Q 1-26

Ex # 5.4

② $\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx$

$= \int_{-3}^{4} 5dx - \frac{1}{2}\int_{-3}^{4} x\,dx$

$= 5\int_{-3}^{4} 1dx - \frac{1}{2}\int_{-3}^{4} x\,dx$

$\because \int 1dx = x + c$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$= 5\left[x\Big|_{-3}^{4}\right] - \frac{1}{2}\left[\frac{x^2}{2}\Big|_{-3}^{4}\right]$

$= 5(4-(-3)) - \frac{1}{4}\left[x^2\Big|_{-3}^{4}\right]$

$= 5(4+3) - \frac{1}{4}\left[(4)^2-(-3)^2\right]$

$= 5(7) - \frac{1}{4}[16-9]$

$= 35 - \frac{7}{4} = \frac{140-7}{4} = \frac{133}{4}$ Ans

Formulas

$\because \int \sin x\,dx = -\cos x + c$

$\because \int \cos x\,dx = \sin x + c$

$\because \int \mathrm{cosec}\,x \cot x\,dx = -\mathrm{cosec}\,x + c$

$\because \int \sec^2 x\,dx = \tan x + c$

$\because \int \mathrm{cosec}^2 x\,dx = -\cot x + c$

$\because \int \sec x \tan x\,dx = \sec x + c$

(10) $\int_0^{\pi} (1+\cos x)\,dx$

$= \int_0^{\pi} 1\,dx + \int_0^{\pi} \cos x\,dx$

$$\boxed{\begin{aligned} &\because \int 1\,dx = x + c \\ &\because \int \cos x\,dx = \sin x + c \end{aligned}}$$

$= \left[x \Big|_0^{\pi}\right] + \left[\sin x\right]_0^{\pi}$

$= (\pi - 0) + (\sin\pi - \sin(0))$

$= \pi + 0 = \pi$ Ans

(14) $\int_0^{\pi/3} 4 \sec u \tan u\,du$

$= 4\int_0^{\pi/3} \sec u \tan u\,du$

$\because \int \sec x \tan x\,dx = \sec x + c$

$= 4\left[\sec x \Big|_0^{\pi/3}\right]$

$= 4\left[\sec\frac{\pi}{3} - \sec(0)\right]$

$= 4[2-1]$

$= 4(1) = 4$ Ans

(18) $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

$= 4\int_{-\pi/3}^{-\pi/4} \sec^2 t + \pi \int_{-\frac{\pi}{3}}^{\pi/4} \frac{1}{t^2}\,dt$

$\because \int \sec^2 x\,dx = \tan x + c$

$\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$= 4\left[\tan x \Big|_{-\pi/3}^{-\pi/4}\right] + \pi \int_{-\pi/3}^{-\pi/4} t^{-2}\,dt$

$= 4\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{3}\right)\right] + \pi\left[\frac{t^{-2+1}}{-2+1}\Big|_{-\pi/3}^{-\pi/4}\right]$

$= 4\left[-1-(-\sqrt{3})\right] + \pi\left[\frac{t^{-1}}{-1}\Big|_{-\pi/3}^{-\pi/4}\right]$

$= 4\left[-1+\sqrt{3}\right] - \pi\left(\frac{1}{\left(-\frac{\pi}{4}\right)} - \frac{1}{\left(-\frac{\pi}{3}\right)}\right)$

$= -4+4\sqrt{3} - \pi\left[\frac{-4}{\pi} + \frac{3}{\pi}\right]$

$= -4+4\sqrt{3} - \cancel{\pi}\left[\frac{-4+3}{\cancel{\pi}}\right]$

$= -4+4\sqrt{3}+1$

$= 4\sqrt{3}-3$ Ans

Home work $\boxed{5, 11, 12, 13, 22, 23}$

5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY

Sir Isaac Newton
(1642–1727)

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we present the integral version of the Mean Value Theorem, which is another important theorem of integral calculus and used to prove the Fundamental Theorem.

Mean Value Theorem for Definite Integrals

In the previous section, we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\,dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

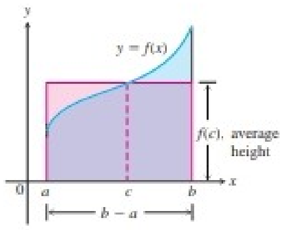


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.3, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.6) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b - a))\int_a^b f(x)\,dx$ at some point c in $[a, b]$. ■

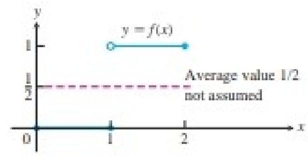


FIGURE 5.17 A discontinuous function need not assume its average value.

The continuity of f is important here. It is possible that a discontinuous function never equals its average value (Figure 5.17).

EXAMPLE 1 Applying the Mean Value Theorem for Integrals

Find the average value of $f(x) = 4 - x$ on $[0, 3]$ and where f actually takes on this value at some point in the given domain.

Solution

$$\begin{aligned}
\text{av}(f) &= \frac{1}{b - a}\int_a^b f(x)\,dx \\
&= \frac{1}{3 - 0}\int_0^3 (4 - x)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x\,dx\right) \\
&= \frac{1}{3}\left(4(3 - 0) - \left(\frac{3^2}{2} - \frac{0^2}{2}\right)\right) && \text{Section 5.3, Eqs. (1) and (2)} \\
&= 4 - \frac{3}{2} = \frac{5}{2}.
\end{aligned}$$

The average value of $f(x) = 4 - x$ over $[0, 3]$ is $5/2$. The function assumes this value when $4 - x = 5/2$ or $x = 3/2$. (Figure 5.18) ■

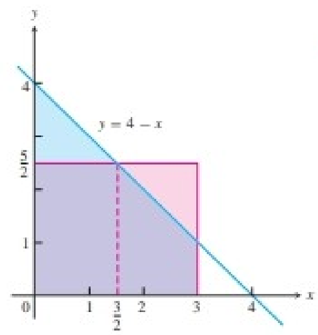


FIGURE 5.18 The area of the rectangle with base $[0, 3]$ and height $5/2$ (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

In Example 1, we actually found a point c where f assumed its average value by setting $f(x)$ equal to the calculated average value and solving for x. It's not always possible to solve easily for the value c. What else can we learn from the Mean Value Theorem for integrals? Here's an example.

EXAMPLE 2 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx = \frac{1}{b - a}\cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$. ■

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2). Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned}
F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt - 0 \\
&= \int_a^b f(t)\,dt.
\end{aligned}$$

■

The theorem says that to calculate the definite integral of f over $[a, b]$ all we need to do is:

1. Find an antiderivative F of f, and
2. Calculate the number $\int_a^b f(x)\,dx = F(b) - F(a)$.

The usual notation for $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 5 Evaluating Integrals

You Try It

(a) $\displaystyle\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

(b) $\displaystyle\int_{-\pi/4}^0 \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^0 = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$

You Try It

(c) $$\begin{aligned}
\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\
&= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\
&= [8 + 1] - [5] = 4.
\end{aligned}$$

■

The process used in Example 5 was much easier than a Riemann sum computation.

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{dF}{dx} = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, the equation

$$\int_a^x \frac{dF}{dt}\,dt = \int_a^x f(t)\,dt = F(x) - F(a)$$

says that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integra-

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–26.

1. $\int_{-2}^{0}(2x+5)\,dx$
2. $\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx$
3. $\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx$
4. $\int_{-2}^{2}(x^3-2x+3)\,dx$
5. $\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$
6. $\int_{0}^{5}x^{3/2}\,dx$
7. $\int_{1}^{32}x^{-6/5}\,dx$
8. $\int_{-2}^{-1}\frac{2}{x^2}\,dx$
9. $\int_{0}^{\pi}\sin x\,dx$
10. $\int_{0}^{\pi}(1+\cos x)\,dx$
11. $\int_{0}^{\pi/3}2\sec^2 x\,dx$
12. $\int_{\pi/6}^{5\pi/6}\csc^2 x\,dx$
13. $\int_{\pi/4}^{3\pi/4}\csc\theta\cot\theta\,d\theta$
14. $\int_{0}^{\pi/3}4\sec u\tan u\,du$
15. $\int_{\pi/2}^{0}\frac{1+\cos 2t}{2}\,dt$
16. $\int_{-\pi/3}^{\pi/3}\frac{1-\cos 2t}{2}\,dt$
17. $\int_{-\pi/2}^{\pi/2}(8y^2+\sin y)\,dy$
18. $\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$
19. $\int_{1}^{-1}(r+1)^2\,dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}}(t+1)(t^2+4)\,dt$
21. $\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-\frac{1}{u^5}\right)du$
22. $\int_{1/2}^{1}\left(\frac{1}{v^3}-\frac{1}{v^4}\right)dv$
23. $\int_{1}^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds$
24. $\int_{9}^{4}\frac{1-\sqrt{u}}{\sqrt{u}}\,du$
25. $\int_{-4}^{4}|x|\,dx$
26. $\int_{0}^{\pi}\frac{1}{2}(\cos x+|\cos x|)\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 27–30

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

27. $\frac{d}{dx}\int_{0}^{\sqrt{x}}\cos t\,dt$
28. $\frac{d}{dx}\int_{1}^{\sin x}3t^2\,dt$
29. $\frac{d}{dt}\int_{0}^{t^4}\sqrt{u}\,du$
30. $\frac{d}{d\theta}\int_{0}^{\tan\theta}\sec^2 y\,dy$

Find dy/dx in Exercises 31–36.

31. $y=\int_{0}^{x}\sqrt{1+t^2}\,dt$
32. $y=\int_{1}^{x}\frac{1}{t}\,dt,\quad x>0$
33. $y=\int_{\sqrt{x}}^{0}\sin(t^2)\,dt$
34. $y=\int_{0}^{x^2}\cos\sqrt{t}\,dt$
35. $y=\int_{0}^{\sin x}\frac{dt}{\sqrt{1-t^2}},\quad |x|<\frac{\pi}{2}$
36. $y=\int_{\tan x}^{0}\frac{dt}{1+t^2}$

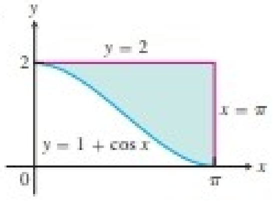
Area

In Exercises 37–42, find the total area between the region and the x-axis.

37. $y=-x^2-2x,\quad -3\le x\le 2$
38. $y=3x^2-3,\quad -2\le x\le 2$
39. $y=x^3-3x^2+2x,\quad 0\le x\le 2$
40. $y=x^3-4x,\quad -2\le x\le 2$
41. $y=x^{1/3},\quad -1\le x\le 8$
42. $y=x^{1/3}-x,\quad -1\le x\le 8$

Find the areas of the shaded regions in Exercises 43–46.

43.



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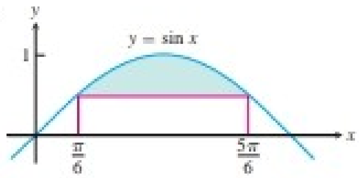

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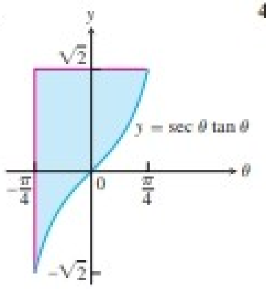

46.