Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of Ax and AB are linear combinations of n vectors—the columns of A. This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at "spaces" of vectors. Without seeing ***vector spaces*** and especially their ***subspaces***, you haven't understood everything about $Ax = b$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We are looking inside the calculations, to find the mathematics. The author's job is to make it clear. The chapter ends with the "*Fundamental Theorem of Linear Algebra*".

We begin with the most important vector spaces. They are denoted by $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, ... Each space $\mathbf{R}^n$ consists of a whole collection of vectors. $\mathbf{R}^5$ contains all column vectors with five components. This is called "5-dimensional space".

DEFINITION ***The space*** $\mathbf{R}^n$ ***consists of all column vectors*** v ***with*** n ***components.***

The components of v are real numbers, which is the reason for the letter $\mathbf{R}$. A vector whose n components are complex numbers lies in the space $\mathbf{C}^n$.

The vector space $\mathbf{R}^2$ is represented by the usual xy plane. Each vector v in $\mathbf{R}^2$ has two components. The word "*space*" asks us to think of all those vectors—the whole plane. Each vector gives the x and y coordinates of a point in the plane: $v = (x, y)$.

Similarly the vectors in $\mathbf{R}^3$ correspond to points (x, y, z) in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line (like the x axis). As before, we print vectors as a column between brackets, or along a line using commas and parentheses:

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \text{ is in } \mathbf{R}^2, \quad (1, 1, 0, 1, 1) \text{ is in } \mathbf{R}^5, \quad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ is in } \mathbf{C}^2.$$

The great thing about linear algebra is that it deals easily with five-dimensional space. We don't draw the vectors, we just need the five numbers (or n numbers).

To multiply v by 7, multiply every component by 7. Here 7 is a "scalar". To add vectors in $\mathbf{R}^5$, add them a component at a time. The two essential vector operations go on *inside the vector space*, and they produce ***linear combinations***:

We can add any vectors in $\mathbf{R}^n$, and we can multiply any vector v by any scalar c.

"Inside the vector space" means that ***the result stays in the space.*** If v is the vector in $\mathbf{R}^4$ with components $1, 0, 0, 1$, then $2v$ is the vector in $\mathbf{R}^4$ with components $2, 0, 0, 2$. (In this case 2 is the scalar.) A whole series of properties can be verified in $\mathbf{R}^n$. The commutative law is $v + w = w + v$; the distributive law is $c(v + w) = cv + cw$. There is a unique "zero vector" satisfying $\mathbf{0} + v = v$. Those are three of the eight conditions listed at the start of the problem set.

These eight conditions are required of every vector space. There are vectors other than column vectors, and vector spaces other than $\mathbf{R}^n$, and all vector spaces have to obey the eight reasonable rules.

A real vector space is a set of "vectors" *together with rules for vector addition and for multiplication by real numbers.* The addition and the multiplication must produce vectors that are in the space. And the eight conditions must be satisfied (which is usually no problem). Here are three vector spaces other than $\mathbf{R}^n$:

M	The vector space of ***all real* 2 *by* 2 *matrices.***
F	The vector space of ***all real functions*** $f(x)$.
Z	The vector space that consists only of a ***zero vector.***

In **M** the "vectors" are really matrices. In **F** the vectors are functions. In **Z** the only addition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$. In each case we can add: matrices to matrices, functions to functions, zero vector to zero vector. We can multiply a matrix by 4 or a function by 4 or the zero vector by 4. The result is still in **M** or **F** or **Z**. The eight conditions are all easily checked.

The function space **F** is infinite-dimensional. A smaller function space is **P**, or $\mathbf{P}_n$, containing all polynomials $a_0 + a_1x + \cdots + a_nx^n$ of degree n.

The space **Z** is zero-dimensional (by any reasonable definition of dimension). It is the smallest possible vector space. We hesitate to call it $\mathbf{R}^0$, which means no components—you might think there was no vector. The vector space **Z** contains exactly *one vector* (zero). No space can do without that zero vector. Each space has its own zero vector—the zero matrix, the zero function, the vector $(0, 0, 0)$ in $\mathbf{R}^3$.

Subspaces

At different times, we will ask you to think of matrices and functions as vectors. But at all times, the vectors that we need most are ordinary column vectors. They are vectors with n components—but *maybe not all* of the vectors with n components. There are important vector spaces ***inside*** $\mathbf{R}^n$. Those are ***subspaces*** of $\mathbf{R}^n$.

Start with the usual three-dimensional space $\mathbf{R}^3$. Choose a plane through the origin $(0, 0, 0)$. ***That plane is a vector space in its own right.*** If we add two vectors in the plane, their sum is in the plane. If we multiply an in-plane vector by 2 or -5, it is still in the plane.

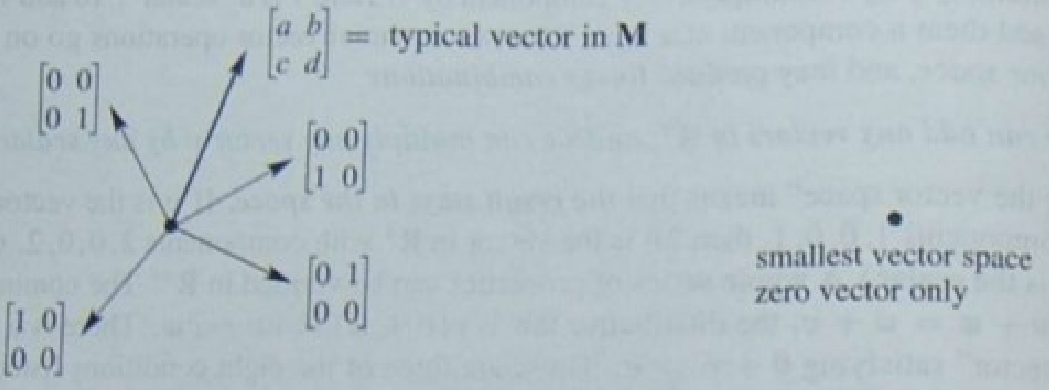


Figure 3.1: "Four-dimensional" matrix space **M**. The "zero-dimensional" space **Z**.

A plane in three-dimensional space is not $\mathbf{R}^2$ (even if it looks like $\mathbf{R}^2$). The vectors have three components and they belong to $\mathbf{R}^3$. The plane is a vector space ***inside*** $\mathbf{R}^3$.

This illustrates one of the most fundamental ideas in linear algebra. The plane going through $(0, 0, 0)$ is a ***subspace*** of the full vector space $\mathbf{R}^3$.

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If*** v ***and*** w ***are vectors in the subspace and*** c ***is any scalar, then***

(i) $v + w$ is in the subspace

(ii) cv is in the subspace.

In other words, the set of vectors is "closed" under addition $v + w$ and multiplication cv (and cw). Those operations leave us in the subspace. We can also subtract, because $-w$ is in the subspace and its sum with v is $v - w$. In short, ***all linear combinations stay in the subspace.***

All these operations follow the rules of the host space, so the eight required conditions are automatic. We just have to check the requirements for a subspace, so that we can take linear combinations.

First fact: ***Every subspace contains the zero vector.*** The plane in $\mathbf{R}^3$ has to go through $(0, 0, 0)$. We mention this separately, for extra emphasis, but it follows directly from rule (ii). Choose $c = 0$, and the rule requires $0v$ to be in the subspace.

Planes that don't contain the origin fail those tests. When v is on such a plane, $-v$ and $0v$ are *not* on the plane. A plane that misses the origin is not a subspace.

Lines through the origin are also subspaces. When we multiply by 5, or add two vectors on the line, we stay on the line. But the line must go through $(0, 0, 0)$.

Another subspace is all of $\mathbf{R}^3$. The whole space is a subspace (*of itself*). Here is a list of all the possible subspaces of $\mathbf{R}^3$:

(**L**) Any line through $(0, 0, 0)$ $\quad$ ($\mathbf{R}^3$) The whole space
(**P**) Any plane through $(0, 0, 0)$ $\quad$ (**Z**) The single vector $(0, 0, 0)$

If we try to keep only *part* of a plane or line, the requirements for a subspace don't hold. Look at these examples in $\mathbf{R}^2$.

Example 1 Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So rule (**ii**) is violated when we try to multiply by $c = -1$. ***The quarter-plane is not a subspace.***

Example 2 Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement (**ii**) is satisfied; we can multiply by any c. But rule (**i**) now fails. The sum of $\boldsymbol{v} = (2, 3)$ and $\boldsymbol{w} = (-3, -2)$ is $(-1, 1)$, which is outside the quarter-planes. ***Two quarter-planes don't make a subspace.***

Rules (**i**) and (**ii**) involve vector addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication by scalars like c and d. The rules can be combined into a single requirement—*the rule for subspaces*:

A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

Example 3 Inside the vector space **M** of all 2 by 2 matrices, here are two subspaces:

(**U**) All upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ (**D**) All diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

Add any two matrices in **U**, and the sum is in **U**. Add diagonal matrices, and the sum is diagonal. In this case **D** is also a subspace of **U**! Of course the zero matrix is in these subspaces, when a, b, and d all equal zero.

To find a smaller subspace of diagonal matrices, we could require $a = d$. The matrices are multiples of the identity matrix I. The sum $2I + 3I$ is in this subspace, and so is 3 times $4I$. The matrices cI form a "line of matrices" inside **M** and **U** and **D**.

Is the matrix I a subspace by itself? Certainly not. Only the zero matrix is. Your mind will invent more subspaces of 2 by 2 matrices—write them down for Problem 5.

The Column Space of A

The most important subspaces are tied directly to a matrix A. We are trying to solve $A\boldsymbol{x} = \boldsymbol{b}$. If A is not invertible, the system is solvable for some $\boldsymbol{b}$ and not solvable for other $\boldsymbol{b}$. We want to describe the good right sides $\boldsymbol{b}$—the vectors that *can* be written as A times some vector $\boldsymbol{x}$. Those $\boldsymbol{b}$'s form the "*column space*" of A.

Remember that $A\boldsymbol{x}$ is a combination of the columns of A. To get every possible $\boldsymbol{b}$, we use every possible $\boldsymbol{x}$. So start with the columns of A, and ***take all their linear combinations. This produces the column space of A.*** **It is a vector space made up of column vectors.**

$\boldsymbol{C}(A)$ contains not just the n columns of A, but all their combinations $A\boldsymbol{x}$.

DEFINITION The *column space* consists of ***all linear combinations of the columns***. The combinations are all possible vectors Ax. They fill the column space $C(A)$.

This column space is crucial to the whole book, and here is why. ***To solve*** $Ax = b$ ***is to express b as a combination of the columns***. The right side b has to be *in the column space* produced by A on the left side, or no solution!

The system $Ax = b$ is solvable if and only if b is in the column space of A.

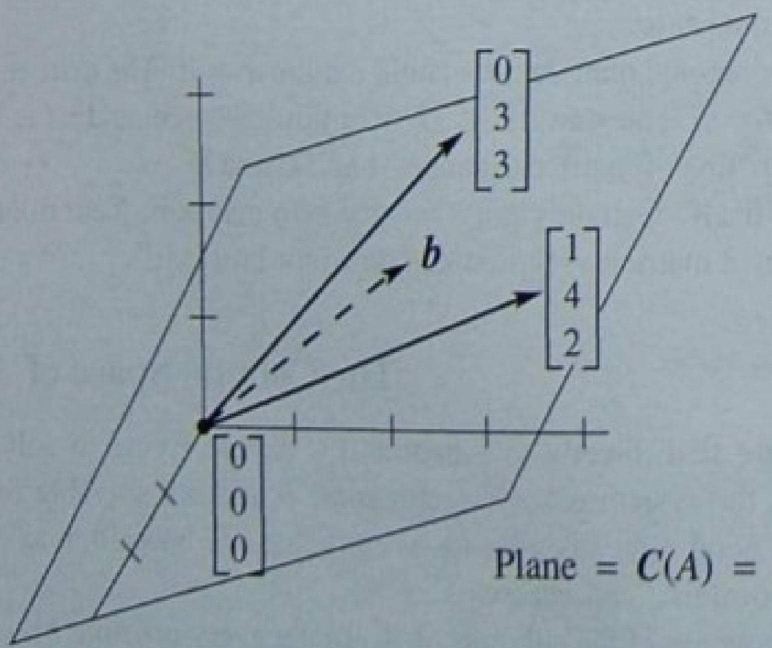
When b is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution x to the system $Ax = b$.

Suppose A is an m by n matrix. Its columns have m components (not n). So the columns belong to $\mathbf{R}^m$. ***The column space of A is a subspace of*** $\mathbf{R}^m$ ***(not*** $\mathbf{R}^n$***)***. The set of all column combinations Ax satisfies rules (**i**) and (**ii**) for a subspace: When we add linear combinations or multiply by scalars, we still produce combinations of the columns. The word "subspace" is justified *by taking all linear combinations*.

Here is a 3 by 2 matrix A, whose column space is a subspace of $\mathbf{R}^3$. The column space of A is a plane in Figure 3.2.

Example 4

$$Ax \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{which is} \quad x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Figure 3.2: The column space $C(A)$ is a plane containing the two columns. $Ax = b$ is solvable when b is on that plane. Then b is a combination of the columns.

The column space of all combinations of the two columns *fills up a plane in* $\mathbf{R}^3$. We drew one particular $\boldsymbol{b}$ (a combination of the columns). This $\boldsymbol{b} = A\boldsymbol{x}$ lies on the plane. The plane has zero thickness, so most right sides $\boldsymbol{b}$ in $\mathbf{R}^3$ are *not* in the column space. For most $\boldsymbol{b}$ there is no solution to our 3 equations in 2 unknowns.

Of course $(0, 0, 0)$ is in the column space. The plane passes through the origin. There is certainly a solution to $A\boldsymbol{x} = \mathbf{0}$. That solution, always available, is $\boldsymbol{x} =$ ____.

To repeat, the attainable right sides $\boldsymbol{b}$ are exactly the vectors in the column space. One possibility is the first column itself—take $x_1 = 1$ and $x_2 = 0$. Another combination is the second column—take $x_1 = 0$ and $x_2 = 1$. The new level of understanding is to see *all* combinations—the whole subspace is generated by those two columns.

Notation The column space of A is denoted by $\boldsymbol{C}(A)$. Start with the columns and take all their linear combinations. We might get the whole $\mathbf{R}^m$ or only a subspace.

Important Instead of columns in $\mathbf{R}^m$, we could start with any set $\mathbf{S}$ of vectors in a vector space $\mathbf{V}$. To get a subspace $\mathbf{SS}$ of $\mathbf{V}$, we take *all combinations* of the vectors in that set:

$$\begin{aligned} \mathbf{S} &= \text{set of vectors in } \mathbf{V} \text{ (probably } \textit{not} \text{ a subspace)} \\ \mathbf{SS} &= \text{all combinations of vectors in } \mathbf{S} \end{aligned}$$

$$\mathbf{SS} = \text{all } c_1\boldsymbol{v}_1 + \cdots + c_N\boldsymbol{v}_N = \textbf{the subspace of V “spanned” by S}$$

When $\mathbf{S}$ is the set of columns, $\mathbf{SS}$ is the column space. When there is only one nonzero vector $\boldsymbol{v}$ in $\mathbf{S}$, the subspace $\mathbf{SS}$ is the line through $\boldsymbol{v}$. *Always* $\mathbf{SS}$ *is the smallest subspace containing* $\mathbf{S}$. This is a fundamental way to create subspaces and we will come back to it.

The subspace SS is the “span” of S, containing all combinations of vectors in S.

Example 5 Describe the column spaces (they are subspaces of $\mathbf{R}^2$) for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The column space of I is the *whole space* $\mathbf{R}^2$. Every vector is a combination of the columns of I. In vector space language, $\boldsymbol{C}(I)$ is $\mathbf{R}^2$.

The column space of A is only a line. The second column $(2, 4)$ is a multiple of the first column $(1, 2)$. Those vectors are different, but our eye is on vector ***spaces***. The column space contains $(1, 2)$ and $(2, 4)$ and all other vectors $(c, 2c)$ along that line. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is only solvable when $\boldsymbol{b}$ is on the line.

For the third matrix (with three columns) the column space $\boldsymbol{C}(B)$ is all of $\mathbf{R}^2$. Every $\boldsymbol{b}$ is attainable. The vector $\boldsymbol{b} = (5, 4)$ is column 2 plus column 3, so $\boldsymbol{x}$ can be $(0, 1, 1)$. The same vector $(5, 4)$ is also 2(column 1) + column 3, so another possible $\boldsymbol{x}$ is $(2, 0, 1)$. This matrix has the same column space as I—any $\boldsymbol{b}$ is allowed. But now $\boldsymbol{x}$ has extra components and there are more solutions—more combinations that give $\boldsymbol{b}$.

The next section creates a vector space $\boldsymbol{N}(A)$, to describe all the solutions of $A\boldsymbol{x} = \mathbf{0}$. This section created the column space $\boldsymbol{C}(A)$, to describe all the attainable right sides $\boldsymbol{b}$.

■ REVIEW OF THE KEY IDEAS ■

1. $\mathbf{R}^n$ contains all column vectors with n real components.

2. $\mathbf{M}$ (2 by 2 matrices) and $\mathbf{F}$ (functions) and $\mathbf{Z}$ (zero vector alone) are vector spaces.

3. A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all their combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

4. The combinations of the columns of A form the ***column space*** $\mathbf{C}(A)$. Then the column space is "spanned" by the columns.

5. $A\boldsymbol{x} = \boldsymbol{b}$ has a solution exactly when $\boldsymbol{b}$ is in the column space of A.

■ WORKED EXAMPLES ■

3.1 A We are given three different vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. Construct a matrix so that the equations $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ are solvable, but $A\boldsymbol{x} = \boldsymbol{b}_3$ is not solvable. How can you decide if this is possible? How could you construct A?

Solution We want to have $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the column space of A. Then $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ will be solvable. *The quickest way is to make* $\boldsymbol{b}_1$ *and* $\boldsymbol{b}_2$ *the two columns of* A. Then the solutions are $\boldsymbol{x} = (1, 0)$ and $\boldsymbol{x} = (0, 1)$.

Also, we don't want $A\boldsymbol{x} = \boldsymbol{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the question is:

$$\text{Is } A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{b}_3 \text{ solvable?} \qquad \text{Is } \boldsymbol{b}_3 \text{ a combination of } \boldsymbol{b}_1 \text{ and } \boldsymbol{b}_2\text{?}$$

If the answer is *no*, we have the desired matrix A. If the answer is *yes*, then it is *not possible* to construct A. When the column space contains $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, it will have to contain all their linear combinations. So $\boldsymbol{b}_3$ would necessarily be in that column space and $A\boldsymbol{x} = \boldsymbol{b}_3$ would necessarily be solvable.

3.1 B Describe a subspace $\mathbf{S}$ of each vector space $\mathbf{V}$, and then a subspace $\mathbf{SS}$ of $\mathbf{S}$.

$\mathbf{V}_1$ = all combinations of $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$
$\mathbf{V}_2$ = all vectors perpendicular to $\boldsymbol{u} = (1, 2, 1)$, so $\boldsymbol{u} \cdot \boldsymbol{v} = 0$
$\mathbf{V}_3$ = all symmetric 2 by 2 matrices (a subspace of $\mathbf{M}$)
$\mathbf{V}_4$ = all solutions to the equation $d^4y/dx^4 = 0$ (a subspace of $\mathbf{F}$)

Describe each $\mathbf{V}$ two ways: *All combinations of ...., all solutions of the equations ....*

Solution $\mathbf{V}_1$ starts with three vectors. A subspace $\mathbf{S}$ comes from all combinations of the first two vectors $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$. A subspace $\mathbf{SS}$ of $\mathbf{S}$ comes from all multiples $(c, c, 0, 0)$ of the first vector. So many possibilities.

A subspace $\mathbf{S}$ of $\mathbf{V}_2$ is the line through $(1, -1, 1)$. This line is perpendicular to $\boldsymbol{u}$. The vector $\boldsymbol{x} = (0, 0, 0)$ is in $\mathbf{S}$ and all its multiples $c\boldsymbol{x}$ give the smallest subspace $\mathbf{SS} = \mathbf{Z}$.

The diagonal matrices are a subspace $\mathbf{S}$ of the symmetric matrices. The multiples cI are a subspace $\mathbf{SS}$ of the diagonal matrices.

$\mathbf{V}_4$ contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $d^4y/dx^4 = 0$. The quadratic polynomials give a subspace $\mathbf{S}$. The linear polynomials are one choice of $\mathbf{SS}$. The constants could be $\mathbf{SSS}$.

In all four parts we could take $\mathbf{S} = \mathbf{V}$ itself, and $\mathbf{SS} =$ the zero subspace $\mathbf{Z}$.

Each $\mathbf{V}$ can be described as *all combinations of ....* and as *all solutions of ....*:

$\mathbf{V}_1 =$ all combinations of the 3 vectors $\quad \mathbf{V}_1 =$ all solutions of $v_1 - v_2 = 0$

$\mathbf{V}_2 =$ all combinations of $(1, 0, -1)$ and $(1, -1, 1)$ are solutions of $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

$\mathbf{V}_3 =$ all combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. $\quad \mathbf{V}_3 =$ all solutions $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $b = c$

$\mathbf{V}_4 =$ all combinations of $1, x, x^2, x^3 \quad \mathbf{V}_4 =$ all solutions to $d^4y/dx^4 = 0$.

Problem Set 3.1

The first problems 1–8 are about vector spaces in general. The vectors in those spaces are not necessarily column vectors. In the definition of a *vector space*, vector addition $x + y$ and scalar multiplication cx must obey the following eight rules:

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) There is a unique "zero vector" such that $x + \mathbf{0} = x$ for all x

(4) For each x there is a unique vector $-x$ such that $x + (-x) = \mathbf{0}$

(5) 1 times x equals x

(6) $(c_1c_2)x = c_1(c_2x)$

(7) $c(x + y) = cx + cy$

(8) $(c_1 + c_2)x = c_1x + c_2x$.

1 Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

2 Suppose the multiplication cx is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2). With the usual addition in $\mathbf{R}^2$, are the eight conditions satisfied?

Questions 19–27 are about column spaces $C(A)$ and the equation $Ax = b$.

19 Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

20 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

21 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of (B or C ?) is also a combination of the columns of A. Which two matrices have the same column _____ ?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

22 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{x} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$?

24 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

25 Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, *then* $\boldsymbol{b} + \boldsymbol{b}^*$ is in $\boldsymbol{C}(A)$.

26 If A is any 5 by 5 invertible matrix, then its column space is _____. Why?

27 True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

3.2 The Nullspace of A: Solving $Ax = 0$

This section is about the subspace containing all solutions to $Ax = \mathbf{0}$. The m by n matrix A can be square or rectangular. *One immediate solution is* $x = \mathbf{0}$. For invertible matrices this is the only solution. For other matrices, not invertible, there are nonzero solutions to $Ax = \mathbf{0}$. *Each solution* x *belongs to the nullspace of* A.

Elimination will find all solutions and identify this very important subspace.

> ***The nullspace of A consists of all solutions to $Ax = \mathbf{0}$.*** **These vectors x are in $\mathbf{R}^n$.** The nullspace containing all solutions of $Ax = \mathbf{0}$ is denoted by $N(A)$.

Check that the solution vectors form a subspace. Suppose x and y are in the nullspace (this means $Ax = \mathbf{0}$ and $Ay = \mathbf{0}$). The rules of matrix multiplication give $A(x + y) = \mathbf{0} + \mathbf{0}$. The rules also give $A(cx) = c\mathbf{0}$. The right sides are still zero. Therefore $x + y$ and cx are also in the nullspace $N(A)$. Since we can add and multiply without leaving the nullspace, it is a subspace.

To repeat: The solution vectors x have n components. They are vectors in $\mathbf{R}^n$, so *the nullspace is a subspace of* $\mathbf{R}^n$. The column space $C(A)$ is a subspace of $\mathbf{R}^m$.

If the right side b is not zero, the solutions of $Ax = b$ do *not* form a subspace. The vector $x = \mathbf{0}$ is only a solution if $b = \mathbf{0}$. When the set of solutions does not include $x = \mathbf{0}$, it cannot be a subspace. Section 3.4 will show how the solutions to $Ax = b$ (if there are any solutions) are shifted away from the origin by one particular solution.

Example 1 $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [1 \;\; 2 \;\; 3]$. This equation $Ax = \mathbf{0}$ produces a plane through the origin $(0, 0, 0)$. The plane is a subspace of $\mathbf{R}^3$. *It is the nullspace of* A.

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular!

Solution Apply elimination to the linear equations $Ax = \mathbf{0}$:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 + 2x_2 &= 0 \\ \mathbf{0} &= \mathbf{0} \end{aligned}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $N(A)$. It contains all solutions (x_1, x_2).

To describe this line of solutions, here is an efficient way. Choose one point on the line (one "***special solution***"). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. The special solution s is $(-2, 1)$:

Special solution The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

This is the best way to describe the nullspace, by computing special solutions to $Ax = \mathbf{0}$. This example has one special solution and the nullspace is a line.

The nullspace consists of all combinations of the special solutions.

The plane $x + 2y + 3z = 0$ in Example 1 had *two* special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has the special solutions } s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Those vectors s_1 and s_2 lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. All vectors on the plane are combinations of s_1 and s_2.

Notice what is special about s_1 and s_2. They have ones and zeros in the last two components. *Those components are "free" and we choose them specially.* Then the first components -2 and -3 are determined by the equation $Ax = \mathbf{0}$.

The first column of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ contains the *pivot*, so the first component of x is *not free*. The free components correspond to columns without pivots. This description of special solutions will be completed after one more example.

The special choice (one or zero) is only for the free variables.

Example 3 Describe the nullspaces of these three matrices A, B, C:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $Ax = \mathbf{0}$ has only the zero solution $x = \mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $x = \mathbf{0}$ in $\mathbf{R}^2$. This comes from elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}.$$

A is invertible. There are no special solutions. All columns of this A have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $Bx = \mathbf{0}$ again require $x = \mathbf{0}$. The last two equations would also force $x = \mathbf{0}$. When we add extra equations, the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors x in the nullspace.

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector x has *four* components. Elimination will produce pivots in the first two columns of C, but the last two columns are "free". They don't have pivots:

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\uparrow \ \uparrow \ \uparrow \ \uparrow$$

pivot columns free columns

For the free variables x_3 and x_4, we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are determined by the equation $Ux = \mathbf{0}$. We get two special solutions in the nullspace of C (which is also the nullspace of U). The special solutions are s_1 and s_2:

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{pivot} \\ \leftarrow \quad \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \quad \text{variables} \end{matrix}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U! We can continue to make this matrix simpler, in two ways:

1. *Produce zeros above the pivots,* **by eliminating upward.**
2. *Produce ones in the pivots,* **by dividing the whole row by its pivot.**

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easiest to see when we reach the ***reduced row echelon form*** R. It has I in the pivot columns:

Reduced form R

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

$\uparrow \; \uparrow$ now the pivot columns contain I

I subtracted row 2 of U from row 1, and then multiplied row 2 by $\frac{1}{2}$. The original two equations have simplified to $x_1 + 2x_3 = 0$ and $x_2 + 2x_4 = 0$.

The first special solution is still $s_1 = (-2, 0, 1, 0)$, and s_2 is also unchanged. Special solutions are much easier to find from the reduced system $Rx = \mathbf{0}$.

Before moving to m by n matrices A and their nullspaces $N(A)$ and special solutions, allow me to repeat one comment. For many matrices, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. Their nullspaces $N(A) = \mathbf{Z}$ contain only that zero vector. The only combination of the columns that produces $b = \mathbf{0}$ is then the "zero combination" or "trivial combination". The solution is trivial (just $x = \mathbf{0}$) but the idea is not trivial.

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are **independent**. No combination of columns gives the zero vector (except the zero combination). All columns have pivots, and no columns are free. You will see this idea of independence again ...

Solving $Ax = 0$ by Elimination

This is important. ***A is rectangular and we still use elimination***. We solve m equations in n unknowns when $b = \mathbf{0}$. After A is simplified by row operations, we read off the solution (or solutions). Remember the two stages (forward and back) in solving $Ax = \mathbf{0}$:

1. **Forward elimination** takes A to a triangular U (or its reduced form R).

2. **Back substitution** in $Ux = \mathbf{0}$ or $Rx = \mathbf{0}$ produces $\boldsymbol{x}$.

You will notice a difference in back substitution, when A and U have fewer than n pivots. *We are allowing all matrices in this chapter*, not just the nice ones (which are square matrices with inverses).

Pivots are still nonzero. The columns below the pivots are still zero. But it might happen that a column has no pivot. That free column doesn't stop the calculation. ***Go on to the next column***. The first example is a 3 by 4 matrix with two pivots:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}.$$

Certainly $a_{11} = 1$ is the first pivot. Clear out the 2 and 3 below that pivot:

$$A \rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{matrix} \\ \text{(subtract } 2 \times \text{ row 1)} \\ \text{(subtract } 3 \times \text{ row 1)} \end{matrix}$$

The second column has a zero in the pivot position. We look below the zero for a nonzero entry, ready to do a row exchange. *The entry below that position is also zero.* Elimination can do nothing with the second column. This signals trouble, which we expect anyway for a rectangular matrix. There is no reason to quit, and we go on to the third column.

The second pivot is 4 (but it is in the third column). Subtracting row 2 from row 3 clears out that column below the pivot. **The pivot columns are** 1 **and** 3:

$$\textbf{Triangular } U: \quad U = \begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \textit{Only two pivots} \\ \textit{The last equation} \\ \textit{became } 0 = 0 \end{matrix}$$

The fourth column also has a zero in the pivot position—but nothing can be done. There is no row below it to exchange, and forward elimination is complete. The matrix has three rows, four columns, and *only two pivots*. The original $Ax = \mathbf{0}$ seemed to involve three different equations, but the third equation is the sum of the first two. It is automatically satisfied $(0 = 0)$ when the first two equations are satisfied. Elimination reveals the inner truth about a system of equations. Soon we push on from U to R.

Now comes back substitution, to find all solutions to $Ux = \mathbf{0}$. With four unknowns and only two pivots, there are many solutions. The question is how to write them all down. A good method is to separate the ***pivot variables*** from the ***free variables***.

P	The ***pivot*** variables are x_1 and x_3.	**Columns 1 and 3 contain pivots.**
F	The ***free*** variables are x_2 and x_4.	**Columns 2 and 4 have no pivots.**

The free variables x_2 and x_4 can be given any values whatsoever. Then back substitution finds the pivot variables x_1 and x_3. (In Chapter 2 no variables were free. When A is invertible, all variables are pivot variables.) The simplest choices for the free variables are ones and zeros. Those choices give the *special solutions*.

Special solutions to $x_1 + x_2 + 2x_3 + 3x_4 = 0$ and $4x_3 + 4x_4 = 0$

- Set $x_2 = 1$ and $x_4 = 0$. By back substitution $x_3 = 0$. Then $x_1 = -1$.
- Set $x_2 = 0$ and $x_4 = 1$. By back substitution $x_3 = -1$. Then $x_1 = -1$.

These special solutions solve $Ux = \mathbf{0}$ and therefore $Ax = \mathbf{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions.*

Complete solution to $Ax = 0$

$$\boldsymbol{x} = x_2 \underset{\textbf{special}}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\textbf{special}}{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} = \underset{\textbf{complete}}{\begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}}. \qquad (1)$$

Please look again at that answer. It is the main goal of this section. The vector $s_1 = (-1, 1, 0, 0)$ is the special solution when $x_2 = 1$ and $x_4 = 0$. The second special solution has $x_2 = 0$ and $x_4 = 1$. ***All solutions are linear combinations of s_1 and s_2.*** The special solutions are in the nullspace $\boldsymbol{N}(A)$, and their combinations fill out the whole nullspace.

The MATLAB code **nullbasis** computes these special solutions. They go into the columns of a ***nullspace matrix*** N. The complete solution to $Ax = \mathbf{0}$ is a combination of those columns. Once we have the special solutions, we have the whole nullspace.

There is a special solution for each free variable. If no variables are free—this means there are n pivots—then the only solution to $Ux = \mathbf{0}$ and $Ax = \mathbf{0}$ is the trivial solution $\boldsymbol{x} = \mathbf{0}$. All variables are pivot variables. In that case the nullspaces of A and U contain only the zero vector. With no free variables, and pivots in every column, the output from **nullbasis** is an empty matrix. The nullspace with n pivots is $\mathbf{Z}$.

Example 4 Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of U has no pivot. So x_2 is free. The special solution has $x_2 = 1$. Back substitution into $9x_3 = 0$ gives $x_3 = 0$. Then $x_1 + 5x_2 = 0$ or $x_1 = -5$. The solutions to $U\boldsymbol{x} = \mathbf{0}$ are multiples of one special solution:

$$\boldsymbol{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of U is a line in $\mathbf{R}^3$.
It contains multiples of the special solution $\boldsymbol{s} = (-5, 1, 0)$.
One variable is free, and $N = $ **nullbasis** (U) has one column $\boldsymbol{s}$.

In a minute elimination will get zeros above the pivots and ones in the pivots. By continuing elimination on U, the 7 is removed and the pivot changes from 9 to 1.

A short wide matrix ($n > m$) always has nonzero vectors in its nullspace. There must be at least $n - m$ free variables, since the number of pivots cannot exceed m. (The matrix only has m rows, and a row never has two pivots.) Of course a row might have *no* pivot—which means an extra free variable. But here is the point: When there is a free variable, it can be set to 1. Then the equation $Ax = \mathbf{0}$ has a nonzero solution.

To repeat: There are at most m pivots. With $n > m$, the system $Ax = \mathbf{0}$ has a nonzero solution. Actually there are infinitely many solutions, since any multiple cx is also a solution. The nullspace contains at least a line of solutions. With two free variables, there are two special solutions and the nullspace is even larger.

The nullspace is a subspace. Its "dimension" is the number of free variables. This central idea—the ***dimension*** of a subspace—is defined and explained in this chapter.

The Reduced Row Echelon Matrix R

From an echelon matrix U we go one more step. Continue with a 3 by 4 example:

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can divide the second row by 4. Then both pivots equal 1. *We can subtract* 2 *times this new row* $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ *from the row above.* ***The reduced row echelon matrix R has zeros above the pivots as well as below***:

Reduced row echelon matrix $\quad R = \text{rref}(A) = \begin{bmatrix} \mathbf{1} & 1 & \mathbf{0} & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad$ **Pivot rows contain I**

R has 1's as pivots. Zeros above pivots come from upward elimination.

Important *If A is invertible, its reduced row echelon form is the identity matrix* $R = I$. This is the ultimate in row reduction. Of course the nullspace is then **Z**.

The zeros in R make it easy to find the special solutions (the same as before):

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = 0$.

Those numbers -1 and 0 are sitting in column 2 of R (with plus signs).

2. Set $x_2 = 0$ and $x_4 = 1$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = -1$.

Those numbers -1 and -1 are sitting in column 4 (with plus signs).

By reversing signs we can read off the special solutions directly from R. The nullspace $N(A) = N(U) = N(R)$ *contains all combinations of the special solutions:*

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\textbf{\textit{complete solution of }} Ax = \mathbf{0}).$$

The next section of the book moves firmly from U to the row reduced form R. The MATLAB command $[R, pivcol] = \text{rref}(A)$ produces R and also a list of the pivot columns.

■ REVIEW OF THE KEY IDEAS ■

1. The nullspace $N(A)$ is a subspace of $\mathbf{R}^n$. It contains all solutions to $Ax = \mathbf{0}$.
2. Elimination produces an echelon matrix U, and then a row reduced R, with pivot columns and free columns.
3. Every free column of U or R leads to a special solution. The free variable equals 1 and the other free variables equal 0. Back substitution solves $Ax = \mathbf{0}$.
4. The complete solution to $Ax = \mathbf{0}$ is a combination of the special solutions.
5. If $n > m$ then A has at least one column without pivots, giving a special solution. So there are nonzero vectors x in the nullspace of this rectangular A.

■ WORKED EXAMPLES ■

3.2 A Create a 3 by 4 matrix whose special solutions to $Ax = \mathbf{0}$ are s_1 and s_2:

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix} \qquad \begin{array}{l} \text{pivot columns 1 and 3} \\ \text{free variables } x_2 \text{ and } x_4 \end{array}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required nullspace $N(A) =$ all combinations of s_1 and s_2.

Solution The reduced matrix R has pivots $= 1$ in columns 1 and 3. There is no third pivot, so the third row of R is all zeros. The free columns 2 and 4 will be combinations of the pivot columns:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad Rs_1 = \mathbf{0} \quad \text{and} \quad Rs_2 = \mathbf{0}.$$

The entries $3, 2, 6$ in R are the negatives of $-3, -2, -6$ in the special solutions!

R is only one matrix (one possible A) with the required nullspace. We could do any elementary operations on R—exchange rows, multiply a row by any $c \neq 0$, subtract any multiple of one row from another. **R can be multiplied** *(on the left)* **by any invertible matrix, without changing its nullspace.**

Every 3 by 4 matrix has at least one special solution. *These matrices have two.*

3.2 B Find the special solutions and describe the *complete solution* to $Ax = \mathbf{0}$ for

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} A_2 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is R in each case?

Solution $A_1x = \mathbf{0}$ has four special solutions. They are the columns s_1, s_2, s_3, s_4 of the 4 by 4 identity matrix. The nullspace is all of $\mathbf{R}^4$. The complete solution to $A_1x = \mathbf{0}$ is any $x = c_1s_1 + c_2s_2 + c_3s_3 + c_4s_4$ in $\mathbf{R}^4$. There are no pivot columns; all variables are free; the reduced R is the same zero matrix as A_1.

$A_2x = \mathbf{0}$ has only one special solution $s = (-2, 1)$. The multiples $x = cs$ give the complete solution. The first column of A_2 is its pivot column, and x_2 is the free variable. The row reduced matrices R_2 for A_2 and R_3 for $A_3 = [A_2 \;\; A_2]$ have 1's in the pivot:

$$A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \to R_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} A_2 & A_2 \end{bmatrix} \to R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that R_3 has only one pivot column (the first column). All the variables x_2, x_3, x_4 are free. There are three special solutions to $A_3x = \mathbf{0}$ (and also $R_3x = \mathbf{0}$):

$$s_1 = (-2, 1, 0, 0) \;\; s_2 = (-1, 0, 1, 0) \;\; s_3 = (-2, 0, 0, 1) \;\; \textbf{Complete } x = c_1s_1 + c_2s_2 + c_3s_3.$$

With r pivots, A has $n - r$ free variables. $Ax = \mathbf{0}$ has $n - r$ special solutions.

Problem Set 3.2

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

1 Reduce these matrices to their ordinary echelon forms U:

$$\text{(a)}\; A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\; B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

3 By combining the special solutions in Problem 2, describe every solution to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. The nullspace contains only $x = \mathbf{0}$ when there are no ____.

4 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false*: The nullspace of R equals the nullspace of U.

5 By row operations reduce each matrix to its echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

(a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$.

6 For the same A and B, find the special solutions to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is ____.

7 In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors x that satisfy those equations as combinations of the special solutions.

8 Reduce the echelon forms U in Problem 5 to R. For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

Questions 9–17 are about free variables and pivot variables.

9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

10 Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

11 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) 2, 4, 5

(b) 1, 3, 6, 7

(c) 4 and 6.

12 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) 2, 4, 5, 6

(b) 1, 3, 6, 7, 8.

13 Suppose column 4 of a 3 by 5 matrix is all zero. Then x_4 is certainly a ____ variable. The special solution for this variable is the vector $x =$ ____.

14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then ____ is a free variable. Find the special solution for this variable.

3.3 The Rank and the Row Reduced Form

The numbers m and n give the size of a matrix—but not necessarily the *true size* of a linear system. An equation like $0 = 0$ should not count. If there are two identical rows in A, the second one disappears in elimination. Also if row 3 is a combination of rows 1 and 2, then row 3 will become all zeros in the triangular U and the reduced echelon form R. We don't want to count rows of zeros. ***The true size of A is given by its rank***:

DEFINITION ***The rank of A is the number of pivots. This number is r.***

That definition is computational, and I would like to say more about the rank r. The matrix will eventually be reduced to r nonzero rows. Start with a 3 by 4 example.

Four columns
How many pivots?

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}. \tag{1}$$

The first two columns are $(1, 1, 1)$ and $(1, 2, 3)$, going in different directions. Those will be pivot columns. The third column $(2, 2, 2)$ is a multiple of the first. We won't see a pivot in that third column. The fourth column $(4, 5, 6)$ is a combination of the first three (their sum). That column will also be without a pivot.

The fourth column is actually a combination $3(1, 1, 1) + (1, 2, 3)$ of the two pivot columns. ***Every "free column" is a combination of earlier pivot columns.*** It is the *special solutions* $\boldsymbol{s}$ that tell us those combinations of pivot columns:

Column 3 = **2** (column 1)	$s_1 = (-2, 0, 1, 0)$	$As_1 = \mathbf{0}$
Column 4 = **3** (column 1) + **1** (column 2)	$s_2 = (-3, -1, 0, 1)$	$As_2 = \mathbf{0}$

With nice numbers we can see the right combinations. The systematic way to find $\boldsymbol{s}$ is by elimination! This will change the columns but it won't change the combinations, because $A\boldsymbol{x} = \mathbf{0}$ is equivalent to $U\boldsymbol{x} = \mathbf{0}$ and also $R\boldsymbol{x} = \mathbf{0}$. I will go from A to U and then to R:

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 1 & 2 & 4 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

U already shows the two pivots in the pivot columns. **The rank of A (and U) is 2.** Continuing to R we see the combinations of pivot columns that produce the free columns:

$$U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[\text{row 1} - \text{row 2}]{\text{Subtract}} \quad R = \begin{bmatrix} \mathbf{1} & 0 & 2 & 3 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Clearly the $(3, 1, 0)$ column equals 3 (column 1) + column 2. Moving all columns to the "left side" will reverse signs to -3 and -1, which go in the special solution $\boldsymbol{s}$:

$$-3\,(\text{column } 1) - (\text{column } 2) + (\text{column } 4) = \mathbf{0} \qquad \boldsymbol{s} = (-3, -1, 0, 1).$$

Rank One

Matrices of ***rank one*** have only *one pivot*. When elimination produces zero in the first column, it produces zero in all the columns. *Every row is a multiple of the pivot row*. At the same time, every column is a multiple of the pivot column!

$$\textbf{Rank one matrix} \qquad A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The column space of a rank one matrix is "one-dimensional". Here all columns are on the line through $\boldsymbol{u} = (1, 2, 3)$. The columns of A are $\boldsymbol{u}$ and $3\boldsymbol{u}$ and $10\boldsymbol{u}$. Put those numbers into the row $\boldsymbol{v}^{\mathrm{T}} = [\ 1 \quad 3 \quad 10\ ]$ and you have the special rank one form $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \textbf{column times row} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \qquad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 3 \quad 10] \tag{3}$$

With rank one, the solutions to $A\boldsymbol{x} = \boldsymbol{0}$ are easy to understand. That equation $\boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x}) = \boldsymbol{0}$ leads us to $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x} = 0$. All vectors $\boldsymbol{x}$ in the nullspace must be orthogonal to $\boldsymbol{v}$ in the row space. This is the geometry: *row space = line, nullspace = perpendicular plane.* Now describe the special solutions with numbers:

Pivot row $[1 \ 3 \ 10]$
Pivot variable x_1
Free variables x_2 **and** x_3

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

The nullspace contains all combinations of s_1 and s_2. This produces the plane $x + 3y + 10z = 0$, perpendicular to the row $(1, 3, 10)$. **Nullspace** (plane) **perpendicular to row space** (line).

Example 1 When all rows are multiples of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ and } [\,6\,] \text{ all have rank 1.}$$

For those matrices, the reduced row echelon $R = \textbf{rref}\,(A)$ can be checked by eye:

$$R = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [\,1\,] \text{ have only one pivot.}$$

Our second definition of rank will be at a higher level. It deals with entire rows and entire columns—vectors and not just numbers. The matrices A and U and R have r *independent* rows (the pivot rows). They also have r *independent* columns (the pivot columns). Section 3.5 says what it means for rows or columns to be independent.

A third definition of rank, at the top level of linear algebra, will deal with *spaces* of vectors. ***The rank r is the "dimension" of the column space.*** It is also the dimension of the row space. The great thing is that r also reveals the dimension of the nullspace.

The Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else. The r pivot columns taken together contain an r by r identity matrix I. It sits above $m - r$ rows of zeros. The numbers of the pivot columns are in the list *pivcol*.

The pivot columns of A are probably not obvious from A itself. But their column numbers are given by the *same list pivcol*. The r columns of A that eventually have pivots (in U and R) are the pivot columns of A. This example has $pivcol = (1, 3)$:

Pivot Columns $$A = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{1} & 3 & \mathbf{1} & 6 & -4 \end{bmatrix} \text{ yields } R = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}.$$

The column spaces of A and R are different! All columns of this R end with zeros. Elimination subtracts rows 1 and 2 of A from row 3, to produce that zero row in R:

$\boldsymbol{EA = R}$
$\boldsymbol{A = E^{-1}R}$
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The r pivot columns of A are also the first r columns of E^{-1}. The r by r identity matrix inside R just picks out the first r columns of E^{-1} as columns of $A = E^{-1}R$.

One more fact about pivot columns. Their definition has been purely computational, based on R. Here is a direct mathematical description of the pivot columns of A:

> ***The pivot columns are not combinations of earlier columns.*** The free columns *are* combinations of earlier columns. These combinations are the special solutions!

A pivot column of R (with 1 in the pivot row) cannot be a combination of earlier columns (with 0's in that row). The same column of A can't be a combination of earlier columns, because $A\boldsymbol{x} = \boldsymbol{0}$ *exactly when* $R\boldsymbol{x} = \boldsymbol{0}$.

Now we look at the special solution $\boldsymbol{x}$ from each free column.

The Special Solutions

Each special solution to $A\boldsymbol{x} = \boldsymbol{0}$ and $R\boldsymbol{x} = \boldsymbol{0}$ has one free variable equal to 1. The other free variables in $\boldsymbol{x}$ are all zero. The solutions come directly from the echelon form R:

Free columns
Free variables
in boldface
$$R\boldsymbol{x} = \begin{bmatrix} 1 & \mathbf{3} & 0 & \mathbf{2} & \mathbf{-1} \\ 0 & \mathbf{0} & 1 & \mathbf{4} & \mathbf{-3} \\ 0 & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_1 \\ \boldsymbol{x_2} \\ x_3 \\ \boldsymbol{x_4} \\ \boldsymbol{x_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set the first free variable to $x_2 = 1$ with $x_4 = x_5 = 0$. The equations give the pivot variables $x_1 = -3$ and $x_3 = 0$. The special solution is $\boldsymbol{s}_1 = (-3, 1, 0, 0, 0)$.

The next special solution has $x_4 = 1$. The other free variables are $x_2 = x_5 = 0$. The solution is $s_2 = (-2, 0, -4, 1, 0)$. Notice -2 and -4 in R, with plus signs.

The third special solution has $x_5 = 1$. With $x_2 = 0$ and $x_4 = 0$ we find $s_3 = (1, 0, 3, 0, 1)$. The numbers $x_1 = 1$ and $x_3 = 3$ are in column 5 of R, again with opposite signs. This is a general rule as we soon verify. The nullspace matrix N contains the three special solutions in its columns, so $AN =$ *zero matrix*:

Nullspace matrix
$n - r = 5 - 2$
3 special solutions

$$N = \begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \\ \text{free} \end{matrix}$$

The linear combinations of these three columns give all vectors in the nullspace. This is the complete solution to $Ax = \mathbf{0}$ (and $Rx = \mathbf{0}$). Where R had the identity matrix (2 by 2) in its pivot columns, N has the identity matrix (3 by 3) in its free rows.

There is a special solution for every free variable. Since r columns have pivots, that leaves $n - r$ free variables. This is the key to $Ax = \mathbf{0}$ and the nullspace:

> $Ax = \mathbf{0}$ ***has r pivots and $n - r$ free variables***: n columns minus r pivot columns. The ***nullspace matrix*** N contains the $n - r$ special solutions. Then $AN = 0$.

When we introduce the idea of "independent" vectors, we will show that the special solutions are independent. You can see in N that no column is a combination of the other columns. The beautiful thing is that the count is exactly right:

$Ax = \mathbf{0}$ has r independent equations so it has n − r independent solutions.

The special solutions are easy for $Rx = \mathbf{0}$. Suppose that the first r columns are the pivot columns. Then the reduced row echelon form looks like

$$R = \begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{matrix} r \text{ pivot rows} \\ m - r \text{ zero rows} \end{matrix} \tag{4}$$

r pivot columns $\quad n - r$ free columns

> The pivot variables in the $n - r$ special solutions come by changing F to $-F$:
>
> **Nullspace matrix** $$N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{matrix} r \text{ pivot variables} \\ n - r \text{ free variables} \end{matrix} \tag{5}$$

Check $RN = 0$. The first block row of RN is (I times $-F$) + (F times I) = zero. The columns of N solve $Rx = \mathbf{0}$. When the free part of $Rx = \mathbf{0}$ moves to the right side, the left side just holds the identity matrix:

$$Rx = \mathbf{0} \quad \text{means} \quad I\begin{bmatrix}\text{pivot}\\ \text{variables}\end{bmatrix} = -F\begin{bmatrix}\text{free}\\ \text{variables}\end{bmatrix}. \tag{6}$$

In each special solution, the free variables are a column of I. Then the pivot variables are a column of $-F$. Those special solutions give the nullspace matrix N.

The idea is still true if the pivot columns are mixed in with the free columns. Then I and F are mixed together. You can still see $-F$ in the solutions. Here is an example where $I = [1]$ comes first and $F = \begin{bmatrix}2 & 3\end{bmatrix}$ comes last.

Example 2 The special solutions of $Rx = x_1 + 2x_2 + 3x_3 = 0$ are the columns of N:

$$R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad N = \begin{bmatrix}-F\\ I\end{bmatrix} = \begin{bmatrix}-2 & -3\\ 1 & 0\\ 0 & 1\end{bmatrix}.$$

The rank is one. There are $n - r = 3 - 1$ special solutions $(-2, 1, 0)$ and $(-3, 0, 1)$.

Final Note How can I write confidently about R not knowing which steps MATLAB will take? A could be reduced to R in different ways. Very likely you and Mathematica and Maple would do the elimination differently. The key is that ***the final R is always the same.*** *The original A completely determines the I and F and zero rows in R.*

For proof I will determine the pivot columns (which locate I) and free columns (which contain F) in an "algebra way"—two rules that have nothing to do with any particular elimination steps. Here are those rules:

1. The pivot columns *are not* combinations of earlier columns of A.
2. The free columns *are* combinations of earlier columns (F tells the combinations).

A small example with rank one will show two E's that produce the correct $EA = R$:

$$A = \begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix} \quad \text{reduces to} \quad R = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = \text{rref}(A) \quad \text{and no other } R.$$

You could multiply row 1 of A by $\frac{1}{2}$, and subtract row 1 from row 2:

Two steps give E
$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1/2 & 0\\ -1/2 & 1\end{bmatrix} = E.$$

Or you could exchange rows in A, and then subtract 2 times row 1 from row 2:

Two different steps give E_{new}
$$\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix} = E_{\text{new}}.$$

Multiplication gives $EA = R$ and also $E_{\text{new}}A = R$. ***Different E's but the same R.***

Codes for Row Reduction

There is no way that rref will ever come close in importance to lu. The Teaching Code elim for this book uses rref. Of course rref(R) would give R again!

MATLAB: $[R, pivcol] =$ rref(A) Teaching Code: $[E, R] =$ elim(A)

The extra output *pivcol* gives the numbers of the pivot columns. They are the same in A and R. The extra output E in the Teaching Code is an m by m ***elimination matrix*** that puts the original A (whatever it was) into its row reduced form R:

$$E\,A = R.$$

The square matrix E is the product of elementary matrices E_{ij} and also P_{ij} and D^{-1}. P_{ij} exchanges rows. The diagonal D^{-1} divides rows by their pivots to produce 1's.

If we want E, we can apply row reduction to the matrix $\begin{bmatrix} A & I \end{bmatrix}$ with $n + m$ columns. All the elementary matrices that multiply A (to produce R) will also multiply I (to produce E). The whole augmented matrix is being multiplied by E:

$$E\,[A \;\; I] \quad = \quad [R \;\; E] \tag{7}$$

This is exactly what "Gauss-Jordan" did in Chapter 2 to compute A^{-1}. ***When A is square and invertible, its reduced row echelon form is I.*** Then $EA = R$ becomes $EA = I$. In this invertible case, E is A^{-1}. This chapter is going further, to every A.

■ REVIEW OF THE KEY IDEAS ■

1. The rank r of A is the number of pivots (which are 1's in $R =$ rref(A)).
2. The r pivot columns of A and R are in the same list *pivcol*.
3. Those r pivot columns are not combinations of earlier columns.
4. The $n - r$ free columns *are* combinations of earlier columns (pivot columns).
5. Those combinations (using $-F$ taken from R) give the $n - r$ special solutions to $Ax = \mathbf{0}$ and $Rx = \mathbf{0}$. They are the $n - r$ columns of the nullspace matrix N.

■ WORKED EXAMPLES ■

3.3 A Find the reduced echelon form of A. What is the rank? What is the special solution to $Ax = \mathbf{0}$?

Second differences $-1, 2, -1$
Notice $A_{11} = A_{44} = 1$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution Add row 1 to row 2. Then add row 2 to row 3. Then add row 3 to row 4:

First differences 1, −1 $$U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now add row 3 to row 2. Then add row 2 to row 1:

Reduced form $$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

The rank is $r = 3$. There is one free variable ($n - r = 1$). *The special solution is* $s = (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. *Every row adds to* 0. Notice $-F = (1, 1, 1)$ in the pivot variables of s.

3.3 B Factor these rank one matrices into $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{(find } d \text{ from } a, b, c \text{ if } a \neq 0)$$

Split this rank two matrix into $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$ = (3 by 2) times (2 by 4) using R:

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E^{-1}R.$$

Solution For the 3 by 3 matrix A, all rows are multiples of $\boldsymbol{v}^{\mathrm{T}} = [1 \ \ 2 \ \ 3]$. All columns are multiples of the column $\boldsymbol{u} = (1, 2, 3)$. This symmetric matrix has $\boldsymbol{u} = \boldsymbol{v}$ and A is $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. Every rank one symmetric matrix will have this form or else $-\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

If the 2 by 2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has rank one, it must be singular. In Chapter 5, its determinant is $ad - bc = 0$. In this chapter, row 2 is c/a times row 1.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 \\ c/a \end{bmatrix} [a \quad b] = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}. \quad \text{So} \quad d = \frac{bc}{a}.$$

The 3 by 4 matrix of rank two is a sum of *two matrices of rank one*. All columns of A are combinations of the pivot columns 1 and 2. All rows are combinations of the nonzero rows of R. The pivot columns are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and those rows are $\boldsymbol{v}_1^{\mathrm{T}}$ and $\boldsymbol{v}_2^{\mathrm{T}}$. Then A is $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$, multiplying r columns of E^{-1} times r rows of R:

Columns times rows $$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} [1 \quad 0 \quad 0 \quad 1] + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [0 \quad 1 \quad 0 \quad 1]$$

3.3 C Find the row reduced form R and the rank r of A and B (*those depend on* c). Which are the pivot columns of A? What are the special solutions and the matrix N?

Find special solutions $$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & c \\ c & c \end{bmatrix}.$$

Solution The matrix A has rank $r = 2$ ***except if*** $c = 4$. The pivots are in columns 1 and 3. The second variable x_2 is free. Notice the form of R:

$$c \neq 4 \quad R = \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad c = 4 \quad R = \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two pivots leave one free variable x_2. But when $c = 4$, the only pivot is in column 1 (rank one). The second and third variables are free, producing two special solutions:

$$c \neq 4 \quad \text{Special solution with } x_2 = 1 \text{ goes into} \quad N = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$$c = 4 \quad \text{Another special solution goes into} \quad N = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 2 by 2 matrix $\left[\begin{smallmatrix} c & c \\ c & c \end{smallmatrix}\right]$ has rank $r = 1$ ***except if*** $c = 0$, when the rank is zero!

$$c \neq 0 \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{Nullspace = line}$$

The matrix has *no pivot columns* if $c = 0$. Then both variables are free:

$$c = 0 \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Nullspace} = \mathbf{R}^2.$$

Problem Set 3.3

1 Which of these rules gives a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.

(b) The number of columns minus the total number of rows.

(c) The number of columns minus the number of free columns.

(d) The number of 1's in the matrix R.

2 Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix with all entries equal to 4.

(b) The 3 by 4 matrix with $a_{ij} = i + j - 1$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

3 Find the reduced R for each of these (block) matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

4 Suppose all the pivot variables come *last* instead of first. Describe all four blocks in the reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N containing the special solutions?

5 (Silly problem) Describe all 2 by 3 matrices A_1 and A_2, with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is is true that $R_1 = A_1$ and $R_2 = A_2$ in this case? Does $R_1 - R_2$ equal $\mathbf{rref}(A_1 - A_2)$?

6 If A has r pivot columns, how do you know that A^T has r pivot columns? Give a 3 by 3 example with different column numbers in *pivcol* for A and A^T.

7 What are the special solutions to $R\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}^T R = \mathbf{0}$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problems 8–11 are about matrices of rank $r = 1$.

8 Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

9 If A is an m by n matrix with $r = 1$, its columns are multiples of one column and its rows are multiples of one row. The column space is a _____ in $\mathbf{R}^m$. The nullspace is a _____ in $\mathbf{R}^n$. The nullspace matrix N has shape _____.

10 Choose vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $A = \boldsymbol{u}\boldsymbol{v}^T$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

$A = \boldsymbol{u}\boldsymbol{v}^T$ is the natural form for every matrix that has rank $r = 1$.

11 If A is a rank one matrix, the second row of U is _____. Do an example.

Problems 12–14 are about r by r invertible matrices inside A.

12 *If A has rank r, then it has an r by r submatrix S that is invertible.* Remove $m-r$ rows and $n-r$ columns to find an invertible submatrix S inside A, B, and C. You could keep the pivot rows and pivot columns:

$$A=\begin{bmatrix}1&2&3\\1&2&4\end{bmatrix} \qquad B=\begin{bmatrix}1&2&3\\2&4&6\end{bmatrix} \qquad C=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}.$$

13 Suppose P contains only the r pivot columns of an m by n matrix. Explain why this m by r submatrix P has rank r.

14 Transpose P in problem 13. Then find the r pivot columns of P^{T}. Transposing back, **this produces an r by r invertible submatrix S inside P and A:**

$$\text{For } A=\begin{bmatrix}1&2&3\\2&4&6\\2&4&7\end{bmatrix} \text{ find } P \text{ (3 by 2) and then the invertible } S \text{ (2 by 2).}$$

Problems 15–20 show that rank(AB) is not greater than rank(A) or rank(B).

15 Find the ranks of AB and AC (rank one matrix times rank one matrix):

$$A=\begin{bmatrix}1&2\\2&4\end{bmatrix} \text{ and } B=\begin{bmatrix}2&1&4\\3&1.5&6\end{bmatrix} \text{ and } C=\begin{bmatrix}1&b\\c&bc\end{bmatrix}.$$

16 The rank one matrix $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ times the rank one matrix $\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is $\boldsymbol{u}\boldsymbol{z}^{\mathrm{T}}$ times the number _____. This product $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ also has rank one unless _____ $=0$.

17 (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so **rank(AB) $\le$ rank(B)**.

(b) Find A_1 and A_2 so that rank(A_1B) $=1$ and rank(A_2B) $=0$ for $B=\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$.

18 Problem 17 proved that rank(AB) $\le$ rank(B). Then the same reasoning gives **rank($B^{\mathrm{T}}A^{\mathrm{T}}$)** $\le$ rank(A^{T}). How do you deduce that **rank(AB) $\le$ rank A**?

19 *(Important)* Suppose A and B are n by n matrices, and $AB=I$. Prove from rank(AB) $\le$ rank(A) that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA=I$ *(which is not so obvious!)*.

20 If A is 2 by 3 and B is 3 by 2 and $AB=I$, show from its rank that $BA\ne I$. Give an example of A and B with $AB=I$. For $m<n$, a right inverse is not a left inverse.

21 Suppose A and B have the *same* reduced row echelon form R.

(a) Show that A and B have the same nullspace and the same row space.

(b) We know $E_1 A = R$ and $E_2 B = R$. So A equals an ______ matrix times B.

22 Express A and then B as the sum of two rank one matrices:

$$\textbf{rank} = 2 \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

23 Answer the same questions as in Worked Example **3.3 C** for

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

24 What is the nullspace matrix N (containing the special solutions) for A, B, C?

$$A = [\, I \;\; I \,] \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = [\, I \;\; I \;\; I \,].$$

25 *Neat fact* ***Every m by n matrix of rank r reduces to*** $(m$ by $r)$ ***times*** $(r$ by $n)$:

$$A = (\text{pivot columns of } A)\,(\text{first } r \text{ rows of } R) = (\mathbf{COL})(\mathbf{ROW}).$$

Write the 3 by 4 matrix A in equation (1) at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R.

Challenge Problems

26 Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the matrix Z (its shape and all its entries) that comes from *transposing the reduced row echelon form of* R' (prime means transpose):

$$R = \text{rref}(A) \quad \text{and} \quad Z = (\text{rref}(R'))'.$$

27 Suppose R is m by n of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

(a) What are the shapes of those four blocks?

(b) Find a *right-inverse* B with $RB = I$ if $r = m$.

(c) Find a *left-inverse* C with $CR = I$ if $r = n$.

(d) What is the reduced row echelon form of R^{T} (with shapes)?

(e) What is the reduced row echelon form of $R^{\mathrm{T}}R$ (with shapes)?

Prove that $R^{\mathrm{T}}R$ has the same nullspace as R. Later we show that $A^{\mathrm{T}}A$ always has the same nullspace as A (a valuable fact).

28 Suppose you allow elementary *column* operations on A as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank r?

3.4 The Complete Solution to $Ax = b$

The last sections totally solved $Ax = \mathbf{0}$. Elimination converted the problem to $Rx = \mathbf{0}$. The free variables were given special values (one and zero). Then the pivot variables were found by back substitution. We paid no attention to the right side b because it started and ended as zero. The solution x was in the nullspace of A.

Now b is not zero. Row operations on the left side must act also on the right side. $Ax = b$ is reduced to a simpler system $Rx = d$. One way to organize that is to ***add b as an extra column of the matrix***. I will *"augment"* A with the right side $(b_1, b_2, b_3) = (1, 6, 7)$ and reduce the bigger matrix $\begin{bmatrix} A & b \end{bmatrix}$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \begin{matrix} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 1 & 3 & 1 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} A & b \end{bmatrix}.$$

The augmented matrix is just $\begin{bmatrix} A & b \end{bmatrix}$. When we apply the usual elimination steps to A, we also apply them to b. That keeps all the equations correct.

In this example we subtract row 1 from row 3 and then subtract row 2 from row 3. This produces a *complete row of zeros* in R, and it changes b to a new right side $d = (1, 6, 0)$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \quad \begin{matrix} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R & d \end{bmatrix}.$$

That very last zero is crucial. The third equation has become $0 = 0$ and the equations can be solved. In the original matrix A, the first row plus the second row equals the third row. If the equations are consistent, this must be true on the right side of the equations also! The all-important property on the right side was $1 + 6 = 7$.

Here are the same augmented matrices for a general $b = (b_1, b_2, b_3)$:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = \begin{bmatrix} R & d \end{bmatrix}$$

Now we get $0 = 0$ in the third equation provided $b_3 - b_1 - b_2 = 0$. This is $b_1 + b_2 = b_3$.

One Particular Solution

For an easy solution x, *choose the free variables to be* $x_2 = x_4 = 0$. Then the two nonzero equations give the two pivot variables $x_1 = 1$ and $x_3 = 6$. Our particular solution to $Ax = b$ (and also $Rx = d$) is $x_p = (1, 0, 6, 0)$. This particular solution is my favorite: *free variables = zero, pivot variables from* d. The method always works.

For a solution to exist, zero rows in R must also be zero in d. Since I is in the pivot rows and pivot columns of R, the pivot variables in $x_{\text{particular}}$ come from d:

$$Rx_p = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 \\ \mathbf{0} & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{6} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{6} \\ 0 \end{bmatrix} \qquad \begin{array}{l} \textbf{Pivot variables 1, 6} \\ \textbf{Free variables } 0, 0 \end{array}$$

Notice how we *choose* the free variables (as zero) and *solve* for the pivot variables. After the row reduction to R, those steps are quick. When the free variables are zero, the pivot variables for x_p are already seen already seen in the right side vector d.

$x_{\text{particular}}$	***The particular solution solves***	$Ax_p = b$
$x_{\text{nullspace}}$	***The $n - r$ special solutions solve***	$Ax_n = \mathbf{0}$.

That particular solution is $(1, 0, 6, 0)$. The two special (nullspace) solutions to $Rx = \mathbf{0}$ come from the two free columns of R, by reversing signs of $3, 2$, and 4. *Please notice how I write the complete solution $x_p + x_n$ to $Ax = b$:*

Complete solution
one x_p
many x_n

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Question Suppose A is a square invertible matrix, $m = n = r$. What are x_p and x_n?
Answer The particular solution is the one and *only* solution $A^{-1}b$. There are no special solutions or free variables. $R = I$ has no zero rows. The only vector in the nullspace is $x_n = \mathbf{0}$. The complete solution is $x = x_p + x_n = A^{-1}b + \mathbf{0}$.

This was the situation in Chapter 2. We didn't mention the nullspace in that chapter. $N(A)$ contained only the zero vector. Reduction goes from $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1}b \end{bmatrix}$. The original $Ax = b$ is reduced all the way to $x = A^{-1}b$ which is d. This is a special case here, but square invertible matrices are the ones we see most often in practice. So they got their own chapter at the start of the book.

For small examples we can reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} R & d \end{bmatrix}$. For a large matrix, MATLAB does it better. One particular solution (not necessarily ours) is $A\backslash b$ from backslash. Here is an example with *full column rank*. Both columns have pivots.

Example 1 Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This condition puts b in the column space of A. Find the complete $x = x_p + x_n$.

Solution Use the augmented matrix, with its extra column $\boldsymbol{b}$. Subtract row 1 of $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ from row 2, and add 2 times row 1 to row 3 to reach $\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix}.$$

The last equation is $0 = 0$ provided $b_3 + b_1 + b_2 = 0$. This is the condition to put $\boldsymbol{b}$ in the column space; then $A\boldsymbol{x} = \boldsymbol{b}$ will be solvable. The rows of A add to the zero row. So for consistency (these are equations!) the entries of $\boldsymbol{b}$ must also add to zero.

This example has no free variables since $n - r = 2 - 2$. Therefore no special solutions. The nullspace solution is $\boldsymbol{x}_n = \boldsymbol{0}$. The particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and $R\boldsymbol{x} = \boldsymbol{d}$ is at the top of the augmented column $\boldsymbol{d}$:

$$\textbf{Only solution} \qquad \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $b_3 + b_1 + b_2$ is not zero, there is no solution to $A\boldsymbol{x} = \boldsymbol{b}$ ($\boldsymbol{x}_p$ doesn't exist).

This example is typical of an extremely important case: A has *full column rank*. Every column has a pivot. *The rank is $r = n$.* The matrix is tall and thin ($m \geq n$). Row reduction puts I at the top, when A is reduced to R with rank n:

$$\textbf{Full column rank} \quad R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} \tag{1}$$

There are no free columns or free variables. The nullspace matrix is empty!

We will collect together the different ways of recognizing this type of matrix.

Every matrix A with **full column rank** ($r = n$) has all these properties:

1. All columns of A are pivot columns.
2. There are no free variables or special solutions.
3. The nullspace $\boldsymbol{N}(A)$ contains only the zero vector $\boldsymbol{x} = \boldsymbol{0}$.
4. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution (it might not) then it has only *one solution*.

In the essential language of the next section, **this A has *independent columns*.** $A\boldsymbol{x} = \boldsymbol{0}$ only happens when $\boldsymbol{x} = \boldsymbol{0}$. In Chapter 4 we will add one more fact to the list: *The square matrix $A^{\mathrm{T}}A$ is invertible when the rank is n.*

In this case the nullspace of A (and R) has shrunk to the zero vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is *unique* (if it exists). There will be $m - n$ (here $3 - 2$) zero rows in R. So there are $m - n$ conditions in order to have $0 = 0$ in those rows, and $\boldsymbol{b}$ in the column space. With full column rank, $A\boldsymbol{x} = \boldsymbol{b}$ has *one* solution or *no* solution ($m > n$ is overdetermined).

Problem Set 3.4

1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $Ax = b$ to be solvable. Together these two conditions put b into the _____ space (two planes give a line):

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} [2 \quad 1 \quad 3] = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

Questions 3–15 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Use the augmented matrix with last column b.

3 Write the complete solution as x_p plus any multiple of s in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include b as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find x in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

3.5 Independence, Basis and Dimension

This important section is about the true size of a subspace. There are n columns in an m by n matrix. But the true "dimension" of the column space is not necessarily n. The dimension is measured by counting ***independent columns***—and we have to say what that means. We will see that *the true dimension of the column space is the rank* r.

The idea of independence applies to any vectors $v_1, \ldots, v_n$ in any vector space. Most of this section concentrates on the subspaces that we know and use—especially the column space and the nullspace of A. In the last part we also study "vectors" that are not column vectors. They can be matrices and functions; they can be linearly independent (or dependent). First come the key examples using column vectors.

The goal is to understand a ***basis***: **independent vectors that "span the space"**.

Every vector in the space is a unique combination of the basis vectors.

We are at the heart of our subject, and we cannot go on without a basis. The four essential ideas in this section (with first hints at their meaning) are:

1. Independent vectors	*(no extra vectors)*
2. Spanning a space	*(enough vectors to produce the rest)*
3. Basis for a space	*(not too many or too few)*
4. Dimension of a space	*(the number of vectors in a basis)*

Linear Independence

Our first definition of independence is not so conventional, but you are ready for it.

DEFINITION The columns of A are *linearly independent* when the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. ***No other combination*** Ax ***of the columns gives the zero vector.***

The columns are independent when the nullspace $N(A)$ contains only the zero vector. Let me illustrate linear independence (and dependence) with three vectors in $\mathbf{R}^3$:

1. If three vectors are *not* in the same plane, they are independent. No combination of v_1, v_2, v_3 in Figure 3.4 gives zero except $0v_1 + 0v_2 + 0v_3$.

2. If three vectors w_1, w_2, w_3 are *in the same plane*, they are dependent.

This idea of independence applies to 7 vectors in 12-dimensional space. If they are the columns of A, and independent, the nullspace only contains $x = \mathbf{0}$. None of the vectors is a combination of the other six vectors.

Now we choose different words to express the same idea. The following definition of independence will apply to any sequence of vectors in any vector space. When the vectors are the columns of A, the two definitions say exactly the same thing.

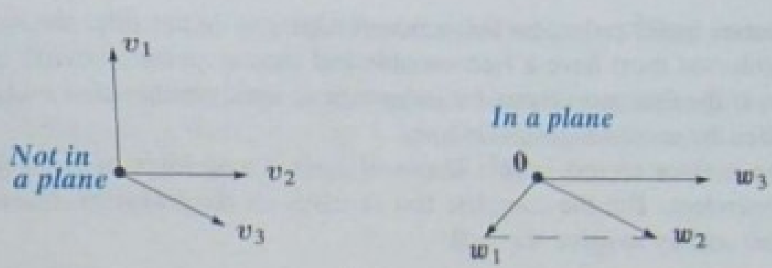


Figure 3.4: Independent vectors v_1, v_2, v_3. Only $0v_1 + 0v_2 + 0v_3$ gives the vector $\mathbf{0}$. Dependent vectors w_1, w_2, w_3. The combination $w_1 - w_2 + w_3$ is $(0, 0, 0)$.

DEFINITION The sequence of vectors $v_1, \ldots, v_n$ is ***linearly independent*** if the only combination that gives the zero vector is $0v_1 + 0v_2 + \cdots + 0v_n$.

Linear independence

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = \mathbf{0} \quad \text{only happens when all } x\text{'s are zero.} \tag{1}$$

If a combination gives $\mathbf{0}$, when the x's are not all zero, the vectors are *dependent*.

Correct language: "The sequence of vectors is linearly independent." *Acceptable shortcut*: "The vectors are independent." *Unacceptable*: "The matrix is independent."

A sequence of vectors is either dependent or independent. They can be combined to give the zero vector (with nonzero x's) or they can't. So the key question is: Which combinations of the vectors give zero? We begin with some small examples in $\mathbf{R}^2$:

(a) The vectors $(1, 0)$ and $(0, 1)$ are independent.

(b) The vectors $(1, 0)$ and $(1, 0.00001)$ are independent.

(c) The vectors $(1, 1)$ and $(-1, -1)$ are *dependent*.

(d) The vectors $(1, 1)$ and $(0, 0)$ are *dependent* because of the zero vector.

(e) In $\mathbf{R}^2$, any three vectors (a, b) and (c, d) and (e, f) are *dependent*.

Geometrically, $(1, 1)$ and $(-1, -1)$ are on a line through the origin. They are dependent. To use the definition, find numbers x_1 and x_2 so that $x_1(1, 1) + x_2(-1, -1) = (0, 0)$. This is the same as solving $Ax = \mathbf{0}$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } x_1 = 1 \text{ and } x_2 = 1.$$

The columns are dependent exactly when *there is a nonzero vector in the nullspace*.

If one of the v's is the zero vector, independence has no chance. Why not?

Three vectors in $\mathbf{R}^2$ cannot be independent! One way to see this: the matrix A with those three columns must have a free variable and then a special solution to $Ax = \mathbf{0}$. Another way: If the first two vectors are independent, some combination will produce the third vector. See the second highlight below.

Now move to three vectors in $\mathbf{R}^3$. If one of them is a multiple of another one, these vectors are dependent. But the complete test involves all three vectors at once. We put them in a matrix and try to solve $Ax = \mathbf{0}$.

Example 1 The columns of this A are dependent. $Ax = \mathbf{0}$ has a nonzero solution:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank is only $r = 2$. *Independent columns produce full column rank $r = n = 3$.*

In that matrix the rows are also dependent. Row 1 minus row 3 is the zero row. For a *square matrix*, we will show that dependent columns imply dependent rows.

Question How to find that solution to $Ax = \mathbf{0}$? The systematic way is elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution $x = (-3, 1, 1)$ was exactly the special solution. It shows how the free column (column 3) is a combination of the pivot columns. That kills independence!

Full column rank The columns of A are independent exactly when the rank is $r = n$. There are n pivots and no free variables. Only $x = \mathbf{0}$ is in the nullspace.

One case is of special importance because it is clear from the start. Suppose seven columns have five components each ($m = 5$ is less than $n = 7$). Then the columns *must be dependent*. Any seven vectors from $\mathbf{R}^5$ are dependent. The rank of A cannot be larger than 5. There cannot be more than five pivots in five rows. $Ax = \mathbf{0}$ has at least $7 - 5 = 2$ free variables, so it has nonzero solutions—which means that the columns are dependent.

Any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$.

This type of matrix has more columns than rows—it is short and wide. The columns are certainly dependent if $n > m$, because $Ax = \mathbf{0}$ has a nonzero solution.

The columns might be dependent or might be independent if $n \le m$. Elimination will reveal the r pivot columns. *It is those r pivot columns that are independent.*

Note Another way to describe linear dependence is this: "*One vector is a combination of the other vectors.*" That sounds clear. Why don't we say this from the start? Our definition was longer: "*Some combination gives the zero vector, other than the trivial combination with every $x = 0$.*" We must rule out the easy way to get the zero vector.

That trivial combination of zeros gives every author a headache. If one vector is a combination of the others, that vector has coefficient $x = 1$.

The point is, our definition doesn't pick out one particular vector as guilty. All columns of A are treated the same. We look at $Ax = \mathbf{0}$, and it has a nonzero solution or it hasn't. In the end that is better than asking if the last column (or the first, or a column in the middle) is a combination of the others.

Vectors that Span a Subspace

The first subspace in this book was the column space. Starting with columns $v_1, \ldots, v_n$, the subspace was filled out by including all combinations $x_1v_1 + \cdots + x_nv_n$. *The column space consists of all combinations Ax of the columns.* We now introduce the single word "span" to describe this: The column space is ***spanned*** by the columns.

DEFINITION A set of vectors ***spans*** a space if their linear combinations fill the space.

The columns of a matrix span its column space. They might be dependent.

Example 2 $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space $\mathbf{R}^2$.

Example 3 $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbf{R}^2$.

Example 4 $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbf{R}^2$. So does w_1 by itself.

Think of two vectors coming out from $(0, 0, 0)$ in 3-dimensional space. Generally they span a plane. Your mind fills in that plane by taking linear combinations. Mathematically you know other possibilities: two vectors could span a line, three vectors could span all of $\mathbf{R}^3$, or only a plane. It is even possible that three vectors span only a line, or ten vectors span only a plane. They are certainly not independent!

The columns span the column space. Here is a new subspace—*which is spanned by the rows.* ***The combinations of the rows produce the "row space".***

DEFINITION The ***row space*** of a matrix is the subspace of $\mathbf{R}^n$ spanned by the rows.
The row space of A is $C(A^T)$. It is the column space of A^T.

The rows of an m by n matrix have n components. They are vectors in $\mathbf{R}^n$—or they would be if they were written as column vectors. There is a quick way to fix that: *Transpose the matrix.* Instead of the rows of A, look at the columns of A^T. Same numbers, but now in the column space $C(A^T)$. This row space of A is a subspace of $\mathbf{R}^n$.

Example 5 Describe the column space and the row space of A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}. \text{ Here } m = 3 \text{ and } n = 2.$$

The column space of A is the plane in $\mathbf{R}^3$ spanned by the two columns of A. *The row space of A is spanned by the three rows of A* (which are columns of A^{T}). This row space is all of $\mathbf{R}^2$. Remember: The rows are in $\mathbf{R}^n$ spanning the row space. The columns are in $\mathbf{R}^m$ spanning the column space. Same numbers, different vectors, different spaces.

A Basis for a Vector Space

Two vectors can't span all of $\mathbf{R}^3$, even if they are independent. Four vectors can't be independent, even if they span $\mathbf{R}^3$. We want ***enough independent vectors to span the space*** (and not more). A "***basis***" is just right.

DEFINITION A *basis* for a vector space is a sequence of vectors with two properties:

The basis vectors are linearly independent and they span the space.

This combination of properties is fundamental to linear algebra. Every vector v in the space is a combination of the basis vectors, because they span the space. More than that, the combination that produces v is *unique*, because the basis vectors $v_1, \ldots, v_n$ are independent:

There is one and only one way to write v as a combination of the basis vectors.

Reason: Suppose $v = a_1v_1 + \cdots + a_nv_n$ and also $v = b_1v_1 + \cdots + b_nv_n$. By subtraction $(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$ is the zero vector. From the independence of the v's, each $a_i - b_i = 0$. Hence $a_i = b_i$, and there are not two ways to produce v.

Example 6 The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbf{R}^2$.

$$\text{The basis vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ are independent. They span } \mathbf{R}^2.$$

Everybody thinks of this basis first. The vector $\boldsymbol{i}$ goes across and $\boldsymbol{j}$ goes straight up. The columns of the 3 by 3 identity matrix are the standard basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The columns of the n by n identity matrix give the "**standard basis**" for $\mathbf{R}^n$.

Now we find many other bases (infinitely many). The basis is not unique!

Example 7 (Important) The columns of *every invertible n by n matrix* give a basis for $\mathbf{R}^n$:

Invertible matrix Independent columns Column space is $\mathbf{R}^3$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ **Singular matrix** Dependent columns Column space $\neq \mathbf{R}^3$ $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$. The columns are independent. They span the whole space $\mathbf{R}^n$—because every vector $\boldsymbol{b}$ is a combination of the columns. $A\boldsymbol{x} = \boldsymbol{b}$ can always be solved by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. Do you see how everything comes together for invertible matrices? Here it is in one sentence:

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a ***basis for*** $\mathbf{R}^n$ exactly when they are ***the columns of an n by n invertible matrix***. Thus $\mathbf{R}^n$ has infinitely many different bases.

When the columns are dependent, we keep only the *pivot columns*—the first two columns of B above, with its two pivots. They are independent and they span the column space.

The pivot columns of A are a basis for its column space. The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R.

Example 8 This matrix is not invertible. Its columns are not a basis for anything!

One pivot column
One pivot row ($r = 1$) $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Column 1 of A is the pivot column. That column alone is a basis for its column space. The second column of A would be a different basis. So would any nonzero multiple of that column. There is no shortage of bases. One definite choice is the pivot columns.

Notice that the pivot column $(1, 0)$ of this R ends in zero. That column is a basis for the column space of R, but it doesn't belong to the column space of A. The column spaces of A and R are different. Their bases are different. (Their dimensions are the same.)

The row space of A is the *same* as the row space of R. It contains $(2, 4)$ and $(1, 2)$ and all other multiples of those vectors. As always, there are infinitely many bases to choose from. One natural choice is to pick the nonzero rows of R (rows with a pivot). So this matrix A with rank one has only one vector in the basis:

$$\text{Basis for the column space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \quad \text{Basis for the row space: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The next chapter will come back to these bases for the column space and row space. We are happy first with examples where the situation is clear (and the idea of a basis is still new). The next example is larger but still clear.

Example 9 Find bases for the column and row spaces of this rank two matrix:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 are the pivot columns. They are a basis for the column space (of R!). The vectors in that column space all have the form $b = (x, y, 0)$. The column space of R is the "xy plane" inside the full 3-dimensional xyz space. That plane is not $\mathbf{R}^2$, it is a subspace of $\mathbf{R}^3$. Columns 2 and 3 are also a basis for the same column space. Which pairs of columns of R are *not* a basis for its column space?

The row space of R is a subspace of $\mathbf{R}^4$. The simplest basis for that row space is the two nonzero rows of R. The third row (the zero vector) is in the row space too. But it is not in a *basis* for the row space. The basis vectors must be independent.

Question Given five vectors in $\mathbf{R}^7$, ***how do you find a basis for the space they span?***

First answer Make them the rows of A, and eliminate to find the nonzero rows of R.
Second answer Put the five vectors into the columns of A. Eliminate to find the pivot columns (of A not R). The program **colbasis** uses the column numbers from *pivcol*.

Could another basis have more vectors, or fewer? This is a crucial question with a good answer: *No*. ***All bases for a vector space contain the same number of vectors.***

The number of vectors, in any and every basis, is the "dimension" of the space.

Dimension of a Vector Space

We have to prove what was just stated. There are many choices for the basis vectors, but the *number* of basis vectors doesn't change.

If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$.

Proof Suppose that there are more w's than v's. From $n > m$ we want to reach a contradiction. The v's are a basis, so w_1 must be a combination of the v's. If w_1 equals $a_{11}v_1 + \cdots + a_{m1}v_m$, this is the first column of a matrix multiplication VA:

Each w is a combination of the v's
$$W = \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = VA.$$

We don't know each a_{ij}, but we know the shape of A (it is m by n). The second vector w_2 is also a combination of the v's. The coefficients in that combination fill the second column of A. The key is that A has a row for every v and a column for every w. A is a short wide matrix, since we assumed $n > m$. So $Ax = \mathbf{0}$ ***has a nonzero solution***.

$Ax = 0$ gives $VAx = \mathbf{0}$ which is $Wx = \mathbf{0}$. *A combination of the w's gives zero*! Then the w's could not be a basis—our assumption $n > m$ is **not possible** for two bases.

If $m > n$ we exchange the v's and w's and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$.

The number of basis vectors depends on the space—not on a particular basis. The number is the same for every basis, and it counts the "degrees of freedom" in the space.

The dimension of the space $\mathbf{R}^n$ is n. We now introduce the important word ***dimension*** for other vector spaces too.

DEFINITION The ***dimension of a space*** is the ***number of vectors*** in every basis.

This matches our intuition. The line through $v = (1, 5, 2)$ has dimension one. It is a subspace with this one vector v in its basis. Perpendicular to that line is the plane $x + 5y + 2z = 0$. This plane has dimension 2. To prove it, we find a basis $(-5, 1, 0)$ and $(-2, 0, 1)$. The dimension is 2 because the basis contains two vectors.

The plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$, which has two free variables. Our basis vectors $(-5, 1, 0)$ and $(-2, 0, 1)$ are the "special solutions" to $Ax = \mathbf{0}$. The next section shows that the $n - r$ special solutions always give *a basis for the nullspace*. $C(A)$ has dimension r and the nullspace $N(A)$ has dimension $n - r$.

Note about the language of linear algebra We never say "the rank of a space" or "the dimension of a basis" or "the basis of a matrix". Those terms have no meaning. It is the ***dimension of the column space*** that equals the ***rank of the matrix.***

Bases for Matrix Spaces and Function Spaces

The words "independence" and "basis" and "dimension" are not at all restricted to column vectors. We can ask whether three matrices A_1, A_2, A_3 are independent. When they are in the space of all 3 by 4 matrices, some combination might give the zero matrix. We can also ask the dimension of the full 3 by 4 matrix space. (It is 12.)

In differential equations, $d^2y/dx^2 = y$ has a space of solutions. One basis is $y = e^x$ and $y = e^{-x}$. Counting the basis functions gives the dimension 2 for the space of all solutions. (The dimension is 2 because of the second derivative.)

Matrix spaces and function spaces may look a little strange after $\mathbf{R}^n$. But in some way, you haven't got the ideas of basis and dimension straight until you can apply them to "vectors" other than column vectors.

Matrix spaces The vector space $\mathbf{M}$ contains all 2 by 2 matrices. Its dimension is 4.

$$\textbf{One basis is} \quad A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Those matrices are linearly independent. We are not looking at their columns, but at the whole matrix. Combinations of those four matrices can produce any matrix in $\mathbf{M}$, so they span the space:

$$\begin{array}{l}\textbf{Every } A \textbf{ combines} \\ \textbf{the basis matrices}\end{array} \qquad c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = A.$$

A is zero only if the c's are all zero—this proves independence of A_1, A_2, A_3, A_4.

The three matrices A_1, A_2, A_4 are a basis for a subspace—the upper triangular matrices. Its dimension is 3. A_1 and A_4 are a basis for the diagonal matrices. What is a basis for the symmetric matrices? Keep A_1 and A_4, and throw in $A_2 + A_3$.

To push this further, think about the space of all n by n matrices. One possible basis uses matrices that have only a single nonzero entry (that entry is 1). There are n^2 positions for that 1, so there are n^2 basis matrices:

The dimension of the whole n by n matrix space is n^2.

The dimension of the subspace of *upper triangular* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$.

The dimension of the subspace of *diagonal* matrices is n.

The dimension of the subspace of *symmetric* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$ (why?).

Function spaces The equations $d^2y/dx^2 = 0$ and $d^2y/dx^2 = -y$ and $d^2y/dx^2 = y$ involve the second derivative. In calculus we solve to find the functions $y(x)$:

$$\begin{array}{ll} y'' = 0 & \text{is solved by any linear function } y = cx + d \\ y'' = -y & \text{is solved by any combination } y = c\sin x + d\cos x \\ y'' = y & \text{is solved by any combination } y = ce^x + de^{-x}. \end{array}$$

That solution space for $y'' = -y$ has two basis functions: $\sin x$ and $\cos x$. The space for $y'' = 0$ has x and 1. It is the "nullspace" of the second derivative! The dimension is 2 in each case (these are second-order equations).

The solutions of $y'' = 2$ don't form a subspace—the right side $b = 2$ is not zero. A particular solution is $y(x) = x^2$. The complete solution is $y(x) = x^2 + cx + d$. All those functions satisfy $y'' = 2$. Notice the particular solution plus any function $cx + d$ in the nullspace. A linear differential equation is like a linear matrix equation $Ax = b$. But we solve it by calculus instead of linear algebra.

We end here with the space $\mathbf{Z}$ that contains only the zero vector. The dimension of this space is *zero*. ***The empty set*** (containing no vectors) ***is a basis for*** $\mathbf{Z}$. We can never allow the zero vector into a basis, because then linear independence is lost.

■ REVIEW OF THE KEY IDEAS ■

1. The columns of A are ***independent*** if $\boldsymbol{x} = \mathbf{0}$ is the only solution to $A\boldsymbol{x} = \mathbf{0}$.
2. The vectors $v_1, \ldots, v_r$ ***span*** a space if their combinations fill that space.
3. ***A basis consists of linearly independent vectors that span the space.*** Every vector in the space is a *unique* combination of the basis vectors.
4. All bases for a space have the same number of vectors. This number of vectors in a basis is the ***dimension*** of the space.
5. The pivot columns are one basis for the column space. The dimension is r.

▪ WORKED EXAMPLES ▪

3.5 A Start with the vectors $v_1 = (1,2,0)$ and $v_2 = (2,3,0)$. **(a)** Are they linearly independent? **(b)** Are they a basis for any space? **(c)** What space **V** do they span? **(d)** What is the dimension of **V**? **(e)** Which matrices A have **V** as their column space? **(f)** Which matrices have **V** as their nullspace? **(g)** Describe all vectors v_3 that complete a basis v_1, v_2, v_3 for $\mathbf{R}^3$.

Solution

(a) v_1 and v_2 are independent—the only combination to give $\mathbf{0}$ is $0v_1 + 0v_2$.

(b) Yes, they are a basis for the space they span.

(c) That space **V** contains all vectors $(x, y, 0)$. It is the xy plane in $\mathbf{R}^3$.

(d) The dimension of **V** is 2 since the basis contains two vectors.

(e) This **V** is the column space of any 3 by n matrix A of rank 2, if every column is a combination of v_1 and v_2. In particular A could just have columns v_1 and v_2.

(f) This **V** is the nullspace of any m by 3 matrix B of rank 1, if every row is a multiple of $(0,0,1)$. In particular take $B = [0\ 0\ 1]$. Then $Bv_1 = \mathbf{0}$ and $Bv_2 = \mathbf{0}$.

(g) Any third vector $v_3 = (a, b, c)$ will complete a basis for $\mathbf{R}^3$ provided $c \neq 0$.

3.5 B Start with three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in matrix form as $V = WM$:

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_3 &= \quad\ \ w_2 + cw_3 \end{aligned} \quad \text{which is} \quad \begin{bmatrix} & & \\ v_1 & v_2 & v_3 \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ w_1 & w_2 & w_3 \\ & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix V to see if its columns are linearly independent? If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent. If $c = 1$ show that the v's are linearly *dependent*.

Solution The test on V for independence of its columns was in our first definition: *The nullspace of V must contain only the zero vector.* Then $x = (0,0,0)$ is the only combination of the columns that gives Vx = zero vector.

If $c = 1$ in our problem, we can see *dependence* in two ways. First, $v_1 + v_3$ will be the same as v_2. (If you add $w_1 + w_2$ to $w_2 + w_3$ you get $w_1 + 2w_2 + w_3$ which is v_2.) In other words $v_1 - v_2 + v_3 = \mathbf{0}$—which says that the v's are not independent.

The other way is to look at the nullspace of M. If $c = 1$, the vector $x = (1, -1, 1)$ is in that nullspace, and $Mx = \mathbf{0}$. Then certainly $WMx = \mathbf{0}$ which is the same as $Vx = 0$. So the v's are dependent. This specific $x = (1, -1, 1)$ from the nullspace tells us again that $v_1 - v_2 + v_3 = \mathbf{0}$.

Now suppose $c \neq 1$. Then the matrix M is invertible. So if x is *any nonzero vector* we know that Mx is nonzero. Since the w's are given as independent, we further know that WMx is nonzero. Since $V = WM$, this says that x is *not* in the nullspace of V. In other words v_1, v_2, v_3 are independent.

The general rule is "independent v's from independent w's when M is invertible". And if these vectors are in $\mathbf{R}^3$, they are not only independent—they are a basis for $\mathbf{R}^3$. "Basis of v's from basis of w's when the change of basis matrix M is invertible."

3.5 C (***Important example***) Suppose $v_1, \ldots, v_n$ is a basis for $\mathbf{R}^n$ and the n by n matrix A is invertible. Show that $Av_1, \ldots, Av_n$ is also a basis for $\mathbf{R}^n$.

Solution In *matrix language*: Put the basis vectors $v_1, \ldots, v_n$ in the columns of an invertible(!) matrix V. Then $Av_1, \ldots, Av_n$ are the columns of AV. Since A is invertible, so is AV and its columns give a basis.

In *vector language*: Suppose $c_1Av_1 + \cdots + c_nAv_n = \mathbf{0}$. This is $Av = \mathbf{0}$ with $v = c_1v_1 + \cdots + c_nv_n$. Multiply by A^{-1} to reach $v = \mathbf{0}$. By linear independence of the v's, all $c_i = 0$. This shows that the Av's are independent.

To show that the Av's span $\mathbf{R}^n$, solve $c_1Av_1 + \cdots + c_nAv_n = b$ which is the same as $c_1v_1 + \cdots + c_nv_n = A^{-1}b$. Since the v's are a basis, this must be solvable.

Problem Set 3.5

Questions 1–10 are about linear independence and linear dependence.

1 Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}$ or $Ax = \mathbf{0}$. The v's go in the columns of A.

2 (Recommended) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \; v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \; v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \; v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \; v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \; v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $Ux = \mathbf{0}$ is $x = \mathbf{0}$. Then the upper triangular U has independent columns.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

6 Choose three independent columns of U. Then make two other choices. Do the same for A.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero. Which matrix A in $[\,\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3\,] = [\,\boldsymbol{w}_1 \;\; \boldsymbol{w}_2 \;\; \boldsymbol{w}_3\,]\, A$ is singular?

8 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the sums $\boldsymbol{v}_1 = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 + \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ are *independent*. (Write $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$ in terms of the $\boldsymbol{w}$'s. Find and solve equations for the c's, to show they are zero.)

9 Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because _____.

(b) The two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will be dependent if _____.

(c) The vectors $\boldsymbol{v}_1$ and $(0, 0, 0)$ are dependent because _____.

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–15 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$

(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

12 The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when _____ has a solution. The vector $\boldsymbol{c}$ is in the row space of A when _____ has a solution.

True or false: If the zero vector is in the row space, the rows are dependent.

3.6 Dimensions of the Four Subspaces

The main theorem in this chapter connects ***rank*** and ***dimension***. The ***rank*** of a matrix is the number of pivots. The ***dimension*** of a subspace is the number of vectors in a basis. We count pivots or we count basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A, and the other two from A^T:

Four Fundamental Subspaces

1. The ***row space*** is $C(A^T)$, a subspace of $\mathbf{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbf{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\mathbf{R}^n$.
4. The ***left nullspace*** is $N(A^T)$, a subspace of $\mathbf{R}^m$. This is our new space.

In this book the column space and nullspace came first. We know $C(A)$ and $N(A)$ pretty well. Now the other two subspaces come forward. The row space contains all combinations of the rows. *This is the column space of A^T.*

For the left nullspace we solve $A^T y = 0$—that system is n by m. *This is the nullspace of A^T.* The vectors y go on the *left* side of A when the equation is written as $y^T A = 0^T$. The matrices A and A^T are usually different. So are their column spaces and their nullspaces. But those spaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: ***The row space and column space have the same dimension*** r (the rank of the matrix). The other important fact involves the two nullspaces:

$N(A)$ and $N(A^T)$ have dimensions $n - r$ and $m - r$, to make up the full n and m.

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together (two in $\mathbf{R}^n$ and two in $\mathbf{R}^m$). That completes the "right way" to understand every $Ax = b$. Stay with it—you are doing real mathematics.

The Four Subspaces for R

Suppose A is reduced to its row echelon form R. For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't change!) as we look back at A. The main point is that *the four dimensions are the same for A and R.*

As a specific 3 by 5 example, look at the four subspaces for the echelon matrix R:

$$\begin{matrix} m=3 \\ n=5 \\ r=2 \end{matrix} \quad \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \textbf{pivot rows 1 and 2} \\ \\ \textbf{pivot columns 1 and 4} \end{matrix}$$

The rank of this matrix R is $r = 2$ (*two pivots*). Take the four subspaces in order.

1. The ***row space*** of R has dimension 2, matching the rank.

Reason: The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing new. So rows 1 and 2 span the row space $C(R^T)$.

The pivot rows 1 and 2 are independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows are a basis for the row space.

The dimension of the row space is the rank r. The nonzero rows of R form a basis.

2. The ***column space*** of R also has dimension $r = 2$.

Reason: The pivot columns 1 and 4 form a basis for $C(R)$. They are independent because they start with the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns. Actually the combinations we need are the three special solutions!

Column 2 is 3 (column 1). The special solution is $(-3, 1, 0, 0, 0)$.

Column 3 is 5 (column 1). The special solution is $(-5, 0, 1, 0, 0,)$.

Column 5 is 7 (column 1) + 2 (column 4). That solution is $(-7, 0, 0, -2, 1)$.

The pivot columns are independent, and they span, so they are a basis for $C(R)$.

The dimension of the column space is the rank r. The pivot columns form a basis.

3. The ***nullspace*** has dimension $n - r = 5 - 2$. There are $n - r = 3$ free variables. Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $Rx = \mathbf{0}$. Set a free variable to 1, and solve for x_1 and x_4:

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \begin{array}{l} Rx = \mathbf{0} \text{ has the} \\ \text{complete solution} \\ x = x_2 s_2 + x_3 s_3 + x_5 s_5 \end{array}$$

There is a special solution for each free variable. With n variables and r pivot variables, that leaves $n - r$ free variables and special solutions. $N(R)$ has dimension $n - r$.

The nullspace has dimension $n - r$. The special solutions form a basis.

The special solutions are independent, because they contain the identity matrix in rows 2, 3, 5. All solutions are combinations of special solutions, $x = x_2 s_2 + x_3 s_3 + x_5 s_5$, because this puts x_2, x_3 and x_5 in the correct positions. Then the pivot variables x_1 and x_4 are totally determined by the equations $Rx = \mathbf{0}$.

4. The ***nullspace of R^{T} (left nullspace of R)*** has dimension $m - r = 3 - 2$.

Reason: The equation $R^{\mathrm{T}} y = \mathbf{0}$ looks for combinations of the columns of R^{T} (*the rows of R*) that produce zero. This equation $R^{\mathrm{T}} y = \mathbf{0}$ or $y^{\mathrm{T}} R = \mathbf{0}^{\mathrm{T}}$ is

$$\textbf{Left nullspace} \qquad \begin{array}{r} y_1\,[1,\ 3,\ 5,\ 0,\ 7] \\ +y_2\,[0,\ 0,\ 0,\ 1,\ 2] \\ +y_3\,[0,\ 0,\ 0,\ 0,\ 0] \\ \hline [0,\ 0,\ 0,\ 0,\ 0] \end{array} \qquad (1)$$

The solutions y_1, y_2, y_3 are pretty clear. We need $y_1 = 0$ and $y_2 = 0$. The variable y_3 is free (it can be anything). The nullspace of R^{T} contains all vectors $y = (0, 0, y_3)$. It is the line of all multiples of the basis vector $(0, 0, 1)$.

In all cases R ends with $m - r$ zero rows. Every combination of these $m - r$ rows gives zero. These are the *only* combinations of the rows of R that give zero, because the pivot rows are linearly independent. The left nullspace of R contains all these solutions $y = (0, \cdots, 0, y_{r+1}, \cdots, y_m)$ to $R^{\mathrm{T}} y = 0$.

If A is m by n of rank r, its left nullspace has dimension $m - r$.

To produce a zero combination, y must start with r zeros. This leaves dimension $m - r$.

Why is this a "*left* nullspace"? The reason is that $R^{\mathrm{T}} y = \mathbf{0}$ can be transposed to $y^{\mathrm{T}} R = \mathbf{0}^{\mathrm{T}}$. Now y^{T} is a row vector to the *left* of R. You see the y's in equation (1) multiplying the rows. This subspace came fourth, and some linear algebra books omit it—but that misses the beauty of the whole subject.

In $\mathbf{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n).
In $\mathbf{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m).

So far this is proved for echelon matrices R. Figure 3.5 shows the same for A.

The Four Subspaces for A

We have a job still to do. ***The subspace dimensions for A are the same as for R.*** The job is to explain why. A is now any matrix that reduces to $R = \mathrm{rref}(A)$.

$$\textbf{A reduces to R} \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \qquad \text{Notice } \boldsymbol{C}(A) \neq \boldsymbol{C}(R) \qquad (2)$$