Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

Linear equations $Ax = b$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $du/dt = Au$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra, based on $Ax = \lambda x$. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \dots$ of a matrix. Suppose you need the hundredth power A^{100}. The starting matrix A becomes unrecognizable after a few steps, and A^{100} is very close to $[.6\ \ .6;\ .4\ \ .4]$:

$$\underset{A}{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix}} \quad \cdots \quad \underset{A^{100}}{\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}}$$

A^{100} was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues (here they are 1 and $1/2$) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.

This section will explain how to compute the x's and λ's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use $\det(A - \lambda I) = 0$ to find the eigenvalues for this first example, and then derive it properly in equation (3).

Example 1 The matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda} = \mathbf{1}$ and $\boldsymbol{\lambda} = \frac{1}{2}$. For those numbers, the matrix $A - \lambda I$ becomes *singular* (zero determinant). The eigenvectors x_1 and x_2 are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)x_1 = 0$ is $Ax_1 = x_1$ and the first eigenvector is $(.6, .4)$.

$(A - \frac{1}{2}I)x_2 = 0$ is $Ax_2 = \frac{1}{2}x_2$ and the second eigenvector is $(1, -1)$:

$$x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad Ax_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = x_1 \quad (Ax = x \text{ means that } \lambda_1 = 1)$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}x_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If x_1 is multiplied again by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2}x_2$, and if we multiply again we get $(\frac{1}{2})^2$ times x_2.

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same x_1 and x_2. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100}$ = very small number.

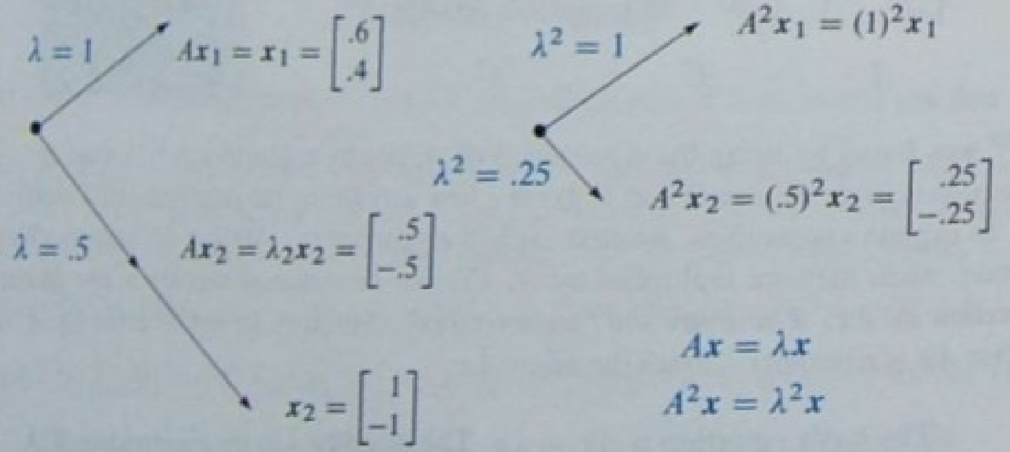

Figure 6.1: The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + (.2)x_2$:

Separate into eigenvectors
$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + (.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \tag{1}$$

Multiplying by A gives $(.7, .3)$, the first column of A^2. Do it separately for x_1 and $(.2)x_2$. Of course $Ax_1 = x_1$. And A multiplies x_2 by its eigenvalue $\frac{1}{2}$:

Multiply each x_i by λ_i $\quad A\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ is $x_1 + \frac{1}{2}(.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}$.

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need these eigenvectors to find A^2. But it is the good way to do 99 multiplications. At every step x_1 is unchanged and x_2 is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad x_1 + (.2)(\frac{1}{2})^{99}x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector x_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector x_2 is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A, the closer its columns approach the steady state.

We mention that this particular A is a ***Markov matrix***. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ (as we found). Its eigenvector $x_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 8.3 shows how Markov matrices appear in applications like Google.

For projections we can spot the steady state ($\lambda = 1$) and the nullspace ($\lambda = 0$).

Example 2 **The projection matrix** $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

Its eigenvectors are $x_1 = (1, 1)$ and $x_2 = (1, -1)$. For those vectors, $Px_1 = x_1$ (steady state) and $Px_2 = \mathbf{0}$ (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special λ's and x's:

1. Each column of $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is **singular**, so $\lambda = 0$ is an eigenvalue.
3. P is **symmetric**, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $Px = 0x$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $Px = x$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

Project each part $\quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **projects onto** $Pv = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Special properties of a matrix lead to special eigenvalues and eigenvectors. That is a major theme of this chapter (it is captured in a table at the very end).

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix R (a reflection and at the same time a permutation) is also special.

Example 3 **The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.**

The eigenvector $(1, 1)$ is unchanged by R. The second eigenvector is $(1, -1)$—its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because *reflection* $= 2($*projection*$) - I$:

$$R = 2P - I \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (2)$$

Here is the point. If $Px = \lambda x$ then $2Px = 2\lambda x$. The eigenvalues are doubled when the matrix is doubled. Now subtract $Ix = x$. The result is $(2P - I)x = (2\lambda - 1)x$. ***When a matrix is shifted by I, each λ is shifted by*** 1. No change in eigenvectors.

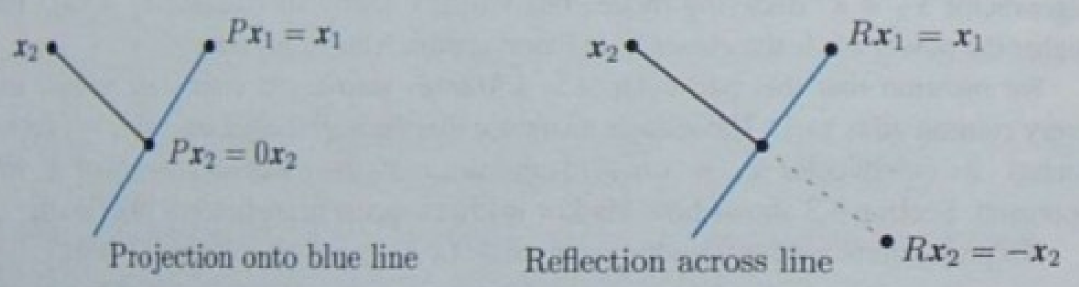

Figure 6.2: Projections P have eigenvalues 1 and 0. Reflections R have $\lambda = 1$ and -1. A typical x changes direction, but not the eigenvectors x_1 and x_2.

Key idea: The eigenvalues of R and P are related exactly as the matrices are related:

The eigenvalues of $R = 2P - I$ are $2(1) - 1 = 1$ and $2(0) - 1 = -1$.

The eigenvalues of R^2 are λ^2. In this case $R^2 = I$. Check $(1)^2 = 1$ and $(-1)^2 = 1$.

The Equation for the Eigenvalues

For projections and reflections we found λ's and x's by geometry: $Px = x$, $Px = \mathbf{0}$, $Rx = -x$. Now we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving $Ax = \lambda x$.

First move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. ***The eigenvectors make up the nullspace of*** $A - \lambda I$. When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)x = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)x = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $A - \lambda I$ must be zero.*** This is how to recognize an eigenvalue λ:

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

Equation for the eigenvalues $\det(A - \lambda I) = 0.$ (3)

This "*characteristic polynomial*" $\det(A - \lambda I)$ involves only λ, not $\boldsymbol{x}$. When A is n by n, equation (3) has degree n. Then A has n eigenvalues (repeats possible!) Each λ leads to $\boldsymbol{x}$:

For each eigenvalue λ solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ or $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find an eigenvector $\boldsymbol{x}$.

Example 4 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is already singular (zero determinant). Find its λ's and $\boldsymbol{x}$'s.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\boldsymbol{x} = 0\boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A - \lambda I) = 0$ is the way to find *all* λ's and $\boldsymbol{x}$'s. Always subtract λI from A:

$$\textit{Subtract } \lambda \textit{ from the diagonal to find} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}. \qquad (4)$$

Take the determinant "$ad - bc$" of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the "ad" part is $\lambda^2 - 5\lambda + 4$. The "bc" part, not containing λ, is 2 times 2.

$$\det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda. \qquad (5)$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0$ yields the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$.

Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ is $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

We need to emphasize: *There is nothing exceptional about $\lambda = 0$.* Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*.

In the example, the shifted matrix $A - 5I$ is singular and 5 is the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.

2. ***Find the roots of this polynomial,*** by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.

3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

A note on the eigenvectors of 2 by 2 matrices. When $A - \lambda I$ is singular, both rows are multiples of a vector (a, b). *The eigenvector is any multiple of* $(b, -a)$. The example had $\lambda = 0$ and $\lambda = 5$:

$$\lambda = 0 : \text{rows of } A - 0I \text{ in the direction } (1, 2); \text{ eigenvector in the direction } (2, -1)$$
$$\lambda = 5 : \text{rows of } A - 5I \text{ in the direction } (-4, 2); \text{ eigenvector in the direction } (2, 4).$$

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB's **eig**(A) divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Similarly some n by n matrices don't have n independent eigenvectors. Without n eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Good News, Bad News

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix.* For this A, the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 7$. That agrees with the sum down the main diagonal (the **trace** is $1 + 6$). These quick checks always work:

The product of the n eigenvalues equals the determinant.
The sum of the n eigenvalues equals the sum of the n diagonal entries.

The sum of the entries on the main diagonal is called the ***trace*** of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textbf{\textit{trace}} = a_{11} + a_{22} + \cdots + a_{nn}. \tag{6}$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ's. But when the computation is wrong, they generally tell us so. To compute the correct λ's, go back to $\det(A - \lambda I) = 0$.

The determinant test makes the *product* of the λ's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the λ's is not the sum of the pivots—as the example showed. The individual λ's have almost nothing to do with the pivots. In this new part of linear algebra, the key equation is really *nonlinear*: λ multiplies $\boldsymbol{x}$.

Why do the eigenvalues of a triangular matrix lie on its diagonal?

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 5 *The* 90° *rotation* $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *has no real eigenvectors. Its eigenvalues are* $\lambda = i$ *and* $\lambda = -i$. *Sum of* λ*'s* = trace = 0. *Product* = *determinant* = 1.

After a rotation, *no vector* Qx *stays in the same direction as* $\boldsymbol{x}$ (except $\boldsymbol{x} = \mathbf{0}$ which is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

To see how i can help, look at Q^2 which is $-I$. If Q is rotation through 90°, then Q^2 is rotation through 180°. Its eigenvalues are -1 and -1. (Certainly $-I\boldsymbol{x} = -1\boldsymbol{x}$.) Squaring Q will square each λ, so we must have $\lambda^2 = -1$. *The eigenvalues of the* 90° *rotation matrix* Q *are* $+i$ *and* $-i$, because $i^2 = -1$.

Those λ's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are i and $-i$. We meet the imaginary number i also in the eigenvectors:

Complex eigenvectors
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Somehow these complex vectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and $-i$ also illustrate two special properties of Q:

1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.

2. Q is a skew-symmetric matrix so each λ is pure imaginary.

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A+4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ which is $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$\boldsymbol{\lambda = 1}: \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3}: \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A+4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda+4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A+4I \text{ has } \begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$$

The trace of A^2 is $5+5$ which agrees with $1+9$. The determinant is $25-16=9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1+2+1=4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $\boldsymbol{x} = (1,1,1)$ gives $A\boldsymbol{x} = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A. The other factors $(1-\lambda)$ and $(3-\lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue $0,1,3$ corresponds to an eigenvector:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad A\boldsymbol{x}_1 = \mathbf{0}\boldsymbol{x}_1 \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad A\boldsymbol{x}_2 = \mathbf{1}\boldsymbol{x}_2 \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad A\boldsymbol{x}_3 = \mathbf{3}\boldsymbol{x}_3 .$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A-\lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0,1,3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] =$ **eig**(A) will produce unit eigenvectors in the columns of the **eigenvector matrix** S. The first one happens to have three minus signs, reversed from $(1,1,1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the ***eigenvalue matrix*** (typed as D but soon called Λ).

Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

6.2 Diagonalizing a Matrix

When x is an eigenvector, multiplication by A is just multiplication by a number λ: $Ax = \lambda x$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.*** This is the matrix form of our key idea. We start right off with that one essential computation.

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an ***eigenvector matrix*** S. Then $S^{-1}AS$ is the ***eigenvalue matrix*** Λ:

Eigenvector matrix S
Eigenvalue matrix Λ

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \qquad (1)$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because of the small λ's (the eigenvalues) on its diagonal.

Proof Multiply A times its eigenvectors, which are the columns of S. The first column of AS is Ax_1. That is $\lambda_1 x_1$. Each column of S is multiplied by its eigenvalue λ_i:

A times S
$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}.$$

The trick is to split this matrix AS into S times Λ:

S times Λ
$$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column x_1, as shown. The diagonalization is complete, and we can write $AS = S\Lambda$ in two good ways:

$$AS = S\Lambda \quad \text{is} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}. \qquad (2)$$

The matrix S has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors $x_1, \ldots, x_n$ was to diagonalize A. Those eigenvectors in S produce $A = S\Lambda S^{-1}$. You will soon see the simplicity and importance and meaning of the nth power $A^n = S\Lambda^n S^{-1}$.

Example 1 This A is triangular so the λ's are on the diagonal: $\lambda = 1$ and $\lambda = 6$.

$$\textbf{Eigenvectors} \quad \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \quad \underset{S^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = S\Lambda S^{-1}$. Then watch $A^2 = S\Lambda S^{-1}S\Lambda S^{-1}$. When you remove $S^{-1}S = I$, this becomes $S\Lambda^2 S^{-1}$. ***Same eigenvectors in S and squared eigenvalues in Λ^2.***

The kth power will be $A^k = S\Lambda^k S^{-1}$ which is easy to compute:

$$\textbf{Powers of } A \qquad \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 6^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}.$$

With $k = 1$ we get A. With $k = 0$ we get $A^0 = I$ (and $\lambda^0 = 1$). *With $k = -1$ we get A^{-1}.* You can see how $A^2 = [1 \ \ 35; \ 0 \ \ 36]$ fits that formula when $k = 2$.

Here are four small remarks before we use Λ again.

Remark 1 Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $x_1, \ldots, x_n$ are independent. ***Any matrix that has no repeated eigenvalues can be diagonalized.***

Remark 2 ***We can multiply eigenvectors by any nonzero constants.*** $Ax = \lambda x$ will remain true. In Example 1, we can divide the eigenvector $(1, 1)$ by $\sqrt{2}$ to produce a unit vector.

Remark 3 The eigenvectors in S come in the same order as the eigenvalues in Λ. To reverse the order in Λ, put $(1, 1)$ before $(1, 0)$ in S:

$$\textbf{New order } 6, 1 \qquad \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\mathbf{new}}$$

To diagonalize A we *must* use an eigenvector matrix. From $S^{-1}AS = \Lambda$ we know that $AS = S\Lambda$. Suppose the first column of S is x. Then the first columns of AS and $S\Lambda$ are Ax and $\lambda_1 x$. For those to be equal, x must be an eigenvector.

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices cannot be diagonalized.* Here are two examples:

$$\textbf{Not diagonalizable} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. Nothing is special about $\lambda = 0$, it is the repetition of λ that counts. All eigenvectors of the first matrix are multiples of $(1, 1)$:

$$\begin{matrix}\textbf{Only one line} \\ \textbf{of eigenvectors}\end{matrix} \qquad Ax = 0x \quad \text{means} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There is no second eigenvector, so the unusual matrix A cannot be diagonalized.

Those matrices are the best examples to test any statement about eigenvectors. In many true-false questions, non-diagonalizable matrices lead to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- ***Invertibility*** is concerned with the ***eigenvalues*** ($\lambda = 0$ or $\lambda \neq 0$).
- ***Diagonalizability*** is concerned with the ***eigenvectors*** (too few or enough for S).

Each eigenvalue has at least one eigenvector! $A - \lambda I$ is singular. If $(A - \lambda I)x = \mathbf{0}$ leads you to $x = \mathbf{0}$, λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$.

Eigenvectors for n different λ's are independent. Then we can diagonalize A.

Independent x from different λ Eigenvectors $x_1, \ldots, x_j$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof Suppose $c_1 x_1 + c_2 x_2 = \mathbf{0}$. Multiply by A to find $c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = \mathbf{0}$. Multiply by λ_2 to find $c_1 \lambda_2 x_1 + c_2 \lambda_2 x_2 = \mathbf{0}$. Now subtract one from the other:

$$\text{Subtraction leaves} \quad (\lambda_1 - \lambda_2) c_1 x_1 = \mathbf{0}. \quad \text{Therefore } c_1 = 0.$$

Since the λ's are different and $x_1 \neq \mathbf{0}$, we are forced to this conclusion that $c_1 = 0$. Similarly $c_2 = 0$. No other combination gives $c_1 x_1 + c_2 x_2 = \mathbf{0}$, so the eigenvectors x_1 and x_2 must be independent.

This proof extends directly to j eigenvectors. Suppose $c_1 x_1 + \cdots + c_j x_j = \mathbf{0}$. Multiply by A, multiply by λ_j, and subtract. This removes x_j. Now multiply by A and by λ_{j-1} and subtract. This removes x_{j-1}. Eventually only x_1 is left:

$$(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_j) c_1 x_1 = \mathbf{0} \quad \text{which forces} \quad c_1 = 0. \tag{3}$$

Similarly every $c_i = 0$. When the λ's are all different, the eigenvectors are independent. A full set of eigenvectors can go into the columns of the eigenvector matrix S.

Example 2 **Powers of A** The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = S\Lambda S^{-1}$ with those eigenvalues in the diagonal Λ:

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = S\Lambda S^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of S. They are also the eigenvectors of A^2. Watch how A^2 has the same S, and ***the eigenvalue matrix of A^2 is Λ^2***:

$$\textbf{Same } S \textbf{ for } A^2 \qquad A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}. \tag{4}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } A \qquad A^k = S\Lambda^k S^{-1} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } k \to \infty \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The limit has the eigenvector x_1 in both columns. We saw this A^∞ on the very first page of the chapter. Now we see it coming, from powers like $A^{100} = S\Lambda^{100} S^{-1}$.

Question *When does $A^k \to$ zero matrix?* **Answer** *All* $|\lambda| < 1$.

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. ***Every new Fibonacci number is the sum of the two previous F's:***

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ***comes from*** $F_{k+2} = F_{k+1} + F_k$.

These numbers turn up in a fantastic variety of applications. Plants and trees grow in a spiral pattern, and a pear tree has 8 growths for every 3 turns. For a willow those numbers can be 13 and 5. The champion is a sunflower of Daniel O'Connell, which had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12}. Our problem is more basic.

Problem: Find the Fibonacci number F_{100} The slow way is to apply the rule $F_{k+2} = F_{k+1} + F_k$ one step at a time. By adding $F_6 = 8$ to $F_7 = 13$ we reach $F_8 = 21$. Eventually we come to F_{100}. Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$. That is a *one-step* rule for vectors, while Fibonacci gave a two-step rule for scalars. We match those rules by putting two Fibonacci numbers into a vector. Then you will see the matrix A.

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}. \text{ The rule } \begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array} \text{ is } \boldsymbol{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}_k. \tag{5}$$

Every step multiplies by $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100}$

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \ldots,$$

This problem is just right for eigenvalues. Subtract λ from the diagonal of A:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{leads to} \quad \det(A - \lambda I) = \lambda^2 - \lambda - 1.$$

The equation $\lambda^2 - \lambda - 1 = 0$ is solved by the quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$:

$$\textbf{Eigenvalues} \quad \lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2} \approx -.618.$$

These eigenvalues lead to eigenvectors $\boldsymbol{x}_1 = (\lambda_1, 1)$ and $\boldsymbol{x}_2 = (\lambda_2, 1)$. Step 2 finds the combination of those eigenvectors that gives $\boldsymbol{u}_0 = (1, 0)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right) \quad \text{or} \quad \boldsymbol{u}_0 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{6}$$

Problem Set 6.2

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and S.

1 (a) Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = S\Lambda S^{-1}$ then $A^3 = (\ )(\ )(\ )$ and $A^{-1} = (\ )(\ )(\ )$.

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

3 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\ )(\ )(\ )^{-1}$.

4 True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix S is triangular, then S^{-1} is triangular. Prove that A is also triangular.

6 Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 11–14 are about diagonalizability.

11 True or false: If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

12 True or false: If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $S\Lambda S^{-1}$.