\bar{z}_0) = 0$ and $q'(-\bar{z}_0) = -\overline{q'(z_0)} = 4i\bar{z}_0 \neq 0$, the point $-\bar{z}_0$ is also a pole of order 2 of f . Moreover, if B_2 denotes the residue there,

$$B_2 = -\frac{q''(-\bar{z}_0)}{[q'(-\bar{z}_0)]^3} = \frac{\overline{q''(z_0)}}{[q'(z_0)]^3} = \overline{\left\{ \frac{q''(z_0)}{[q'(z_0)]^3} \right\}} = -\bar{B}_1.$$

Thus

$$B_1 + B_2 = B_1 - \bar{B}_1 = 2i \operatorname{Im} B_1 = \frac{1}{8A^2 i} \operatorname{Im} \left[\frac{-a + i(2a^2 + 3)}{z_0} \right].$$

(c) We now integrate $f(z)$ around the simple closed path in the figure below, where $R > |z_0|$ and C_R denotes the semicircular portion of the path. The residue theorem tells us that

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i (B_1 + B_2),$$

or

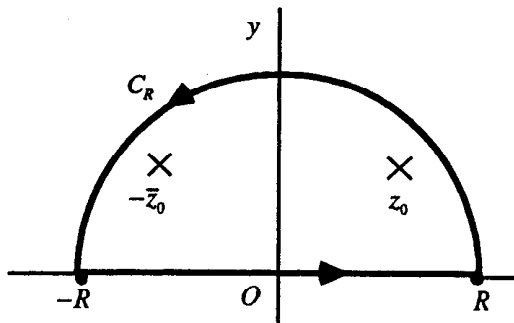
$$\int_{-R}^R \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{4A^2} \operatorname{Im} \left[\frac{-a + i(2a^2 + 3)}{z_0} \right] - \int_{C_R} \frac{dz}{[q(z)]^2}.$$

In order to show that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{[q(z)]^2} = 0,$$

we start with the observation that the polynomial $q(z)$ can be factored into the form

$$q(z) = (z - z_0)(z + z_0)(z - \bar{z}_0)(z + \bar{z}_0).$$



Recall now that $R > |z_0|$. If z is a point on C_R , so that $|z| = R$, then

$$|z \pm z_0| \geq ||z| - |z_0|| = R - |z_0| \quad \text{and} \quad |z \pm \bar{z}_0| \geq ||z| - |\bar{z}_0|| = R - |z_0|.$$

This enables us to see that $|q(z)| \geq (R - |z_0|)^4$ when z is on C_R . Thus

$$\left| \frac{1}{[q(z)]^2} \right| \leq \frac{1}{(R - |z_0|)^8}$$

for such points, and we arrive at the inequality

$$\left| \int_{C_R} \frac{1}{[q(z)]^2} dz \right| \leq \frac{\pi R}{(R - |z_0|)^8} = \frac{\frac{\pi}{R^7}}{\left(1 - \frac{|z_0|}{R}\right)^8},$$

which tells us that the value of this integral does, indeed, tend to 0 as R tends to ∞ . Consequently,

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{4A^2} \text{Im} \left[\frac{-a + i(2a^2 + 3)}{z_0} \right].$$

But the integrand here is even, and

$$\text{Im} \left[\frac{-a + i(2a^2 + 3)}{z_0} \right] = \text{Im} \left[\sqrt{2} \frac{-a + i(2a^2 + 3)}{\sqrt{A+a} + i\sqrt{A-a}} \cdot \frac{\sqrt{A+a} - i\sqrt{A-a}}{\sqrt{A+a} - i\sqrt{A-a}} \right].$$

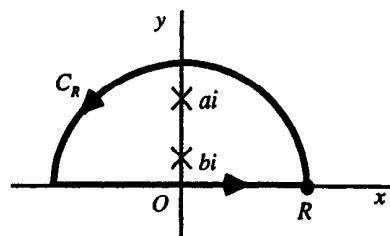
So, the desired result is

$$\int_0^{\infty} \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{8\sqrt{2}A^3} [(2a^2 + 3)\sqrt{A+a} + a\sqrt{A-a}],$$

where $A = \sqrt{a^2 + 1}$.

SECTION 74

1. The problem here is to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2 + a^2)(x^2 + b^2)}$, where $a > b > 0$. To do this, we introduce the function $f(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)}$, whose singularities ai and bi lie inside the simple closed contour shown below, where $R > a$. The other singularities are, of course, in the lower half plane.



According to the residue theorem,

$$\int_{-R}^R \frac{e^{ix} dx}{(x^2 + a^2)(x^2 + b^2)} + \int_{C_R} f(z)e^{iz} dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \text{Res}_{z=ai}[f(z)e^{iz}] = \left. \frac{e^{iz}}{(z+ai)(z^2+b^2)} \right]_{z=ai} = \frac{e^{-a}}{2a(b^2-a^2)i}$$

and

$$B_2 = \text{Res}_{z=bi}[f(z)e^{iz}] = \left. \frac{e^{iz}}{(z^2+a^2)(z+bi)} \right]_{z=bi} = \frac{e^{-b}}{2b(a^2-b^2)i}.$$

That is,

$$\int_{-R}^R \frac{e^{ix} dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) - \int_{C_R} f(z)e^{iz} dz,$$

or

$$\int_{-R}^R \frac{\cos x dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) - \text{Re} \int_{C_R} f(z)e^{iz} dz.$$

Now, if z is a point on C_R ,

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{1}{(R^2 - a^2)(R^2 - b^2)}$$

and $|e^{iz}| = e^{-y} \leq 1$. Hence

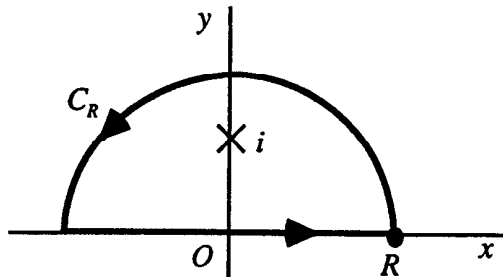
$$\left| \text{Re} \int_{C_R} f(z)e^{iz} dz \right| \leq \left| \int_{C_R} f(z)e^{iz} dz \right| \leq M_R \pi R = \frac{\pi R}{(R^2 - a^2)(R^2 - b^2)} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) \quad (a > b > 0).$$

2. This problem is to evaluate the integral $\int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx$, where $a \geq 0$. The function

$f(z) = \frac{1}{z^2 + 1}$ has the singularities $\pm i$; and so we may integrate around the simple closed contour shown below, where $R > 1$.



We start with

$$\int_{-R}^R \frac{e^{iax}}{x^2+1} dx + \int_{C_R} f(z)e^{iaz} dz = 2\pi i B,$$

where

$$B = \operatorname{Res}_{z=i} [f(z)e^{iaz}] = \left. \frac{e^{iaz}}{z+i} \right|_{z=i} = \frac{e^{-a}}{2i}.$$

Hence

$$\int_{-R}^R \frac{e^{iax}}{x^2+1} dx = \pi e^{-a} - \int_{C_R} f(z)e^{iaz} dz,$$

or

$$\int_{-R}^R \frac{\cos ax}{x^2+1} dx = \pi e^{-a} - \operatorname{Re} \int_{C_R} f(z)e^{iaz} dz,$$

Since

$$|f(z)| \leq M_R \text{ where } M_R = \frac{1}{R^2-1},$$

we know that

$$\left| \operatorname{Re} \int_{C_R} f(z)e^{iaz} dz \right| \leq \left| \int_{C_R} f(z)e^{iaz} dz \right| \leq \frac{\pi R}{R^2-1};$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx = \pi e^{-a}.$$

That is,

$$\int_0^{\infty} \frac{\cos ax}{x^2+1} dx = \frac{\pi}{2} e^{-a} \quad (a \geq 0).$$

4. To evaluate the integral $\int_0^{\infty} \frac{x \sin 2x}{x^2+3} dx$, we first introduce the function

$$f(z) = \frac{z}{z^2+3} = \frac{z}{(z-z_1)(z-\bar{z}_1)},$$

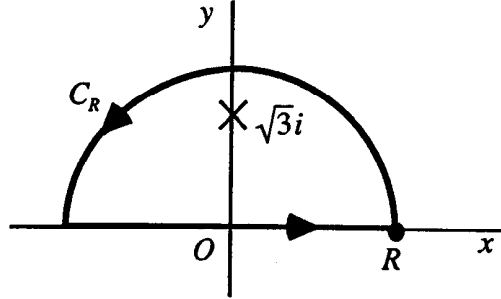
where $z_1 = \sqrt{3}i$. The point z_1 lies above the x axis, and \bar{z}_1 lies below it. If we write

$$f(z)e^{i2z} = \frac{\phi(z)}{z-z_1} \text{ where } \phi(z) = \frac{z \exp(i2z)}{z-\bar{z}_1},$$

we see that z_1 is a simple pole of the function $f(z)e^{i2z}$ and that the corresponding residue is

$$B_1 = \phi(z_1) = \frac{\sqrt{3}i \exp(-2\sqrt{3})}{2\sqrt{3}i} = \frac{\exp(-2\sqrt{3})}{2}.$$

Now consider the simple closed contour shown in the figure below, where $R > \sqrt{3}$.



Integrating $f(z)e^{i2z}$ around the closed contour, we have

$$\int_{-R}^R \frac{x e^{i2x}}{x^2 + 3} dx = 2\pi i B_1 - \int_{C_R} f(z) e^{i2z} dz.$$

Thus

$$\int_{-R}^R \frac{x \sin x}{x^2 + 3} dx = \text{Im}(2\pi i B_1) - \text{Im} \int_{C_R} f(z) e^{i2z} dz.$$

Now, when z is a point on C_R ,

$$|f(z)| \leq M_R, \quad \text{where} \quad M_R = \frac{R}{R^2 - 3} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty;$$

and so, by limit (1), Sec. 74,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{i2z} dz = 0.$$

Consequently, since

$$\left| \text{Im} \int_{C_R} f(z) e^{i2z} dz \right| \leq \left| \int_{C_R} f(z) e^{i2z} dz \right|,$$

we arrive at the result

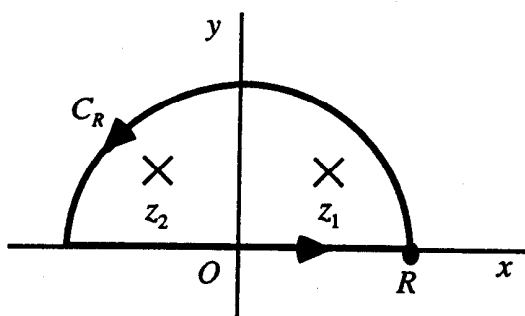
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 3} dx = \pi \exp(-2\sqrt{3}), \quad \text{or} \quad \int_0^{\infty} \frac{x \sin x}{x^2 + 3} dx = \frac{\pi}{2} \exp(-2\sqrt{3}).$$

6. The integral to be evaluated is $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx$, where $a > 0$. We define the function

$f(z) = \frac{z^3}{z^4 + 4}$; and, by computing the fourth roots of -4 , we find that the singularities

$$z_1 = \sqrt{2}e^{i\pi/4} = 1+i \quad \text{and} \quad z_2 = \sqrt{2}e^{i3\pi/4} = \sqrt{2}e^{i\pi/4}e^{i\pi/2} = (1+i)i = -1+i$$

both lie inside the simple closed contour shown below, where $R > \sqrt{2}$. The other two singularities lie below the real axis.



The residue theorem and the method of Theorem 2 in Sec. 69 for finding residues at simple poles tell us that

$$\int_{-R}^R \frac{x^3 e^{iax}}{x^4 + 4} dx + \int_{C_R} f(z) e^{iaz} dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \text{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} = \frac{z_1^3 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4} = \frac{e^{ia(1+i)}}{4} = \frac{e^{-a} e^{ia}}{4}$$

and

$$B_2 = \text{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} = \frac{z_2^3 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4} = \frac{e^{ia(-1+i)}}{4} = \frac{e^{-a} e^{-ia}}{4}.$$

Since

$$2\pi i(B_1 + B_2) = \pi i e^{-a} \left(\frac{e^{ia} + e^{-ia}}{2} \right) = i\pi e^{-a} \cos a,$$

we are now able to write

$$\int_{-R}^R \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a - \text{Im} \int_{C_R} f(z) e^{iaz} dz.$$

Furthermore, if z is a point on C_R , then

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{R^3}{R^4 - 4} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty;$$

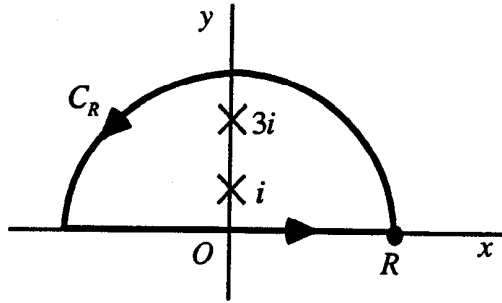
and this means that

$$\left| \operatorname{Im} \int_{C_R} f(z) e^{iaz} dz \right| \leq \left| \int_{C_R} f(z) e^{iaz} dz \right| \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty,$$

according to limit (1), Sec. 74. Finally, then,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a \quad (a > 0).$$

8. In order to evaluate the integral $\int_0^{\infty} \frac{x^3 \sin x dx}{(x^2 + 1)(x^2 + 9)}$, we introduce here the function $f(z) = \frac{z^3}{(z^2 + 1)(z^2 + 9)}$. Its singularities in the upper half plane are i and $3i$, and we consider the simple closed contour shown below, where $R > 3$.



Since

$$\operatorname{Res}_{z=i} [f(z)e^{iz}] = \left. \frac{z^3 e^{iz}}{(z+i)(z^2+9)} \right|_{z=i} = -\frac{1}{16e}$$

and

$$\operatorname{Res}_{z=3i} [f(z)e^{iz}] = \left. \frac{z^3 e^{iz}}{(z^2+1)(z+3i)} \right|_{z=3i} = \frac{9}{16e^3},$$

the residue theorem tells us that

$$\int_{-R}^R \frac{x^3 e^{ix} dx}{(x^2 + 1)(x^2 + 9)} + \int_{C_R} f(z) e^{iz} dz = 2\pi i \left(-\frac{1}{16e} + \frac{9}{16e^3} \right),$$

or

$$\int_{-R}^R \frac{x^3 \sin x dx}{(x^2 + 1)(x^2 + 9)} = \frac{\pi}{8e} \left(\frac{9}{e^2} - 1 \right) - \operatorname{Im} \int_{C_R} f(z) e^{iz} dz.$$

Now if z is a point on C_R , then

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{R}{(R^2-1)(R^2-9)} \quad \text{as } R \rightarrow \infty.$$

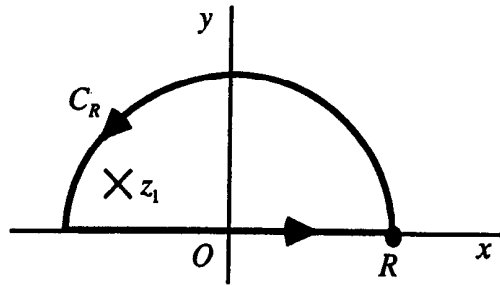
So, in view of limit (1), Sec. 74,

$$\left| \operatorname{Im} \int_{C_R} f(z) e^{iz} dz \right| \leq \left| \int_{C_R} f(z) e^{iz} dz \right| \rightarrow 0 \quad \text{as } R \rightarrow \infty;$$

and this means that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e} \left(\frac{9}{e^2} - 1 \right), \quad \text{or} \quad \int_0^{\infty} \frac{x^3 \sin x dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16e} \left(\frac{9}{e^2} - 1 \right).$$

9. The Cauchy principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5}$ can be found with the aid of the function $f(z) = \frac{1}{z^2+4z+5}$ and the simple closed contour shown below, where $R > \sqrt{5}$. Using the quadratic formula to solve the equation $z^2+4z+5=0$, we find that f has singularities at the points $z_1 = -2+i$ and $\bar{z}_1 = -2-i$. Thus $f(z) = \frac{1}{(z-z_1)(z-\bar{z}_1)}$, where z_1 is interior to the closed contour and \bar{z}_1 is below the real axis.



The residue theorem tells us that

$$\int_{-R}^R \frac{e^{ix} dx}{x^2+4x+5} + \int_{C_R} f(z) e^{iz} dz = 2\pi i B,$$

where

$$B = \operatorname{Res}_{z=z_1} \left[\frac{e^{iz}}{(z-z_1)(z-\bar{z}_1)} \right] = \frac{e^{iz_1}}{(z_1-\bar{z}_1)};$$

and so

$$\int_{-R}^R \frac{\sin x dx}{x^2+4x+5} = \operatorname{Im} \left[\frac{2\pi i e^{iz_1}}{(z_1-\bar{z}_1)} \right] - \operatorname{Im} \int_{C_R} f(z) e^{iz} dz,$$

or

$$\int_{-R}^R \frac{\sin x \, dx}{x^2 + 4x + 5} = -\frac{\pi}{e} \sin 2 - \operatorname{Im} \int_{C_R} f(z) e^{iz} dz.$$

Now, if z is a point on C_R , then $|e^{iz}| = e^{-y} \leq 1$ and

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{1}{(R-|z_1|)(R-|\bar{z}_1|)} = \frac{1}{(R-\sqrt{5})^2}.$$

Hence

$$\left| \operatorname{Im} \int_{C_R} f(z) e^{iz} dz \right| \leq \left| \int_{C_R} f(z) e^{iz} dz \right| \leq M_R \pi R = \frac{\pi R}{(R-\sqrt{5})^2} \rightarrow 0 \text{ as } R \rightarrow \infty,$$

and we may conclude that

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2 + 4x + 5} = -\frac{\pi}{e} \sin 2.$$

10. To find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2 + 4x + 5} dx$, we shall use

the function $f(z) = \frac{z+1}{z^2 + 4z + 5} = \frac{z+1}{(z-z_1)(z-\bar{z}_1)}$, where $z_1 = -2+i$, and $\bar{z}_1 = -2-1$, and the same simple closed contour as in Exercise 9. In this case,

$$\int_{-R}^R \frac{(x+1)e^{ix} \, dx}{x^2 + 4x + 5} + \int_{C_R} f(z) e^{iz} dz = 2\pi i B,$$

where

$$B = \operatorname{Res}_{z=z_1} \left[\frac{(z+1)e^{iz}}{(z-z_1)(z-\bar{z}_1)} \right] = \frac{(z_1+1)e^{iz_1}}{(z_1-\bar{z}_1)} = \frac{(-1+i)e^{-2i}}{2ei}.$$

Thus

$$\int_{-R}^R \frac{(x+1)\cos x}{x^2 + 4x + 5} dx = \operatorname{Re}(2\pi i B) - \int_{C_R} f(z) e^{iz} dz,$$

or

$$\int_{-R}^R \frac{(x+1)\cos x}{x^2 + 4x + 5} dx = \frac{\pi}{e} (\sin 2 - \cos 2) - \int_{C_R} f(z) e^{iz} dz.$$

Finally, we observe that if z is a point on C_R , then

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{R+1}{(R-|z_1|)(R-|\bar{z}_1|)} = \frac{R+1}{(R-\sqrt{5})^2} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

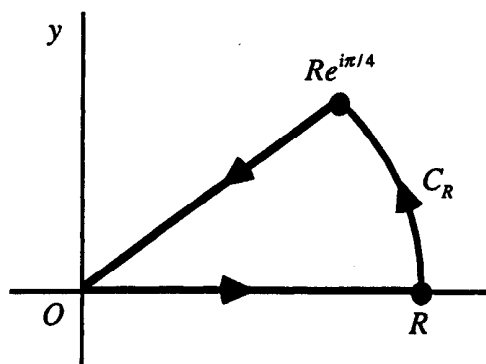
Limit (1), Sec. 74, then tells us that

$$\left| \operatorname{Re} \int_{C_R} f(z) e^{iz} dz \right| \leq \left| \int_{C_R} f(z) e^{iz} dz \right| \rightarrow 0 \text{ as } R \rightarrow \infty,$$

and so

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{(x+1) \cos x}{x^2 + 4x + 5} dx = \frac{\pi}{e} (\sin 2 - \cos 2).$$

12. (a) Since the function $f(z) = \exp(iz^2)$ is entire, the Cauchy-Goursat theorem tells us that its integral around the positively oriented boundary of the sector $0 \leq r \leq R$, $0 \leq \theta \leq \pi/4$ has value zero. The closed path is shown below.



A parametric representation of the horizontal line segment from the origin to the point R is $z = x$ ($0 \leq x \leq R$), and a representation for the segment from the origin to the point $Re^{i\pi/4}$ is $z = re^{i\pi/4}$ ($0 \leq r \leq R$). Thus

$$\int_0^R e^{ix^2} dx + \int_{C_R} e^{iz^2} dz - e^{i\pi/4} \int_0^R e^{-r^2} dr = 0,$$

or

$$\int_0^R e^{ix^2} dx = e^{i\pi/4} \int_0^R e^{-r^2} dr - \int_{C_R} e^{iz^2} dz.$$

By equating real parts and then imaginary parts on each side of this last equation, we see that

$$\int_0^R \cos(x^2) dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \operatorname{Re} \int_{C_R} e^{iz^2} dz$$

and

$$\int_0^R \sin(x^2) dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \operatorname{Im} \int_{C_R} e^{iz^2} dz.$$

(b) A parametric representation for the arc C_R is $z = Re^{i\theta}$ ($0 \leq \theta \leq \pi/4$). Hence

$$\int_{C_R} e^{iz^2} dz = \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} Rie^{i\theta} d\theta = iR \int_0^{\pi/4} e^{-R^2 \sin 2\theta} e^{iR^2 \cos 2\theta} e^{i\theta} d\theta.$$

Since $|e^{iR^2 \cos 2\theta}| = 1$ and $|e^{i\theta}| = 1$, it follows that

$$\left| \int_{C_R} e^{iz^2} dz \right| \leq R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta.$$

Then, by making the substitution $\phi = 2\theta$ in this last integral and referring to the form (3), Sec. 74, of Jordan's inequality, we find that

$$\left| \int_{C_R} e^{iz^2} dz \right| \leq \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \phi} d\phi \leq \frac{R}{2} \cdot \frac{\pi}{2R^2} = \frac{\pi}{4R} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

(c) In view of the result in part (b) and the integration formula

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

it follows from the last two equations in part (a) that

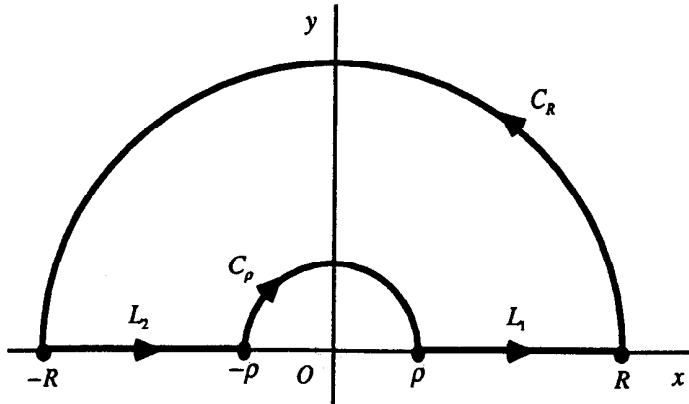
$$\int_0^{\infty} \cos(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^{\infty} \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

SECTION 77

1. The main problem here is to derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \quad (a \geq 0, b \geq 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z) dz + \int_{C_R} f(z) dz + \int_{L_2} f(z) dz + \int_{C_\rho} f(z) dz = 0,$$

or

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = -\int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R),$$

we can see that

$$\begin{aligned} \int_{L_1} f(z) dz + \int_{L_2} f(z) dz &= \int_{L_1} f(z) dz - \int_{-L_2} f(z) dz = \int_{\rho}^R \frac{e^{iar} - e^{ibr}}{r^2} dr + \int_{\rho}^R \frac{e^{-iar} - e^{-ibr}}{r^2} dr \\ &= \int_{\rho}^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2} dr = 2 \int_{\rho}^R \frac{\cos(ar) - \cos(br)}{r^2} dr. \end{aligned}$$

Thus

$$2 \int_{\rho}^R \frac{\cos(ar) - \cos(br)}{r^2} dr = -\int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

In order to find the limit of the first integral on the right here as $\rho \rightarrow 0$, we write

$$\begin{aligned} f(z) &= \frac{1}{z^2} \left[\left(1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \dots \right) - \left(1 + \frac{ibz}{1!} + \frac{(ibz)^2}{2!} + \frac{(ibz)^3}{3!} + \dots \right) \right] \\ &= \frac{i(a-b)}{z} + \dots \quad (0 < |z| < \infty). \end{aligned}$$

From this we see that $z = 0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = -B_0 \pi i = -i(a-b) \pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R \rightarrow \infty$, we note that if z is a point on C_R , then

$$f(z) \leq \frac{|e^{iaz}| + |e^{ibz}|}{|z|^2} = \frac{e^{-ay} + e^{-by}}{R^2} \leq \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{2}{R^2} \pi R = \frac{2\pi}{R} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

It is now clear that letting $\rho \rightarrow 0$ and $R \rightarrow \infty$ yields

$$2 \int_0^{\infty} \frac{\cos(ar) - \cos(br)}{r^2} dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x . Observe that when $a = 0$ and $b = 2$, that result becomes

$$\int_0^{\infty} \frac{1 - \cos(2x)}{x^2} dx = \pi.$$

But $\cos(2x) = 1 - 2\sin^2 x$, and we arrive at

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

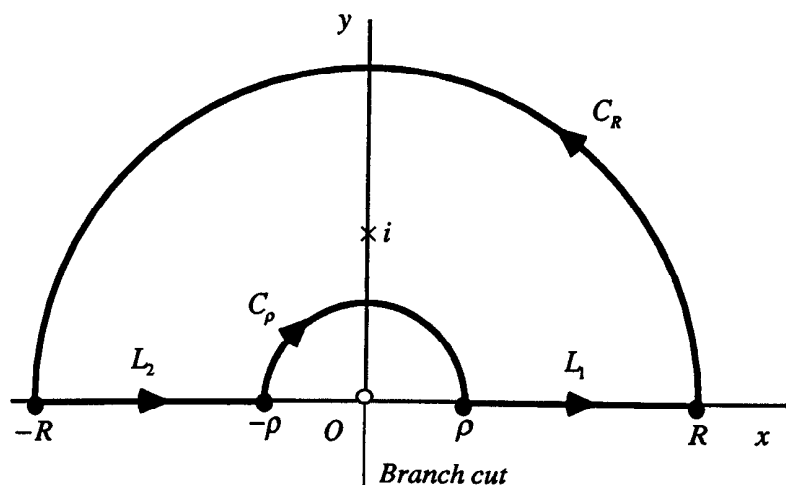
2. Let us derive the integration formula

$$\int_0^{\infty} \frac{x^a}{(x^2+1)^2} dx = \frac{(1-a)\pi}{4\cos(a\pi/2)} \quad (-1 < a < 3),$$

where $x^a = \exp(a \ln x)$ when $x > 0$. We shall integrate the function

$$f(z) = \frac{z^a}{(z^2+1)^2} = \frac{\exp(a \log z)}{(z^2+1)^2} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right),$$

whose branch cut is the origin and the negative imaginary axis, around the simple closed path shown below.



By Cauchy's residue theorem,

$$\int_{L_1} f(z) dz + \int_{C_R} f(z) dz + \int_{L_2} f(z) dz + \int_{C_\rho} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z).$$

That is,

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R),$$

the left-hand side of this last equation can be written

$$\begin{aligned} \int_{L_1} f(z) dz - \int_{-L_2} f(z) dz &= \int_{\rho}^R \frac{e^{a(\ln r + i0)}}{(r^2 + 1)^2} dr - \int_{\rho}^R \frac{e^{a(\ln r + i\pi)}}{(r^2 + 1)^2} e^{i\pi} dr \\ &= \int_{\rho}^R \frac{r^a}{(r^2 + 1)^2} dr + e^{ia\pi} \int_{\rho}^R \frac{r^a}{(r^2 + 1)^2} dr = (1 + e^{ia\pi}) \int_{\rho}^R \frac{r^a}{(r^2 + 1)^2} dr. \end{aligned}$$

Also,

$$\operatorname{Res}_{z=i} f(z) = \phi'(i) \quad \text{where} \quad \phi(z) = \frac{z^a}{(z+i)^2},$$

the point $z = i$ being a pole of order 2 of the function $f(z)$. Straightforward differentiation reveals that

$$\phi'(z) = e^{(a-1)\log z} \left[\frac{a(z+i) - 2z}{(z+i)^3} \right],$$

and from this it follows that

$$\operatorname{Res}_{z=i} f(z) = -ie^{ia\pi/2} \left(\frac{1-a}{4} \right).$$

We now have

$$(1 + e^{ia\pi}) \int_{\rho}^R \frac{r^a}{(r^2 + 1)^2} dr = \frac{\pi(1-a)}{2} e^{ia\pi/2} - \int_{C_{\rho}} f(z) dz - \int_{C_R} f(z) dz.$$

Once we show that

$$\lim_{\rho \rightarrow 0} \int_{C_{\rho}} f(z) dz = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0,$$

we arrive at the desired result:

$$\int_0^{\infty} \frac{r^a}{(r^2 + 1)^2} dr = \frac{\pi(1-a)}{2} \cdot \frac{e^{ia\pi/2}}{1 + e^{ia\pi}} \cdot \frac{e^{-ia\pi/2}}{e^{-ia\pi/2}} = \frac{\pi(1-a)}{4} \cdot \frac{2}{e^{ia\pi/2} + e^{-ia\pi/2}} = \frac{(1-a)\pi}{4 \cos(a\pi/2)}.$$

The first of the above limits is shown by writing

$$\left| \int_{C_{\rho}} f(z) dz \right| \leq \frac{\rho^a}{(1-\rho^2)^2} \pi\rho = \frac{\pi\rho^{a+1}}{(1-\rho^2)^2}$$

and noting that the last term tends to 0 as $\rho \rightarrow 0$ since $a+1 > 0$. As for the second limit,

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{R^a}{(R^2-1)^2} \pi R = \frac{\pi R^{a+1}}{(R^2-1)^2} \cdot \frac{1}{R^4} = \frac{\pi R^{3-a}}{\left(1 - \frac{1}{R^2}\right)^2};$$

and the last term here tends to 0 as $R \rightarrow \infty$ since $3-a > 0$.

3. The problem here is to derive the integration formulas

$$I_1 = \int_0^{\infty} \frac{\sqrt[3]{x} \ln x}{x^2 + 1} dx = \frac{\pi^2}{6} \quad \text{and} \quad I_2 = \int_0^{\infty} \frac{\sqrt[3]{x}}{x^2 + 1} dx = \frac{\pi}{\sqrt{3}}$$

by integrating the function

$$f(z) = \frac{z^{1/3} \log z}{z^2 + 1} = \frac{e^{(1/3)\log z} \log z}{z^2 + 1} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right),$$

around the contour shown in Exercise 2. As was the case in that exercise,

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{e^{(1/3)\log z} \log z}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$, with residue

$$\operatorname{Res}_{z=i} f(z) = \phi(i) = \frac{\pi}{4} e^{i\pi/6}.$$

The parametric representations

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R)$$

can be used to write

$$\int_{L_1} f(z) dz = \int_{\rho}^R \frac{\sqrt[3]{r} \ln r}{r^2 + 1} dr \quad \text{and} \quad \int_{L_2} f(z) dz = e^{i\pi/3} \int_{\rho}^R \frac{\sqrt[3]{r} \ln r + i\pi \sqrt[3]{r}}{r^2 + 1} dr.$$

Thus

$$\int_{\rho}^R \frac{\sqrt[3]{r} \ln r}{r^2 + 1} dr + e^{i\pi/3} \int_{\rho}^R \frac{\sqrt[3]{r} \ln r + i\pi \sqrt[3]{r}}{r^2 + 1} dr = \frac{\pi^2}{2} i e^{i\pi/6} - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

By equating real parts on each side of this equation, we have

$$\begin{aligned} \int_{\rho}^R \frac{\sqrt[3]{r} \ln r}{r^2 + 1} dr + \cos(\pi/3) \int_{\rho}^R \frac{\sqrt[3]{r} \ln r}{r^2 + 1} dr - \pi \sin(\pi/3) \int_{\rho}^R \frac{\sqrt[3]{r}}{r^2 + 1} dr &= -\frac{\pi^2}{2} \sin(\pi/6) \\ &\quad - \operatorname{Re} \int_{C_\rho} f(z) dz - \operatorname{Re} \int_{C_R} f(z) dz; \end{aligned}$$

and equating imaginary parts yields

$$\begin{aligned} \sin(\pi/3) \int_{\rho}^R \frac{\sqrt[3]{r} \ln r}{r^2 + 1} dr + \pi \cos(\pi/3) \int_{\rho}^R \frac{\sqrt[3]{r}}{r^2 + 1} dr &= \frac{\pi^2}{2} \cos(\pi/6) \\ &\quad - \operatorname{Im} \int_{C_\rho} f(z) dz - \operatorname{Im} \int_{C_R} f(z) dz. \end{aligned}$$

Now $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$, $\sin(\pi/6) = \frac{1}{2}$, $\cos(\pi/6) = \frac{\sqrt{3}}{2}$ and it is routine to show that

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

Thus

$$\frac{3}{2} \int_0^{\infty} \frac{\sqrt[3]{r} \ln r}{r^2+1} dr - \frac{\pi\sqrt{3}}{2} \int_0^{\infty} \frac{\sqrt[3]{r}}{r^2+1} dr = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2} \int_0^{\infty} \frac{\sqrt[3]{r} \ln r}{r^2+1} dr + \frac{\pi}{2} \int_0^{\infty} \frac{\sqrt[3]{r}}{r^2+1} dr = \frac{\pi^2\sqrt{3}}{4}.$$

That is,

$$\frac{3}{2} I_1 - \frac{\pi\sqrt{3}}{2} I_2 = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2} I_1 + \frac{\pi}{2} I_2 = \frac{\pi^2\sqrt{3}}{4}.$$

Solving these simultaneous equations for I_1 and I_2 , we arrive at the desired integration formulas.

4. Let us use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right)$$

and the contour in Exercise 2 to show that

$$\int_0^{\infty} \frac{(\ln x)^2}{x^2+1} dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^{\infty} \frac{\ln x}{x^2+1} dx = 0.$$

Integrating $f(z)$ around the closed path shown in Exercise 2, we have

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_p} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^2}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$ and the residue is

$$\operatorname{Res}_{z=i} f(z) = \phi(i) = \frac{(\log i)^2}{2i} = \frac{(\ln 1 + i\pi/2)^2}{2i} = -\frac{\pi^2}{8i}.$$

Also, the parametric representations

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R)$$

enable us to write

$$\int_{L_1} f(z) dz = \int_{\rho}^R \frac{(\ln r)^2}{r^2 + 1} dr \quad \text{and} \quad \int_{L_2} f(z) dz = \int_{\rho}^R \frac{(\ln r + i\pi)^2}{r^2 + 1} dr.$$

Since

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2 \int_{\rho}^R \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_{\rho}^R \frac{dr}{r^2 + 1} + 2\pi i \int_{\rho}^R \frac{\ln r}{r^2 + 1} dr,$$

then,

$$2 \int_{\rho}^R \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_{\rho}^R \frac{dr}{r^2 + 1} + 2\pi i \int_{\rho}^R \frac{\ln r}{r^2 + 1} dr = -\frac{\pi^3}{4} - \int_{C_{\rho}} f(z) dz - \int_{C_R} f(z) dz.$$

Equating real parts on each side of this equation, we have

$$2 \int_{\rho}^R \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_{\rho}^R \frac{dr}{r^2 + 1} = -\frac{\pi^3}{4} - \operatorname{Re} \int_{C_{\rho}} f(z) dz - \operatorname{Re} \int_{C_R} f(z) dz;$$

and equating imaginary parts yields

$$2\pi \int_{\rho}^R \frac{\ln r}{r^2 + 1} dr = \operatorname{Im} \int_{C_{\rho}} f(z) dz - \operatorname{Im} \int_{C_R} f(z) dz.$$

It is straightforward to show that

$$\lim_{\rho \rightarrow 0} \int_{C_{\rho}} f(z) dz = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

Hence

$$2 \int_0^{\infty} \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_0^{\infty} \frac{dr}{r^2 + 1} = -\frac{\pi^3}{4}$$

and

$$2\pi \int_0^{\infty} \frac{\ln r}{r^2 + 1} dr = 0.$$

Finally, inasmuch as (see Exercise 1, Sec. 72),

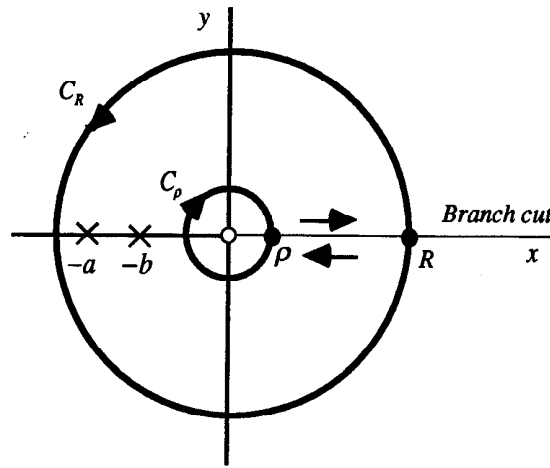
$$\int_0^{\infty} \frac{dr}{r^2 + 1} = \frac{\pi}{2},$$

we arrive at the desired integration formulas.

5. Here we evaluate the integral $\int_0^{\infty} \frac{\sqrt[3]{x}}{(x+a)(x+b)} dx$, where $a > b > 0$. We consider the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3} \log z\right)}{(z+a)(z+b)} \quad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown below, which is similar to the one used in Sec. 77. The numbers ρ and R are small and large enough, respectively, so that the points $z = -a$ and $z = -b$ are between the circles.



A parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ ($\rho \leq r \leq R$), and so the value of the integral of f along that edge is

$$\int_{\rho}^R \frac{\exp\left[\frac{1}{3}(\ln r + i0)\right]}{(r+a)(r+b)} dr = \int_{\rho}^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr.$$

A representation for the lower edge from ρ to R is $z = re^{i2\pi}$ ($\rho \leq r \leq R$). Hence the value of the integral of f along that edge from R to ρ is

$$-\int_{\rho}^R \frac{\exp\left[\frac{1}{3}(\ln r + i2\pi)\right]}{(r+a)(r+b)} dr = -e^{i2\pi/3} \int_{\rho}^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr.$$

According to the residue theorem, then,

$$\int_{\rho}^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr + \int_{C_R} f(z) dz - e^{i2\pi/3} \int_{\rho}^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr + \int_{C_{\rho}} f(z) dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname{Res}_{z=-a} f(z) = \frac{\exp\left[\frac{1}{3}\log(-a)\right]}{-a+b} = -\frac{\exp\left[\frac{1}{3}(\ln a + i\pi)\right]}{a-b} = -\frac{e^{i\pi/3} \sqrt[3]{a}}{a-b}$$

and

$$B_2 = \operatorname{Res}_{z=-b} f(z) = \frac{\exp\left[\frac{1}{3}\log(-b)\right]}{-b+a} = \frac{\exp\left[\frac{1}{3}(\ln b + i\pi)\right]}{-b+a} = \frac{e^{i\pi/3} \sqrt[3]{b}}{a-b}.$$

Consequently,

$$(1 - e^{i2\pi/3}) \int_{\rho}^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr = -\frac{2\pi i e^{i\pi/3} (\sqrt[3]{a} - \sqrt[3]{b})}{a-b} - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Now

$$\left| \int_{C_\rho} f(z) dz \right| \leq \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)} 2\pi\rho = \frac{2\pi\sqrt[3]{\rho}\rho}{(a-\rho)(b-\rho)} \rightarrow 0 \text{ as } \rho \rightarrow 0$$

and

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\sqrt[3]{R}}{(R-a)(R-b)} 2\pi R = \frac{2\pi R^2}{(R-a)(R-b)} \cdot \frac{1}{\sqrt[3]{R^2}} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Hence

$$\begin{aligned} \int_0^{\infty} \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr &= -\frac{2\pi i e^{i\pi/3} (\sqrt[3]{a} - \sqrt[3]{b})}{(1 - e^{i2\pi/3})(a-b)} \cdot \frac{e^{-i\pi/3}}{e^{-i\pi/3}} = \frac{2\pi i (\sqrt[3]{a} - \sqrt[3]{b})}{(e^{i\pi/3} - e^{-i\pi/3})(a-b)} \\ &= \frac{\pi(\sqrt[3]{a} - \sqrt[3]{b})}{\sin(\pi/3)(a-b)} = \frac{\pi(\sqrt[3]{a} - \sqrt[3]{b})}{\frac{\sqrt{3}}{2}(a-b)} = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b}. \end{aligned}$$

Replacing the variable of integration r here by x , we have the desired result:

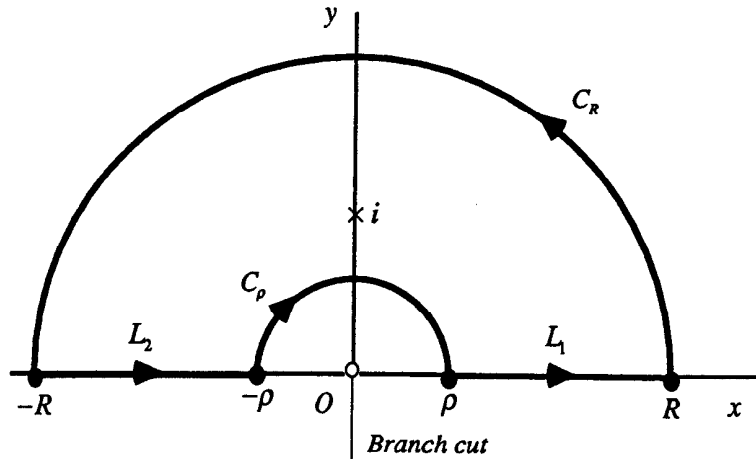
$$\int_0^{\infty} \frac{\sqrt[3]{x}}{(x+a)(x+b)} dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \quad (a > b > 0).$$

6. (a) Let us first use the branch

$$f(z) = \frac{z^{-1/2}}{z^2 + 1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2 + 1} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2 + 1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z) dz + \int_{C_R} f(z) dz + \int_{L_2} f(z) dz + \int_{C_\rho} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R),$$

we may write

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} - i \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} = (1 - i) \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)}.$$

Thus

$$(1 - i) \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Now the point $z = i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname{Res}_{z=i} f(z) = \left. \frac{z^{-1/2}}{z+i} \right|_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{1}{2i} \left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left| \int_{C_\rho} f(z) dz \right| \leq \frac{\pi \rho}{\sqrt{\rho(1-\rho^2)}} = \frac{\pi \sqrt{\rho}}{1-\rho^2} \rightarrow 0 \text{ as } \rho \rightarrow 0$$

and

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi \sqrt{R}}{(R^2-1)} = \frac{\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Finally, then, we have

$$(1-i) \int_0^\infty \frac{dr}{\sqrt{r}(r^2+1)} = \frac{\pi(1-i)}{\sqrt{2}},$$

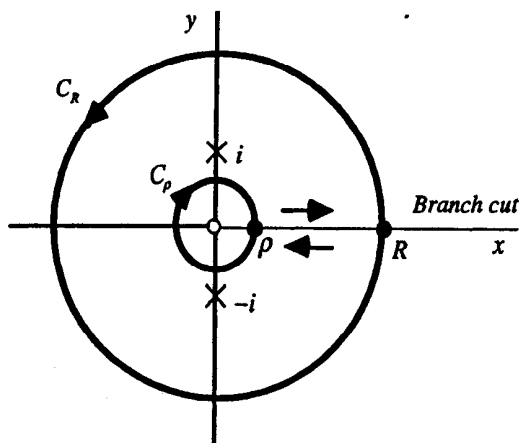
which is the same as

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(b) To evaluate the improper integral $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \quad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 99 in Sec. 77. We stipulate that $\rho < 1$ and $R > 1$, so that the singularities $z = \pm i$ are between C_ρ and C_R .



Since a parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ ($\rho \leq r \leq R$), the value of the integral of f along that edge is

$$\int_{\rho}^R \frac{\exp\left[-\frac{1}{2}(\ln r + i0)\right]}{r^2 + 1} dr = \int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr.$$

A representation for the lower edge from ρ to R is $z = re^{i2\pi}$ ($\rho \leq r \leq R$), and so the value of the integral of f along that edge from R to ρ is

$$-\int_{\rho}^R \frac{\exp\left[-\frac{1}{2}(\ln r + i2\pi)\right]}{r^2 + 1} dr = -e^{-i\pi} \int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr = \int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr.$$

Hence, by the residue theorem,

$$\int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr + \int_{C_R} f(z) dz + \int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr + \int_{C_{\rho}} f(z) dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname{Res}_{z=i} f(z) = \left. \frac{z^{-1/2}}{z+i} \right|_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i}$$

and

$$B_2 = \operatorname{Res}_{z=-i} f(z) = \left. \frac{z^{-1/2}}{z-i} \right|_{z=-i} = \frac{\exp\left[-\frac{1}{2}\log(-i)\right]}{-2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{3\pi}{2}\right)\right]}{-2i} = \frac{e^{-i3\pi/4}}{2i}.$$

That is,

$$2 \int_{\rho}^R \frac{1}{\sqrt{r}(r^2 + 1)} dr = \pi(e^{-i\pi/4} - e^{-i3\pi/4}) - \int_{C_{\rho}} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$\left| \int_{C_{\rho}} f(z) dz \right| \leq \frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{2\pi\sqrt{\rho}}{1-\rho^2} \rightarrow 0 \text{ as } \rho \rightarrow 0$$

and

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{2\pi R}{\sqrt{R}(R^2 - 1)} = \frac{2\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \rightarrow 0 \text{ as } R \rightarrow \infty,$$

we now find that

$$\begin{aligned} \int_0^{\infty} \frac{1}{\sqrt{r}(r^2+1)} dr &= \pi \frac{e^{-i\pi/4} - e^{-i3\pi/4}}{2} = \pi \frac{e^{-i\pi/4} + e^{-i3\pi/4} e^{i\pi}}{2} \\ &= \pi \frac{e^{i\pi/4} + e^{-i\pi/4}}{2} = \pi \cos\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

When x , instead of r , is used as the variable of integration here, we have the desired result:

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

SECTION 78

1. Write

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{1}{5+4\left(\frac{z-z^{-1}}{2i}\right)} \cdot \frac{dz}{iz} = \int_C \frac{dz}{2z^2+5iz-2},$$

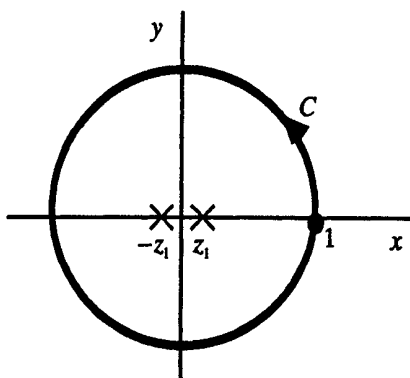
where C is the positively oriented unit circle $|z|=1$. The quadratic formula tells us that the singular points of the integrand on the far right here are $z=-i/2$ and $z=-2i$. The point $z=-i/2$ is a simple pole interior to C ; and the point $z=-2i$ is exterior to C . Thus

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \operatorname{Res}_{z=-i/2} \left[\frac{1}{2z^2+5iz-2} \right] = 2\pi i \left[\frac{1}{4z+5i} \right]_{z=-i/2} = 2\pi i \left(\frac{1}{3i} \right) = \frac{2\pi}{3}.$$

2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_C \frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2} \cdot \frac{dz}{iz} = \int_C \frac{4iz dz}{z^4-6z^2+1},$$

where C is the positively oriented unit circle $|z|=1$. This circle is shown below.



Solving the equation $(z^2)^2 - 6(z^2) + 1 = 0$ for z^2 with the aid of the quadratic formula, we find that the zeros of the polynomial $z^4 - 6z^2 + 1$ are the numbers z such that $z^2 = 3 \pm 2\sqrt{2}$. Those zeros are, then, $z = \pm\sqrt{3+2\sqrt{2}}$ and $z = \pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1 = \sqrt{3-2\sqrt{2}} \quad \text{and} \quad z_2 = -z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname{Res}_{z=z_1} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{4iz_1}{4z_1^3 - 12z_1} = \frac{i}{z_1^2 - 3} = \frac{i}{(3-2\sqrt{2})-3} = -\frac{i}{2\sqrt{2}}$$

and

$$B_2 = \operatorname{Res}_{z=-z_1} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{-4iz_1}{-4z_1^3 + 12z_1} = \frac{i}{z_1^2 - 3} = -\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1 + B_2) = 2\pi i\left(-\frac{i}{\sqrt{2}}\right) = \frac{2\pi}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \sqrt{2}\pi.$$

7. Let C be the positively oriented unit circle $|z|=1$. In view of the binomial formula (Sec. 3)

$$\begin{aligned} \int_0^{\pi} \sin^{2n} \theta d\theta &= \frac{1}{2} \int_{-\pi}^{\pi} \sin^{2n} \theta d\theta = \frac{1}{2} \int_C \left(\frac{z - z^{-1}}{2i} \right)^{2n} \frac{dz}{iz} = \frac{1}{2^{2n+1} (-1)^n i} \int_C \frac{(z - z^{-1})^{2n}}{z} dz \\ &= \frac{1}{2^{2n+1} (-1)^n i} \int_C \sum_{k=0}^n \binom{2n}{k} z^{2n-k} (-z^{-1})^k z^{-1} dz \\ &= \frac{1}{2^{2n+1} (-1)^n i} \sum_{k=0}^n \binom{2n}{k} (-1)^k \int_C z^{2n-2k-1} dz. \end{aligned}$$

Now each of these last integrals has value zero except when $k = n$:

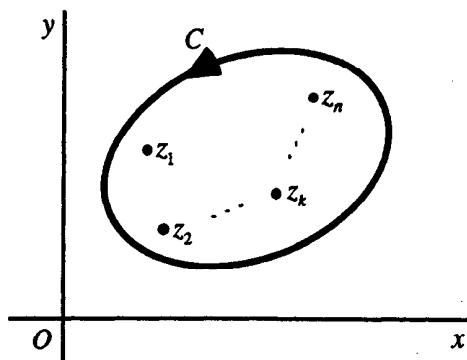
$$\int_C z^{-1} dz = 2\pi i.$$

Consequently,

$$\int_0^\pi \sin^{2n} \theta d\theta = \frac{1}{2^{2n+1}(-1)^n i} \cdot \frac{(2n)!(-1)^n 2\pi i}{(n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2} \pi.$$

SECTION 80

5. We are given a function f that is analytic inside and on a positively oriented simple closed contour C , and we assume that f has no zeros on C . Also, f has n zeros z_k ($k = 1, 2, \dots, n$) inside C , where each z_k is of multiplicity m_k . (See the figure below.)



The object here is to show that

$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^n m_k z_k.$$

To do this, we consider the k th zero and start with the fact that

$$f(z) = (z - z_k)^{m_k} g(z),$$

where $g(z)$ is analytic and nonzero at z_k . From this, it is straightforward to show that

$$\frac{zf'(z)}{f(z)} = \frac{m_k z}{z - z_k} + \frac{zg'(z)}{g(z)} = \frac{m_k(z - z_k) + m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)} = m_k + \frac{zg'(z)}{g(z)} + \frac{m_k z_k}{z - z_k}.$$

Since the term $\frac{zg'(z)}{g(z)}$ here has a Taylor series representation at z_k , it follows that $\frac{zf'(z)}{f(z)}$ has a simple pole at z_k and that

$$\operatorname{Res}_{z=z_k} \frac{zf'(z)}{f(z)} = m_k z_k.$$

An application of the residue theorem now yields the desired result.

6. (a) To determine the number of zeros of the polynomial $z^6 - 5z^4 + z^3 - 2z$ inside the circle $|z|=1$, we write

$$f(z) = -5z^4 \quad \text{and} \quad g(z) = z^6 + z^3 - 2z.$$

We then observe that when z is on the circle,

$$|f(z)| = 5 \quad \text{and} \quad |g(z)| \leq |z|^6 + |z|^3 + 2|z| = 4.$$

Since $|f(z)| > |g(z)|$ on the circle and since $f(z)$ has 4 zeros, counting multiplicities, inside it, the theorem in Sec. 80 tells us that the sum

$$f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$$

also has four zeros, counting multiplicities, inside the circle.

- (b) Let us write the polynomial $2z^4 - 2z^3 + 2z^2 - 2z + 9$ as the sum $f(z) + g(z)$, where

$$f(z) = 9 \quad \text{and} \quad g(z) = 2z^4 - 2z^3 + 2z^2 - 2z.$$

Observe that when z is on the circle $|z|=1$,

$$|f(z)| = 9 \quad \text{and} \quad |g(z)| \leq 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8.$$

Since $|f(z)| > |g(z)|$ on the circle and since $f(z)$ has no zeros inside it, the sum $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ has no zeros there either.

7. Let C denote the circle $|z|=2$.

- (a) The polynomial $z^4 + 3z^3 + 6$ can be written as the sum of the polynomials

$$f(z) = 3z^3 \quad \text{and} \quad g(z) = z^4 + 6.$$

On C ,

$$|f(z)| = 3|z|^3 = 24 \quad \text{and} \quad |g(z)| = |z^4 + 6| \leq |z|^4 + 6 = 22.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 3 zeros, counting multiplicities, inside C , it follows that the original polynomial has 3 zeros, counting multiplicities, inside C .

- (b) The polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ can be written as the sum of the polynomials

$$f(z) = 9z^2 \quad \text{and} \quad g(z) = z^4 - 2z^3 + z - 1.$$

On C ,

$$|f(z)| = 9|z|^2 = 36 \quad \text{and} \quad |g(z)| = |z^4 - 2z^3 + z - 1| \leq |z|^4 + 2|z|^3 + |z| + 1 = 35.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 2 zeros, counting multiplicities, inside C , it follows that the original polynomial has 2 zeros, counting multiplicities, inside C .

(c) The polynomial $z^5 + 3z^3 + z^2 + 1$ can be written as the sum of the polynomials

$$f(z) = z^5 \quad \text{and} \quad g(z) = 3z^3 + z^2 + 1.$$

On C ,

$$|f(z)| = |z|^5 = 32 \quad \text{and} \quad |g(z)| = |3z^3 + z^2 + 1| \leq 3|z|^3 + |z|^2 + 1 = 29.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 5 zeros, counting multiplicities, inside C , it follows that the original polynomial has 5 zeros, counting multiplicities, inside C .

10. The problem here is to give an alternative proof of the fact that any polynomial

$$P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \quad (a_n \neq 0),$$

where $n \geq 1$, has precisely n zeros, counting multiplicities. Without loss of generality, we may take $a_n = 1$ since

$$P(z) = a_n \left(\frac{a_0}{a_n} + \frac{a_1}{a_n}z + \cdots + \frac{a_{n-1}}{a_n}z^{n-1} + z^n \right).$$

Let

$$f(z) = z^n \quad \text{and} \quad g(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1}.$$

Then let R be so large that

$$R > 1 + |a_0| + |a_1| + \cdots + |a_{n-1}|.$$

If z is a point on the circle $C: |z| = R$, we find that

$$\begin{aligned} |g(z)| &\leq |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} = |a_0| + |a_1|R + \cdots + |a_{n-1}|R^{n-1} \\ &< |a_0|R^{n-1} + |a_1|R^{n-1} + \cdots + |a_{n-1}|R^{n-1} = (|a_0| + |a_1| + \cdots + |a_{n-1}|)R^{n-1} \\ &< RR^{n-1} = R^n = |z|^n = |f(z)|. \end{aligned}$$

Since $f(z)$ has precisely n zeros, counting multiplicities, inside C and since R can be made arbitrarily large, the desired result follows.

1. The singularities of the function

$$F(s) = \frac{2s^3}{s^4 - 4}$$

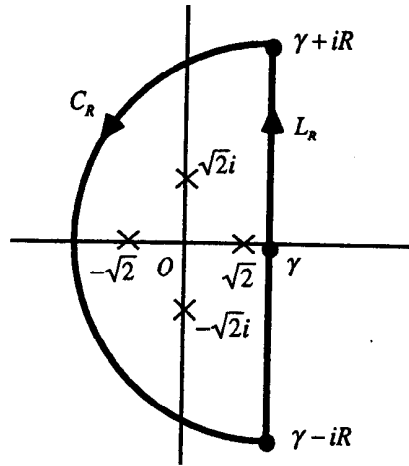
are the fourth roots of 4. They are readily found to be

$$s = \sqrt{2} e^{ik\pi/2} \quad (k = 0, 1, 2, 3),$$

or

$$\sqrt{2}, \quad \sqrt{2}i, \quad -\sqrt{2}, \quad \text{and} \quad -\sqrt{2}i.$$

See the figure below, where $\gamma > \sqrt{2}$ and $R > \sqrt{2} + \gamma$.



The function

$$e^{st} F(s) = \frac{2s^3 e^{st}}{s^4 - 4}$$

has simple poles at the points

$$s_0 = \sqrt{2}, \quad s_1 = \sqrt{2}i, \quad s_2 = -\sqrt{2}, \quad \text{and} \quad s_3 = -\sqrt{2}i;$$

and

$$\sum_{n=0}^3 \operatorname{Res}_{s=s_n} [e^{st} F(s)] = \sum_{n=0}^3 \operatorname{Res}_{s=s_n} \frac{2s^3 e^{st}}{s^4 - 4} = \sum_{n=0}^3 \frac{2s_n^3 e^{s_n t}}{4s_n^3} = \sum_{n=0}^3 \frac{1}{2} e^{s_n t}$$

$$= \frac{1}{2} e^{\sqrt{2}t} + \frac{1}{2} e^{i\sqrt{2}t} + \frac{1}{2} e^{-\sqrt{2}t} + \frac{1}{2} e^{-i\sqrt{2}t}$$

$$= \frac{e^{\sqrt{2}t} + e^{-\sqrt{2}t}}{2} + \frac{e^{i\sqrt{2}t} + e^{-i\sqrt{2}t}}{2}$$

$$= \cosh \sqrt{2}t + \cos \sqrt{2}t.$$

Suppose now that s is a point on C_R , and observe that

$$|s| = |\gamma + Re^{i\theta}| \leq \gamma + R = R + \gamma \quad \text{and} \quad |s| = |\gamma + Re^{i\theta}| \geq |\gamma - R| = R - \gamma > \sqrt{2}.$$

It follows that

$$|2s^3| = 2|s|^3 \leq 2(R + \gamma)^3$$

and

$$|s^4 - 4| \geq ||s|^4 - 4| \geq (R - \gamma)^4 - 4 > 0.$$

Consequently,

$$|F(s)| \leq \frac{2(R + \gamma)^3}{(R - \gamma)^4 - 4} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

This ensures that

$$f(t) = \cosh \sqrt{2}t + \cos \sqrt{2}t.$$

2. The polynomials in the denominator of

$$F(s) = \frac{2s - 2}{(s + 1)(s^2 + 2s + 5)}$$

have zeros at $s = -1$ and $s = -1 \pm 2i$. Let us, then, write

$$e^{st} F(s) = \frac{e^{st}(2s - 2)}{(s + 1)(s - s_1)(s - \bar{s}_1)},$$

where $s_1 = -1 + 2i$. The points -1 , s_1 , and \bar{s}_1 are evidently simple poles of $e^{st} F(s)$ with the following residues:

$$B_1 = \text{Res}_{z=-1} [e^{st} F(s)] = \left. \frac{e^{st}(2s - 2)}{(s - s_1)(s - \bar{s}_1)} \right]_{s=-1} = -e^{-t},$$

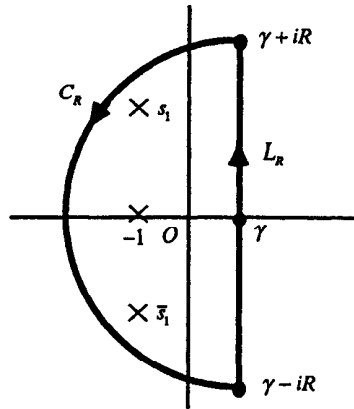
$$B_2 = \text{Res}_{s=s_1} [e^{st} F(s)] = \frac{e^{s_1 t}(2s_1 - 2)}{(s_1 + 1)(s_1 - \bar{s}_1)} = \left(\frac{1}{2} - \frac{i}{2} \right) e^{-t} e^{i2t},$$

$$B_3 = \text{Res}_{s=\bar{s}_1} [e^{st} F(s)] = \frac{e^{\bar{s}_1 t}(2\bar{s}_1 - 2)}{(\bar{s}_1 + 1)(\bar{s}_1 - s_1)} = \overline{\left[\frac{e^{s_1 t}(2s_1 - 2)}{(s_1 + 1)(s_1 - \bar{s}_1)} \right]} = \bar{B}_2 = \left(\frac{1}{2} + \frac{i}{2} \right) e^{-t} e^{-i2t}.$$

It is easy to see that

$$\begin{aligned} B_1 + B_2 + B_3 &= -e^{-t} + \left(\frac{1}{2} - \frac{i}{2}\right)e^{-t}e^{i2t} + \left(\frac{1}{2} + \frac{i}{2}\right)e^{-t}e^{-i2t} \\ &= -e^{-t} + e^{-t} \left(\frac{e^{i2t} - e^{-i2t}}{2i} + \frac{e^{i2t} + e^{-i2t}}{2} \right) = e^{-t}(\sin 2t + \cos 2t - 1). \end{aligned}$$

Now let s be any point on the semicircle shown below, where $\gamma > 0$ and $R > \sqrt{5} + \gamma$.



Since

$$|s| = |\gamma + Re^{i\theta}| \leq \gamma + R = R + \gamma \quad \text{and} \quad |s| = |\gamma + Re^{i\theta}| \geq |\gamma - R| = R - \gamma > \sqrt{5},$$

we find that

$$|2s - 2| \leq 2|s| + 2 \leq 2(R + \gamma) + 2,$$

$$|s + 1| \geq ||s| - 1| \geq (R - \gamma) - 1 > 0,$$

and

$$|s^2 + 2s + 5| = |s - s_1||s - \bar{s}_1| \geq (|s| - |s_1|)^2 \geq [(R - \gamma)^2 - \sqrt{5}]^2 > 0.$$

Thus

$$|F(s)| = \frac{|2s - 2|}{|s + 1||s^2 + 2s + 5|} \leq \frac{2(R + \gamma) + 2}{[(R - \gamma) - 1][(R - \gamma)^2 - \sqrt{5}]^2} \rightarrow 0 \text{ as } R \rightarrow \infty,$$

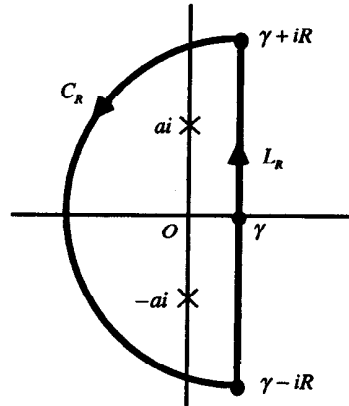
and we may conclude that

$$f(t) = e^{-t}(\sin 2t + \cos 2t - 1).$$

4. The function

$$F(s) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \quad (a > 0)$$

has singularities at $s = \pm ai$. So we consider the simple closed contour shown below, where $\gamma > 0$ and $R > a + \gamma$.



Upon writing

$$F(s) = \frac{\phi(s)}{(s - ai)^2} \quad \text{where} \quad \phi(s) = \frac{s^2 - a^2}{(s + ai)^2},$$

we see that $\phi(s)$ is analytic and nonzero at $s_0 = ai$. Hence s_0 is a pole of order $m = 2$ of $F(s)$. Furthermore, $\overline{F(\bar{s})} = F(s)$ at points where $F(s)$ is analytic. Consequently, \bar{s}_0 is also a pole of order 2 of $F(s)$; and we know from expression (2), Sec. 82, that

$$\operatorname{Res}_{s=s_0} [e^{st} F(s)] + \operatorname{Res}_{s=\bar{s}_0} [e^{st} F(s)] = 2 \operatorname{Re} [e^{iat} (b_1 + b_2 t)],$$

where b_1 and b_2 are the coefficients in the principal part

$$\frac{b_1}{s - ai} + \frac{b_2}{(s - ai)^2}$$

of $F(s)$ at ai . These coefficients are readily found with the aid of the first two terms in the Taylor series for $\phi(s)$ about $s_0 = ai$:

$$F(s) = \frac{1}{(s - ai)^2} \phi(s) = \frac{1}{(s - ai)^2} \left[\phi(ai) + \frac{\phi'(ai)}{1!} (s - ai) + \dots \right]$$

$$= \frac{\phi(ai)}{(s-ai)^2} + \frac{\phi'(ai)}{s-ai} + \dots \quad (0 < |s-ai| < 2a).$$

It is straightforward to show that $\phi(ai) = 1/2$ and $\phi'(ai) = 0$, and we find that $b_1 = 0$ and $b_2 = 1/2$. Hence

$$\operatorname{Res}_{s=s_0} [e^{st} F(s)] + \operatorname{Res}_{s=\bar{s}_0} [e^{st} F(s)] = 2 \operatorname{Re} \left[e^{iat} \left(\frac{1}{2} t \right) \right] = t \cos at.$$

We can, then, conclude that

$$f(t) = t \cos at \quad (a > 0),$$

provided that $F(s)$ satisfies the desired boundedness condition. As for that condition, when z is a point on C_R ,

$$|z| = |\gamma + Re^{i\theta}| \leq \gamma + R = R + \gamma \quad \text{and} \quad |z| = |\gamma + Re^{i\theta}| \geq |\gamma - R| = R - \gamma > a;$$

and this means that

$$|z^2 - a^2| \leq |z|^2 + a^2 \leq (R + \gamma)^2 + a^2 \quad \text{and} \quad |z^2 + a^2| \geq ||z|^2 - a^2| \geq (R - \gamma)^2 - a^2 > 0.$$

Hence

$$|F(z)| \leq \frac{(R + \gamma)^2 + a^2}{[(R - \gamma)^2 - a^2]^2} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.$$

6. We are given

$$F(s) = \frac{\sinh(xs)}{s^2 \cosh s} \quad (0 < x < 1),$$

which has isolated singularities at the points

$$s_0 = 0, \quad s_n = \frac{(2n-1)\pi}{2} i, \quad \text{and} \quad \bar{s}_n = -\frac{(2n-1)\pi}{2} i \quad (n = 1, 2, \dots).$$

This function has the property $\overline{F(s)} = F(\bar{s})$, and so

$$f(t) = \operatorname{Res}_{s=s_0} [e^{st} F(s)] + \sum_{n=1}^{\infty} \left\{ \operatorname{Res}_{s=s_n} [e^{st} F(s)] + \operatorname{Res}_{s=\bar{s}_n} [e^{st} F(s)] \right\}.$$

To find the residue at $s_0 = 0$, we write

$$\frac{\sinh(xs)}{s^2 \cosh s} = \frac{xs + (xs)^3 / 3! + \dots}{s^2 (1 + s^2 / 2! + \dots)} = \frac{x + x^3 s^2 / 6 + \dots}{s + s^3 / 2 + \dots} \quad \left(0 < |s| < \frac{\pi}{2} \right).$$

Division of series then reveals that s_0 is a simple pole of $F(s)$, with residue x ; and, according to expression (3), Sec. 82,

$$\operatorname{Res}_{s=s_0} [e^{st} F(s)] = \operatorname{Res}_{s=s_0} F(s) = x.$$

As for the residues of $F(s)$ at the singular points s_n ($n = 1, 2, \dots$), we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \sinh(xs) \quad \text{and} \quad q(s) = s^2 \cosh s.$$

We note that

$$p(s_n) = i \sin \frac{(2n-1)\pi x}{2} \neq 0 \quad \text{and} \quad q(s_n) = 0;$$

furthermore, since

$$q'(s) = 2s \cosh s + s^2 \sinh s,$$

we find that

$$\begin{aligned} q'(s_n) &= -\frac{(2n-1)^2 \pi^2}{4} i \sin \frac{(2n-1)\pi}{2} = -i \frac{(2n-1)^2 \pi^2}{4} \sin \left(n\pi - \frac{\pi}{2} \right) \\ &= -i \frac{(2n-1)^2 \pi^2}{4} \left(\sin n\pi \cos \frac{\pi}{2} - \cos n\pi \sin \frac{\pi}{2} \right) = \frac{(2n-1)^2 \pi^2}{4} (-1)^n i \neq 0. \end{aligned}$$

In view of Theorem 2 in Sec. 69, then, s_n is a simple pole of $F(s)$, and

$$\operatorname{Res}_{s=s_n} F(s) = \frac{p(s_n)}{q'(s_n)} = \frac{4}{\pi^2} \cdot \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2}.$$

Expression (4), Sec. 82, now gives us

$$\begin{aligned} \operatorname{Res}_{s=s_n} [e^{st} F(s)] + \operatorname{Res}_{s=\bar{s}_n} [e^{st} F(s)] &= 2 \operatorname{Re} \left\{ \frac{4}{\pi^2} \cdot \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2} \exp \left[i \frac{(2n-1)\pi t}{2} \right] \right\} \\ &= \frac{8}{\pi^2} \cdot \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2} \cos \frac{(2n-1)\pi t}{2}. \end{aligned}$$

Summing all of the above residues, we arrive at the final result:

$$f(t) = x + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2} \cos \frac{(2n-1)\pi t}{2}.$$

7. The function

$$F(s) = \frac{1}{s \cosh(s^{1/2})},$$

where it is agreed that the branch cut of $s^{1/2}$ does not lie along the negative real axis, has isolated singularities at $s_0 = 0$ and when $\cosh(s^{1/2}) = 0$, or at the points $s_n = -\frac{(2n-1)^2 \pi^2}{4}$

($n = 1, 2, \dots$). The point s_0 is a simple pole of $F(s)$, as is seen by writing

$$\frac{1}{s \cosh(s^{1/2})} = \frac{1}{s [1 + (s^{1/2})^2 / 2! + (s^{1/2})^4 / 4! + \dots]} = \frac{1}{s + s^2 / 2 + s^3 / 24 + \dots}$$

and dividing this last denominator into 1. In fact, the residue is found to be 1; and expression (3), Sec. 82, tells us that

$$\operatorname{Res}_{s=s_0} [e^{st} F(s)] = \operatorname{Res}_{s=s_0} F(s) = 1.$$

As for the other singularities, we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = 1 \quad \text{and} \quad q(s) = s \cosh(s^{1/2}).$$

Now

$$p(s_n) = 1 \neq 0 \quad \text{and} \quad q(s_n) = 0;$$

also, since

$$q'(s) = \frac{1}{2} s^{1/2} \sinh(s^{1/2}) + \cosh(s^{1/2}),$$

it is straightforward to show that

$$q'(s_n) = -\frac{(2n-1)\pi}{4} \sin\left(n\pi - \frac{\pi}{2}\right) = \frac{(2n-1)\pi}{4} (-1)^n \neq 0.$$

So each point s_n is a simple pole of $F(s)$, and

$$\operatorname{Res}_{s=s_n} F(s) = \frac{p(s_n)}{q'(s_n)} = \frac{4}{\pi} \cdot \frac{(-1)^n}{2n-1}.$$

Consequently, according to expression (3), Sec. 82,

$$\operatorname{Res}_{s=s_n} [e^{st} F(s)] = e^{s_n t} \operatorname{Res}_{s=s_n} F(s) = \frac{4}{\pi} \cdot \frac{(-1)^n}{2n-1} \exp\left[-\frac{(2n-1)^2 \pi^2 t}{4}\right] \quad (n=1,2,\dots).$$

Finally, then,

$$f(t) = \operatorname{Res}_{s=s_0} [e^{st} F(s)] + \sum_{n=1}^{\infty} \operatorname{Res}_{s=s_n} [e^{st} F(s)],$$

or

$$f(t) = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \exp\left[-\frac{(2n-1)^2 \pi^2 t}{4}\right].$$

8. Here we are given the function

$$F(s) = \frac{\coth(\pi s / 2)}{s^2 + 1} = \frac{\cosh(\pi s / 2)}{(s^2 + 1) \sinh(\pi s / 2)},$$

which has the property $\overline{F(s)} = F(\bar{s})$. We consider first the singularities at $s = \pm i$. Upon writing

$$F(s) = \frac{\phi(s)}{s-i} \quad \text{where} \quad \phi(s) = \frac{\cosh(\pi s / 2)}{(s+i) \sinh(\pi s / 2)},$$

we find that, since $\phi(i) = 0$, the point i is a removable singularity of $F(s)$ [see Exercise 3(b), Sec. 65]; and the same is true of the point $-i$. At each of these points, it follows that the residue of $e^{st} F(s)$ is 0. The other singularities occur when $\pi s / 2 = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$), or at the points $s = 2ni$ ($n = 0, \pm 1, \pm 2, \dots$). To find the residues, we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \cosh\left(\frac{\pi s}{2}\right) \quad \text{and} \quad q(s) = (s^2 + 1) \sinh\left(\frac{\pi s}{2}\right)$$

and note that

$$p(2ni) = \cosh(n\pi i) = \cos(n\pi) = (-1)^n \neq 0 \quad \text{and} \quad q(2ni) = 0.$$

Furthermore, since

$$q'(s) = (s^2 + 1) \frac{\pi}{2} \cosh\left(\frac{\pi s}{2}\right) + 2s \sinh\left(\frac{\pi s}{2}\right),$$

we have

$$q'(2ni) = (-4n^2 + 1) \frac{\pi}{2} \cosh(n\pi i) = (-4n^2 + 1) \frac{\pi}{2} \cos(n\pi) = -\frac{\pi(4n^2 - 1)}{2} (-1)^n \neq 0.$$

Thus

$$\operatorname{Res}_{s=2ni} F(s) = \frac{p(2ni)}{q'(2ni)} = -\frac{2}{\pi} \cdot \frac{1}{4n^2 - 1} \quad (n = 0, \pm 1, \pm 2, \dots).$$

Expressions (3) and (4) in Sec. 82 now tell us that

$$\operatorname{Res}_{s=0} [e^{st} F(s)] = \operatorname{Res}_{s=0} F(s) = \frac{2}{\pi}$$

and

$$\operatorname{Res}_{s=2ni} [e^{st} F(s)] + \operatorname{Res}_{s=-2ni} [e^{st} F(s)] = 2 \operatorname{Re} \left[e^{i2nt} \left(-\frac{2}{\pi} \cdot \frac{1}{4n^2 - 1} \right) \right] = -\frac{4}{\pi} \cdot \frac{\cos 2nt}{4n^2 - 1} \quad (n = 1, 2, \dots).$$

The desired function of t is, then,

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}.$$

9. The function

$$F(s) = \frac{\sinh(xs^{1/2})}{s^2 \sinh(s^{1/2})} \quad (0 < x < 1),$$

where it is agreed that the branch cut of $s^{1/2}$ does not lie along the negative real axis, has isolated singularities at $s=0$ and when $\sinh(s^{1/2})=0$, or at the points $s = -n^2\pi^2$ ($n = 1, 2, \dots$). The point $s=0$ is a pole of order 2 of $F(s)$, as is seen by first writing

$$\frac{\sinh(xs^{1/2})}{s^2 \sinh(s^{1/2})} = \frac{xs^{1/2} + (xs^{1/2})^3 / 3! + (xs^{1/2})^5 / 5! + \dots}{s^2 [s^{1/2} + (s^{1/2})^3 / 3! + (s^{1/2})^5 / 5! + \dots]} = \frac{x + x^3 s / 6 + x^5 s^2 / 120 + \dots}{s^2 + s^3 / 6 + s^4 / 120 + \dots}$$

and dividing the series in the denominator into the series in the numerator. The result is

$$\frac{\sinh(xs^{1/2})}{s^2 \sinh(s^{1/2})} = x \frac{1}{s^2} + \frac{1}{6}(x^3 - x) \frac{1}{s} + \dots \quad (0 < |s| < \pi^2).$$

In view of expression (1), Sec. 82, then,

$$\operatorname{Res}_{s=0} [e^{st} F(s)] = \frac{1}{6}(x^3 - x) + xt = \frac{1}{6}x(x^2 - 1) + xt.$$

As for the singularities $s = -n^2\pi^2$ ($n = 1, 2, \dots$), we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \sinh(xs^{1/2}) \quad \text{and} \quad q(s) = s^2 \sinh(s^{1/2}).$$

Observe that $p(-n^2\pi^2) \neq 0$ and $q(-n^2\pi^2) = 0$. Also, since

$$q'(s) = 2s \sinh(s^{1/2}) + \frac{1}{2} s s^{1/2} \cosh(s^{1/2}),$$

it is easy to see that $q'(-n^2\pi^2) \neq 0$. So the points $s = -n^2\pi^2$ ($n = 1, 2, \dots$), are simple poles of $F(s)$, and

$$\operatorname{Res}_{s=-n^2\pi^2} F(s) = \left. \frac{p(s)}{q'(s)} \right|_{s=-n^2\pi^2} = \left. \frac{2 \sinh(xs^{1/2})}{s s^{1/2} \cosh(s^{1/2})} \right|_{s=-n^2\pi^2} = \frac{2}{\pi^3} \cdot \frac{(-1)^{n+1}}{n^3} \sin n\pi x \quad (n = 1, 2, \dots).$$

Thus, in view of expression (3), Sec. 82,

$$\operatorname{Res}_{s=-n^2\pi^2} [e^{st} F(s)] = \frac{2}{\pi^3} \cdot \frac{(-1)^{n+1}}{n^3} e^{-n^2\pi^2 t} \sin n\pi x \quad (n = 1, 2, \dots).$$

Finally, since

$$f(t) = \operatorname{Res}_{s=0} [e^{st} F(s)] + \sum_{n=1}^{\infty} \operatorname{Res}_{s=-n^2\pi^2} [e^{st} F(s)],$$

we arrive at the expression

$$f(t) = \frac{1}{6}x(x^2 - 1) + xt + \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} e^{-n^2\pi^2 t} \sin n\pi x.$$

10. The function

$$F(s) = \frac{1}{s^2} - \frac{1}{s \sinh s}$$

has isolated singularities at the points

$$s_0 = 0 \quad \text{and} \quad s_n = n\pi i, \quad \bar{s}_n = -n\pi i \quad (n = 1, 2, \dots).$$

Now

$$s \sinh s = s \left(s + \frac{1}{6} s^3 + \dots \right) = s^2 + \frac{1}{6} s^4 + \dots \quad (0 < |s| < \infty),$$

and division of this series into 1 reveals that

$$F(s) = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{6} + \dots \right) = -\frac{1}{6} + \dots \quad (0 < |s| < \pi).$$

This shows that $F(s)$ has a removable singularity at s_0 . Evidently, then, $e^{st}F(s)$ must also have a removable singularity there; and so

$$\operatorname{Res}_{s=s_0} [e^{st}F(s)] = 0.$$

To find the residue of $F(s)$ at $s_n = n\pi i$ ($n = 1, 2, \dots$), we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \sinh s - s \quad \text{and} \quad q(s) = s^2 \sinh s$$

and observe that

$$p(n\pi i) = -n\pi i \neq 0, \quad q(n\pi i) = 0, \quad \text{and} \quad q'(n\pi i) = n^2 \pi^2 (-1)^{n+1} \neq 0.$$

Consequently, $F(s)$ has a simple pole at s_n , and

$$\operatorname{Res}_{s=s_n} F(s) = \frac{p(n\pi i)}{q'(n\pi i)} = \frac{-n\pi i}{n^2 \pi^2 (-1)^{n+1}} = \frac{(-1)^n}{n\pi} i \quad (n = 1, 2, \dots).$$

Since $\overline{F(s)} = F(\bar{s})$, the points \bar{s}_n are also simple poles of $F(s)$; and we may write

$$\begin{aligned} \operatorname{Res}_{s=s_n} [e^{st}F(s)] + \operatorname{Res}_{s=\bar{s}_n} [e^{st}F(s)] &= 2 \operatorname{Re} \left[\frac{(-1)^n}{n\pi} i e^{in\pi t} \right] = 2 \operatorname{Re} \left[\frac{(-1)^n}{n\pi} (i \cos n\pi t - \sin n\pi t) \right] \\ &= 2 \frac{(-1)^{n+1}}{n\pi} \sin n\pi t. \end{aligned}$$

Hence the desired result is

$$f(t) = \operatorname{Res}_{s=s_0} [e^{st}F(s)] + \sum_{n=1}^{\infty} \left\{ \operatorname{Res}_{s=s_n} [e^{st}F(s)] + \operatorname{Res}_{s=\bar{s}_n} [e^{st}F(s)] \right\},$$

or

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t.$$

11. We consider here the function

$$F(s) = \frac{\sinh(xs)}{s(s^2 + \omega^2) \cosh s} \quad (0 < x < 1),$$

where $\omega > 0$ and $\omega \neq \omega_n = \frac{(2n-1)\pi}{2}$ ($n = 1, 2, \dots$). The singularities of $F(s)$ are at

$$s = 0, \quad s = \pm \omega i, \quad \text{and} \quad s = \pm \omega_n i \quad (n = 1, 2, \dots).$$

Because the first term in the Maclaurin series for $\sinh(xs)$ is xs , it is easy to see that $s = 0$ is a removable singularity of $e^s F(s)$ and that

$$\operatorname{Res}_{s=s_0} [e^s F(s)] = 0.$$

To find the residue of $F(s)$ at $s = \omega i$, we write

$$F(s) = \frac{\phi(s)}{s - \omega i} \quad \text{where} \quad \phi(s) = \frac{\sinh(xs)}{s(s + \omega i) \cosh s},$$

from which it follows that $s = \omega i$ is simple pole and

$$\operatorname{Res}_{s=\omega i} F(s) = \phi(\omega i) = \frac{\sinh(x\omega i)}{\omega i 2\omega i \cosh(\omega i)} = \frac{i \sin \omega x}{-2\omega^2 \cos \omega}.$$

Since $\overline{F(s)} = F(\bar{s})$, then,

$$\operatorname{Res}_{s=\omega i} [e^s F(s)] + \operatorname{Res}_{s=-\omega i} [e^s F(s)] = 2 \operatorname{Re} \left[\frac{i \sin \omega x}{-2\omega^2 \cos \omega} i e^{i\omega x} \right] = 2 \frac{\sin \omega x}{2\omega^2 \cos \omega} \sin \omega x = \frac{\sin \omega x \sin \omega x}{\omega^2 \cos \omega}.$$

As for the residues at $s = \omega_n i$ ($n = 1, 2, \dots$), we put $F(s)$ in the form

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \sinh(xs) \quad \text{and} \quad q(s) = (s^3 + \omega^2 s) \cosh s.$$

Now $p(\omega_n i) = \sinh(x\omega_n i) = i \sin \omega_n x \neq 0$ and $q(\omega_n i) = 0$. Also, since

$$q'(s) = (s^3 + \omega^2 s) \sinh s + (3s^2 + \omega^2) \cosh s,$$

we find that

$$q'(\omega_n i) = (-\omega_n^3 i + \omega^2 \omega_n i) \sinh(\omega_n i) = -\omega_n (\omega^2 - \omega_n^2) \sin \omega_n \neq 0.$$

Hence we have a simple pole at $s = \omega_n i$, with residue

$$\operatorname{Res}_{s=\omega_n i} F(s) = \frac{p(\omega_n i)}{q'(\omega_n i)} = \frac{i \sin \omega_n x}{-\omega_n (\omega^2 - \omega_n^2) \sin \omega_n}.$$

Consequently,

$$\operatorname{Res}_{s=\omega_n i} [e^{st} F(s)] + \operatorname{Res}_{s=-\omega_n i} [e^{st} F(s)] = 2 \operatorname{Re} \left[\frac{i \sin \omega_n x}{-\omega_n (\omega^2 - \omega_n^2) \sin \omega_n} e^{i\omega_n t} \right] = 2 \frac{\sin \omega_n x \sin \omega_n t}{\omega_n (\omega^2 - \omega_n^2) \sin \omega_n}.$$

But $\sin \omega_n = \sin \left(n\pi - \frac{\pi}{2} \right) = (-1)^{n+1}$, and this means that

$$\operatorname{Res}_{s=\omega_n i} [e^{st} F(s)] + \operatorname{Res}_{s=-\omega_n i} [e^{st} F(s)] = 2 \frac{(-1)^{n+1}}{\omega_n} \cdot \frac{\sin \omega_n x \sin \omega_n t}{\omega^2 - \omega_n^2}.$$

Finally,

$$f(t) = \operatorname{Res}_{s=0} [e^{st} F(s)] + \left\{ \operatorname{Res}_{s=\omega i} [e^{st} F(s)] + \operatorname{Res}_{s=-\omega i} [e^{st} F(s)] \right\} + \sum_{n=1}^{\infty} \left\{ \operatorname{Res}_{s=\omega_n i} [e^{st} F(s)] + \operatorname{Res}_{s=-\omega_n i} [e^{st} F(s)] \right\}.$$

That is,

$$f(t) = \frac{\sin \omega x \sin \omega t}{\omega^2 \cos \omega} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\omega_n} \cdot \frac{\sin \omega_n x \sin \omega_n t}{\omega^2 - \omega_n^2}.$$