Chapter 7

Nilpotent Groups

Recall the commutator is given by

$$[x, y] = x^{-1}y^{-1}xy.$$

Definition 7.1 Let A and B be subgroups of a group G. Define the *commutator subgroup* $[A, B]$ by

$$[A, B] = \langle\, [a, b] \mid a \in A,\ b \in B\,\rangle,$$

the subgroup generated by all commutators $[a, b]$ with $a \in A$ and $b \in B$.

In this notation, the derived series is given recursively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for all i.

Definition 7.2 The *lower central series* $(\gamma_i(G))$ (for $i \geqslant 1$) is the chain of subgroups of the group G defined by

$$\gamma_1(G) = G$$

and

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \qquad \text{for } i \geqslant 1.$$

Definition 7.3 A group G is *nilpotent* if $\gamma_{c+1}(G) = \mathbf{1}$ for some c. The least such c is the *nilpotency class* of G.

It is easy to see that $G^{(i)} \leqslant \gamma_{i+1}(G)$ for all i (by induction on i). Thus if G is nilpotent, then G is soluble. Note also that $\gamma_2(G) = G'$.

Lemma 7.4 (i) *If H is a subgroup of G, then $\gamma_i(H) \leqslant \gamma_i(G)$ for all i.*

(ii) *If $\phi\colon G \to K$ is a surjective homomorphism, then $\gamma_i(G)\phi = \gamma_i(K)$ for all i.*

(iii) $\gamma_i(G)$ *is a characteristic subgroup of* G *for all* i.

(iv) *The lower central series of* G *is a chain of subgroups*

$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \gamma_3(G) \geqslant \cdots .$$

PROOF: (i) Induct on i. Note that $\gamma_1(H) = H \leqslant G = \gamma_1(G)$. If we assume that $\gamma_i(H) \leqslant \gamma_i(G)$, then this together with $H \leqslant G$ gives

$$[\gamma_i(H), H] \leqslant [\gamma_i(G), G]$$

so $\gamma_{i+1}(H) \leqslant \gamma_{i+1}(G)$.

(ii) Induct on i. Note that $\gamma_1(G)\phi = G\phi = K = \gamma_1(K)$. Suppose $\gamma_i(G)\phi = \gamma_i(K)$. If $x \in \gamma_i(G)$ and $y \in G$, then

$$[x, y]\phi = [x\phi, y\phi] \in [\gamma_i(G)\phi, G\phi] = [\gamma_i(K), K] = \gamma_{i+1}(K),$$

so $\gamma_{i+1}(G)\phi = [\gamma_i(G), G]\phi \leqslant \gamma_{i+1}(K)$.

On the other hand, if $a \in \gamma_i(K)$ and $b \in K$, then $a = x\phi$ and $b = y\phi$ for some $x \in \gamma_i(G)$ and $y \in G$. So

$$[a, b] = [x\phi, y\phi] = [x, y]\phi \in [\gamma_i(G), G]\phi = \gamma_{i+1}(G)\phi.$$

Thus $\gamma_{i+1}(K) = [\gamma_i(K), K] \leqslant \gamma_{i+1}(G)\phi$.

(iii) If ϕ is an automorphism of G, then $\phi\colon G \to G$ is a surjective homomorphism, so from (ii)

$$\gamma_i(G)\phi = \gamma_i(G).$$

Thus $\gamma_i(G) \operatorname{char} G$.

(iv) From (iii), $\gamma_i(G) \trianglelefteq G$. Hence if $x \in \gamma_i(G)$ and $y \in G$, then

$$[x, y] = x^{-1}x^y \in \gamma_i(G).$$

Hence

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \leqslant \gamma_i(G) \qquad \text{for all } i.$$

□

We deduce two consequences immediately:

Lemma 7.5 *Subgroups and homomorphic images of nilpotent groups are themselves nilpotent.*

PROOF: Let $\gamma_{c+1}(G) = \mathbf{1}$ and $H \leqslant G$. Then by Lemma 7.4(i), $\gamma_{c+1}(H) \leqslant \gamma_{c+1}(G) = \mathbf{1}$, so $\gamma_{c+1}(H) = \mathbf{1}$ and H is nilpotent.

Let $\phi\colon G \to K$ be a surjective homomorphism. Then Lemma 7.4(ii) gives $\gamma_{c+1}(K) = \gamma_{c+1}(G)\phi = \mathbf{1}\phi = \mathbf{1}$, so K is nilpotent. □

Note, however, that

$$N \trianglelefteqslant G,\ \ G/N \text{ and } N \text{ nilpotent} \quad \not\Rightarrow \quad G \text{ nilpotent.}$$

In this way, nilpotent groups are different to soluble groups.

Example 7.6 Finite p-groups are nilpotent.

Proof: Let G be a finite p-group, say $|G| = p^n$. We proceed by induction on $|G|$. If $|G| = 1$, then $\gamma_1(G) = G = \mathbf{1}$ so G is nilpotent.

Now suppose $|G| > 1$. Apply Corollary 2.41: $\mathrm{Z}(G) \neq \mathbf{1}$. Consider the quotient group $G/\mathrm{Z}(G)$. This is a p-group of order smaller than G, so by induction it is nilpotent, say

$$\gamma_{c+1}(G/\mathrm{Z}(G)) = \mathbf{1}.$$

Let $\pi \colon G \to G/\mathrm{Z}(G)$ be the natural homomorphism. Then by Lemma 7.4(ii),

$$\gamma_{c+1}(G)\pi = \gamma_{c+1}(G/\mathrm{Z}(G)) = \mathbf{1},$$

so $\gamma_{c+1}(G) \leqslant \ker \pi = \mathrm{Z}(G)$. Thus

$$\gamma_{c+2}(G) = [\gamma_{c+1}(G), G] \leqslant [\mathrm{Z}(G), G] = \mathbf{1},$$

so G is nilpotent. □

The example illustrates that the centre has a significant role in the study of nilpotent groups. We make two further definitions:

Definition 7.7 The *upper central series* of G, denoted $(\mathrm{Z}_i(G))$ for $i \geqslant 0$, is the chain of subgroups defined by

$$\begin{aligned} \mathrm{Z}_0(G) &= \mathbf{1}; \\ \mathrm{Z}_{i+1}(G)/\mathrm{Z}_i(G) &= \mathrm{Z}(G/\mathrm{Z}_i(G)) \qquad \text{for } i \geqslant 0. \end{aligned}$$

Suppose that $\mathrm{Z}_i(G) \trianglelefteqslant G$. Then $\mathrm{Z}(G/\mathrm{Z}_i(G))$ is a normal subgroup of $G/\mathrm{Z}_i(G)$, so corresponds to a normal subgroup $\mathrm{Z}_{i+1}(G)$ of G containing $\mathrm{Z}_i(G)$ by the Correspondence Theorem. In this way we define a chain of subgroups

$$\mathbf{1} = \mathrm{Z}_0(G) \leqslant \mathrm{Z}_1(G) \leqslant \mathrm{Z}_2(G) \leqslant \cdots,$$

each of which is normal in G. Here $\mathrm{Z}_1(G) = \mathrm{Z}(G)$.

Definition 7.8 A *central series* for a group G is a chain of subgroups

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_n = \mathbf{1}$$

such that G_i is a normal subgroup of G and $G_{i-1}/G_i \leqslant \mathrm{Z}(G/G_i)$ for all i.

Lemma 7.9 *Let*

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_n = \mathbf{1}$$

be a central series for G*. Then for all* i*:*

$$\gamma_{i+1}(G) \leqslant G_i \qquad \text{and} \qquad \mathrm{Z}_i(G) \geqslant G_{n-i}.$$

PROOF: First observe that $\gamma_1(G) = G = G_0$. Suppose that $\gamma_i(G) \leqslant G_{i-1}$ for some i. If $x \in \gamma_i(G)$ and $y \in G$, then

$$G_i x \in G_{i-1}/G_i \leqslant \mathrm{Z}(G/G_i),$$

so $G_i x$ commutes with $G_i y$. Therefore

$$G_i[x,y] = (G_i x)^{-1}(G_i y)^{-1}(G_i x)(G_i y) = G_i,$$

so $[x,y] \in G_i$. Hence

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \leqslant G_i.$$

Thus, by induction, the first inclusion holds.

Now, $\mathrm{Z}_0(G) = \mathbf{1} = G_n$. Suppose that $\mathrm{Z}_i(G) \geqslant G_{n-i}$. Since (G_i) is a central series for G,

$$G_{n-i-1}/G_{n-i} \leqslant \mathrm{Z}(G/G_{n-i}).$$

Thus if $x \in G_{n-i-1}$ and $y \in G$, then

$$G_{n-i}x \text{ and } G_{n-i}y \text{ commute;} \quad \text{i.e., } [x,y] \in G_{n-i}.$$

Hence $[x,y] \in \mathrm{Z}_i(G)$, so $\mathrm{Z}_i(G)x$ and $\mathrm{Z}_i(G)y$ commute. Since y is an arbitrary element of G, we deduce that

$$\mathrm{Z}_i(G)x \in \mathrm{Z}(G/\mathrm{Z}_i(G)) = \mathrm{Z}_{i+1}(G)/\mathrm{Z}_i(G)$$

for all $x \in G_{n-i-1}$. Thus $G_{n-i-1} \leqslant \mathrm{Z}_{i+1}(G)$ and the second inclusion holds by induction. □

We have now established the link between a general central series and the behaviour of the lower and the upper central series.

Theorem 7.10 *The following conditions are equivalent for a group* G*:*

(i) $\gamma_{c+1}(G) = \mathbf{1}$ *for some* c*;*

(ii) $\mathrm{Z}_c(G) = G$ *for some* c*;*

(iii) G *has a central series.*

Thus these are equivalent conditions for a group to be nilpotent.

PROOF: If G has a central series (G_i) of length n, then Lemma 7.9 gives

$$\gamma_{n+1}(G) \leqslant G_n = \mathbf{1} \qquad \text{and} \qquad \mathrm{Z}_n(G) \geqslant G_0 = G.$$

Hence (iii) implies both (i) and (ii).

If $\mathrm{Z}_c(G) = G$, then

$$G = \mathrm{Z}_c(G) \geqslant \mathrm{Z}_{c-1}(G) \geqslant \cdots \geqslant \mathrm{Z}_1(G) \geqslant \mathrm{Z}_0(G) = \mathbf{1}$$

is a central series for G (as $\mathrm{Z}_{i+1}(G)/\mathrm{Z}_i(G) = \mathrm{Z}(G/\mathrm{Z}_i(G))$). Thus (ii) implies (iii).

If $\gamma_{c+1}(G) = \mathbf{1}$, then

$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots \geqslant \gamma_{c+1}(G) = \mathbf{1}$$

is a central series for G. (For if $x \in \gamma_{i-1}(G)$ and $y \in G$, then $[x, y] \in \gamma_i(G)$, so $\gamma_i(G)x$ and $\gamma_i(G)y$ commute for all such x and y; thus $\gamma_{i-1}(G)/\gamma_i(G) \leqslant \mathrm{Z}(G/\gamma_i(G))$.) Hence (i) implies (iii). □

Further examination of this proof and Lemma 7.9 shows that

$$\gamma_{c+1}(G) = \mathbf{1} \qquad \text{if and only if} \qquad \mathrm{Z}_c(G) = G.$$

Thus for a nilpotent group, the lower central series and the upper central series have the same length.

Our next goal is to develop further equivalent conditions for finite groups to be nilpotent.

Proposition 7.11 *Let G be a nilpotent group. Then every proper subgroup of G is properly contained in its normaliser:*

$$H < \mathrm{N}_G(H) \qquad \textit{whenever } H < G.$$

PROOF: Let

$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots \geqslant \gamma_{c+1}(G) = \mathbf{1}$$

be the lower central series. Then $\gamma_{c+1}(G) \leqslant H$ but $\gamma_1(G) \nleqslant H$. Choose i as small as possible so that $\gamma_i(G) \leqslant H$. Then $\gamma_{i-1}(G) \nleqslant H$. Now

$$[\gamma_{i-1}(G), H] \leqslant [\gamma_{i-1}(G), G] = \gamma_i(G) \leqslant H,$$

so

$$x^{-1}hxh^{-1} = [x, h^{-1}] \in H \qquad \text{for } x \in \gamma_{i-1}(G) \text{ and } h \in H.$$

Therefore

$$x^{-1}hx \in H \qquad \text{for } x \in \gamma_{i-1}(G) \text{ and } h \in H.$$

We deduce that $H^x = H$ for all $x \in \gamma_{i-1}(G)$, so that $\gamma_{i-1}(G) \leqslant \mathrm{N}_G(H)$. Therefore, since $\gamma_{i-1}(G) \nleqslant H$, we deduce $\mathrm{N}_G(H) > H$. □

Let us now analyse how nilpotency affects the Sylow subgroups of a finite group. This links into the previous proposition via the following lemma.

Lemma 7.12 *Let G be a finite group and let P be a Sylow p-subgroup of G for some prime p. Then*

$$\mathrm{N}_G(\mathrm{N}_G(P)) = \mathrm{N}_G(P).$$

PROOF: Let $H = \mathrm{N}_G(P)$. Then $P \trianglelefteqslant H$, so P is the unique Sylow p-subgroup of H. (Note that as it is a Sylow p-subgroup of G and $P \leqslant H$, it is also a Sylow p-subgroup of H, as it must have the largest possible order for a p-subgroup of H.) Let $g \in \mathrm{N}_G(H)$. Then

$$P^g \leqslant H^g = H,$$

so P^g is also a Sylow p-subgroup of H and we deduce $P^g = P$; that is, $g \in \mathrm{N}_G(P) = H$. Thus $\mathrm{N}_G(H) \leqslant H$, so we deduce

$$\mathrm{N}_G(H) = H,$$

as required. □

We can now characterise finite nilpotent groups as being built from p-groups in the most simple way.

Theorem 7.13 *Let G be a finite group. The following conditions on G are equivalent:*

(i) *G is nilpotent;*

(ii) *every Sylow subgroup of G is normal;*

(iii) *G is a direct product of p-groups (for various primes p).*

PROOF: (i) ⇒ (ii): Let G be nilpotent and P be a Sylow p-subgroup of G (for some prime p). Let $H = \mathrm{N}_G(P)$. By Lemma 7.12, $\mathrm{N}_G(H) = H$. Hence, by Proposition 7.11, $H = G$. That is, $\mathrm{N}_G(P) = G$ and so $P \trianglelefteqslant G$.

(ii) ⇒ (iii): Let $p_1, p_2, \dots, p_k$ be the distinct prime factors of $|G|$, say

$$|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k},$$

and assume that G has a normal Sylow p_i-subgroup P_i for $i = 1, 2, \dots, k$.

Claim: $P_1P_2\dots P_j \cong P_1 \times P_2 \times \cdots \times P_j$ for all j.

Certainly this claim holds for $j = 1$. Assume it holds for some j, and consider $N = P_1P_2\dots P_j \cong P_1 \times \cdots \times P_j \trianglelefteq G$ and $P_{j+1} \trianglelefteq G$. Then $|N|$ is coprime to $|P_{j+1}|$. Hence $N \cap P_{j+1} = \mathbf{1}$ and therefore NP_{j+1} satisfies the conditions to be an (internal) direct product. Thus

$$NP_{j+1} \cong N \times P_{j+1} \cong P_1 \times P_2 \times \cdots \times P_j \times P_{j+1},$$

and by induction the claim holds.

In particular, note

$$|P_1P_2\dots P_k| = |P_1 \times P_2 \times \cdots \times P_k| = |P_1| \cdot |P_2| \cdot \ldots \cdot |P_k| = |G|,$$

so

$$G = P_1P_2\dots P_k \cong P_1 \times P_2 \times \cdots \times P_k.$$

(iii) $\Rightarrow$ (i): Suppose $G = P_1 \times P_2 \times \cdots \times P_k$, a direct product of non-trivial p-groups. Then

$$\mathrm{Z}(G) = \mathrm{Z}(P_1) \times \mathrm{Z}(P_2) \times \cdots \times \mathrm{Z}(P_k) \neq \mathbf{1}$$

(by Corollary 2.41). Then

$$G/\mathrm{Z}(G) = P_1/\mathrm{Z}(P_1) \times P_2/\mathrm{Z}(P_2) \times \cdots \times P_k/\mathrm{Z}(P_k)$$

is a direct product of p-groups of smaller order. By induction, $G/\mathrm{Z}(G)$ is nilpotent, say $\gamma_c(G/\mathrm{Z}(G)) = \mathbf{1}$. Now apply Lemma 7.4(ii) to the natural map $\pi\colon G \to G/\mathrm{Z}(G)$ to see that $\gamma_c(G)\pi = \gamma_c(G/\mathrm{Z}(G)) = \mathbf{1}$. Thus $\gamma_c(G) \leqslant \ker \pi = \mathrm{Z}(G)$ and hence

$$\gamma_{c+1}(G) = [\gamma_c(G), G] \leqslant [\mathrm{Z}(G), G] = \mathbf{1}.$$

Therefore G is nilpotent. □

This tells us that the study of finite nilpotent groups reduces to understanding p-groups. We finish by introducing the Frattini subgroup, which is of significance in many parts of group theory.

Definition 7.14 A *maximal subgroup* of a group G is a subgroup $M < G$ such that there is no subgroup H with $M < H < G$.

Thus a maximal subgroup is a proper subgroup which is largest amongst the proper subgroups.

If G is a nilpotent group, then Proposition 7.11 tells us that

$$M < \mathrm{N}_G(M) \leqslant G,$$

for any maximal subgroup M of G. The maximality of M forces $\mathrm{N}_G(M) = G$; that is, $M \trianglelefteq G$. Thus:

Lemma 7.15 *Let G be a nilpotent group. Then every maximal subgroup of G is normal in G.* □

Definition 7.16 The *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all its maximal subgroups:

$$\Phi(G) = \bigcap_{\substack{M \text{ maximal} \\ \text{in } G}} M.$$

(If G is an (infinite) group with no maximal subgroups, then $\Phi(G) = G$.)

If we apply an automorphism to a maximal subgroup, we map it to another maximal subgroup. Hence the automorphism group permutes the maximal subgroups of G.

Lemma 7.17 *If G is a group, then the Frattini subgroup $\Phi(G)$ is a characteristic subgroup of G.* □

Our final theorem characterising nilpotent finite groups is:

Theorem 7.18 *Let G be a finite group. The following are equivalent:*

(i) *G is nilpotent;*

(ii) *$H < \mathrm{N}_G(H)$ for all $H < G$;*

(iii) *every maximal subgroup of G is normal;*

(iv) *$\Phi(G) \geqslant G'$;*

(v) *every Sylow subgroup of G is normal;*

(vi) *G is a direct product of p-groups.*

Proof: We have already proved that (i) ⇒ (ii) (Proposition 7.11), (ii) ⇒ (iii) (see the proof of Lemma 7.15) and (v) ⇒ (vi) ⇒ (i).

(iii) ⇒ (iv): Let M be a maximal subgroup of G. By assumption, $M \trianglelefteq G$. Since M is maximal, the Correspondence Theorem tells us that G/M has no non-trivial proper subgroups. It follows that G/M is cyclic and so is abelian. Lemma 6.16 gives

$$G' \leqslant M.$$

Hence

$$G' \leqslant \bigcap_{M \max G} M = \Phi(G).$$

(iv) $\Rightarrow$ (v): Let P be a Sylow p-subgroup of G and let $N = P\,\Phi(G)$ (which is a subgroup of G, since $\Phi(G) \trianglelefteqslant G$ by Lemma 7.17). Let $x \in N$ and $g \in G$. Then

$$x^{-1}x^g = [x, g] \in G' \leqslant \Phi(G) \leqslant N.$$

Hence $x^g \in N$ for all $x \in N$ and $g \in G$, so $N \trianglelefteqslant G$. Now P is a Sylow p-subgroup of N (since it is the largest possible p-subgroup of G, so is certainly largest amongst p-subgroups of N). Apply the Frattini Argument (Lemma 6.35):

$$\begin{aligned} G &= \mathrm{N}_G(P)\,N \\ &= \mathrm{N}_G(P)\,P\,\Phi(G) \\ &= \mathrm{N}_G(P)\,\Phi(G) \qquad \text{(as } P \leqslant \mathrm{N}_G(P)\text{)}. \end{aligned}$$

From this we deduce that $G = \mathrm{N}_G(P)$: for suppose $\mathrm{N}_G(P) \neq G$. Then $\mathrm{N}_G(P) \leqslant M < G$ for some maximal subgroup M of G. By definition, $\Phi(G) \leqslant M$, so

$$\mathrm{N}_G(P)\,\Phi(G) \leqslant M < G,$$

a contradiction. Hence $\mathrm{N}_G(P) = G$ and so $P \trianglelefteqslant G$.

This completes all remaining stages in the proof. □

Theorem 7.19 *Let G be a finite group. Then the Frattini subgroup $\Phi(G)$ is nilpotent.*

Proof: Let P be a Sylow p-subgroup of $\Phi(G)$. The Frattini Argument (Lemma 6.35) gives

$$G = \mathrm{N}_G(P)\,\Phi(G).$$

If $\mathrm{N}_G(P) \neq G$, then there is a maximal proper subgroup M of G with $\mathrm{N}_G(P) \leqslant M < G$. By definition, $\Phi(G) \leqslant M$. Hence

$$\mathrm{N}_G(P)\,\Phi(G) \leqslant M < G,$$

contrary to above. Therefore $\mathrm{N}_G(P) = G$. Hence $P \trianglelefteqslant G$, and so in particular $P \trianglelefteqslant \Phi(G)$. Therefore $\Phi(G)$ is nilpotent by Theorem 7.13. □

We have used one property of the Frattini subgroup twice now, so it is worth drawing attention to it.

Definition 7.20 A subset S of a group G is a *set of non-generators* if it can always be removed from a set of generators for G without affecting the property of generating G.

Thus S is a set of non-generators if

$$G = \langle X, S \rangle \qquad \text{implies} \qquad G = \langle X \rangle$$

for all subsets $X \subseteq G$.

Lemma 7.21 *The Frattini subgroup $\Phi(G)$ is a set of non-generators for a finite group G.*

PROOF: Let $G = \langle X, \Phi(G) \rangle$. If $\langle X \rangle \neq G$, then there exists a maximal subgroup M of G such that $\langle X \rangle \leqslant M < G$. By definition of the Frattini subgroup, $\Phi(G) \leqslant M$. Hence $X \cup \Phi(G) \subseteq M$, so $\langle X, \Phi(G) \rangle \leqslant M < G$ which contradicts the assumption. Therefore $G = \langle X \rangle$ and so we deduce $\Phi(G)$ is a set of non-generators for G. □

Theorem 7.22 *Let G be a finite group. Then G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

PROOF: By Lemma 7.5, a homomorphic image of a nilpotent group is nilpotent. Consequently if G is nilpotent, then $G/\Phi(G)$ is nilpotent.

Conversely suppose $G/\Phi(G)$ is nilpotent. Let P be a Sylow p-subgroup of G. Then $P\Phi(G)/\Phi(G)$ is a Sylow p-subgroup of $G/\Phi(G)$. Hence

$$P\Phi(G)/\Phi(G) \trianglelefteqslant G/\Phi(G),$$

as $G/\Phi(G)$ is nilpotent. Therefore

$$P\,\Phi(G) \trianglelefteqslant G$$

by the Correspondence Theorem. Now P is a Sylow p-subgroup of $P\,\Phi(G)$ (as even G has no larger p-subgroups), so we apply the Frattini Argument (Lemma 6.35) to give

$$G = \mathrm{N}_G(P) \cdot P\,\Phi(G).$$

Therefore

$$G = \mathrm{N}_G(P)\,\Phi(G)$$

(as $P \leqslant \mathrm{N}_G(P)$). Now as $\Phi(G)$ is a set of non-generators for G (see Lemma 7.21), we deduce

$$G = \mathrm{N}_G(P).$$

Thus $P \trianglelefteqslant G$. Hence G is nilpotent by Theorem 7.13. □