Elementary Matrix:-

"An $n \times n$ square matrix is called an elementary matrix if it can be obtained from the $n \times n$ Identity matrix I using a single elementary row operation."

Example:-

Now we have given an Identity matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now multiply 3 by row 2

we get

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Now this is an elementary matrix with only single row operation.

Now we have again an Identity matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now multiply 3 by column 1

we get

$$E = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

so this is an Elementary matrix which is formed by single column operation

Now we have discussed the properties of the elementary matrix.

Properties:-

We have two properties of the elementary matrix

(i) All elementary matrices are invertible an inverse exist.

(ii) The inverse of an elementary matrix is also an elementary matrix.

Elementary Matrix operation:-

Elementary matrix operations play an important role in many matrix algebra applications, such as finding the inverse of a matrix and solving simultaneous linear equations.

Elementary operations:-

There are the three kinds of the elementary matrix operations.

(ii) Interchanging two rows or two columns.

(iii) Multiply each element in a row or column by a non-zero number.

(iiii) Multiply a row or column by a non-zero number and add the result to another row or column.

When these operations are performed on the rows they are called elementary row operations; and when they are performed on the columns, they are called elementary column operations.

Elementary operation Notation

In many references, you will encounter a compact notation to describe elementary operations.

Row operations:-

1) Interchanging rows i and J. $R_i \longleftrightarrow R_J$

ii) Multiply row i by s $\quad sR_i \rightarrow R_i$
Where $s \neq 0$

iii) Add s times row i $\quad sR_i + R_j \rightarrow R_j$
to row j.

Column operations:-

i) Interchange column $\quad C_i \leftrightarrow C_j$
i and j.

ii) Multiply column i by $\quad sC_i \rightarrow C_i$
s where $s \neq 0$

iii) Add s times column i $\quad sC_i + C_j \rightarrow C_j$
to column j.

Elementary operators:-

Each type of elementary operation may be performed by matrix multiplication, using square matrices called elementary operators.

For example, suppose you want to interchange rows 1 and 2 of Matrix A. To accomplish this you could premultiply A by E to produce B as shown.

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

E A

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 0+2 & 0+4 & 0+6 \\ 0+1 & 0+3 & 0+5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = B$$

Here E is an Elementary operator. It operates on A to produce the desired interchanged rows in B.

How to perform Elementary row operations:

To perform an elementary row operation on A an rxc matrix, take the following steps.

1) To find E, the elementary row operator, apply the operation to an rxr Identity matrix.

2) To carry out the elementary row operation premultiply the A by E.

We will illustrate this process below for the each of the three types of elementary row operations.

- Interchanging two rows.

Suppose we want to interchange the second and third rows of A, a 3x2 matrix. To create the elementary row operator E, we interchange the second and third rows of the Identity matrix I_3.

$$\underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

Then, to interchange the second and third rows of A, we premultiply A by E, as shown.

$$R_2 \longleftrightarrow R_3 = \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}}$$

$$R_2 \leftrightarrow R_3 = \begin{bmatrix} 1\times0+0\times2+0\times4 & 1\times1+0\times3+0\times5 \\ 0\times0+0\times2+1\times4 & 0\times1+0\times3+1\times5 \\ 0\times0+1\times2+0\times4 & 0\times1+1\times3+0\times5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 = \begin{bmatrix} 0 & 1 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$$

- Multiply a row by a number:

Suppose we want to multiply each element in the second row of matrix A by 7. To create the elementary row operator E, we multiply each element in the second row of the Identity matrix I_2 by 7.

$$\underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}}$$

Then to multiply each element in the second row of A by 7, we premultiply A by E.

$$7R_2 \rightarrow R_2 = \underset{E}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}}$$

$$7R_2 \to R_2 = \begin{bmatrix} 1\times0+0\times3 & 1\times1+0\times4 & 1\times2+0\times5 \\ 0\times0+7\times3 & 0\times1+7\times4 & 0\times2+7\times5 \end{bmatrix}$$

$$7R_2 \to R_2 = \begin{bmatrix} 0 & 1 & 2 \\ 21 & 28 & 35 \end{bmatrix}$$

- Multiply a row and add it to another row:

Assume A is a 2x2 matrix. Suppose we want to multiply each element in the first row of A by 3: and we want to add that result to the second row of A. For this operation, creating the elementary row operator is a two-step process. First, we multiply each element in the first row of the identity matrix I_2 by 3. Next we add the result of that multiplication to second row of I_2 to produce E.

$$\underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0+3\times1 & 1+3\times0 \end{bmatrix} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

Then, to multiple each element in the first row of A by 3 and add the result to the second row, we premultiply A by E.

$$3R_1 + R_2 \rightarrow R_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

E A

$$3R_1 + R_2 \rightarrow R_2 = \begin{bmatrix} 1\times0+0\times2 & 1\times1+0\times3 \\ 3\times0+1\times2 & 3\times1+1\times3 \end{bmatrix}$$

$$3R_1 + R_2 \rightarrow R_2 = \begin{bmatrix} 0 & 1 \\ 2 & 6 \end{bmatrix}$$

How to perform Elementary Column operations :-

To perform an elementary column operation on A an r×c matrix, take the following steps.

1. To find E, the elementary column operator, apply the operation to an c×c Identity matrix.

2. To carry out the elementary column operation. Postmultiply A by E.

Let's work through an elementary column operation to illustrate the process. For example, suppose we want to interchange the first and second column of A, a 3x2 matrix. To create the elementary column operator E, we interchange the first and second columns of the Identity matrix I_2.

$$\underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \underset{E}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Then to interchange the first and second columns of A, we Postmultiply A by E as shown

$$C_1 \leftrightarrow C_2 = \underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}} \underset{E}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$C_1 \leftrightarrow C_2 = \begin{bmatrix} 0\times0+1\times1 & 0\times1+1\times0 \\ 2\times0+3\times1 & 2\times1+3\times0 \\ 4\times0+5\times1 & 4\times1+5\times0 \end{bmatrix}$$

$$C_1 \leftrightarrow C_2 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$$

Note that the process for the performing an elementary column operation on an $r \times c$ matrix is very similar to the process for performing an elementary row operation. The main differences are

- To operate on the $r \times c$ matrix A, the row operator E is created from an $r \times r$ identity matrix; whereas the column operator E is created from an $c \times c$ identity matrix.

- To perform a row operation, A is Premultiplied by E Whereas to perform a column operation, A is postmultiplied by E.

Now we check some example Which is not the elementary matrix.

Q. Determine if the following matrix is an elementary matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:-

Before solving this matrix we have Ist recall that the elementary matrix is formed by Identity matrix only by single row or single column operation on the Identity matrix.

So now ~~take~~ an Identity matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now interchange two rows R_2 and R_3

$$R_2 \leftrightarrow R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

so this is not our desired result.

Now interchange two columns C_2 & C_3.

$$C_2 \leftrightarrow C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now again this is not our required result.

Now multiply 7 by R_3 we get

$$7R_3 \rightarrow R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

so this is not possible only single row or column operation.

Now add R_2 in R_1.

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now again this is not our

desired result.
so we cannot get our desired result by only single row or column operation on the Identity matrix.
so this is not an elementary matrix.

Now we discuss the importance of elementary matrix or use of elementary matrix.

Applications:-

Elementary row operations are used in the Gaussian elemination to reduce a matrix to row echelon form. They are also used in Guass-Jordan elimination to further reduce the matrix to redused row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Elementary matrices are important because they can be used to stimulate the elementary row transformations. If we want to perform an elementary row transformation on a matrix A, it is enough to pre-multiply A by the elementary matrix obtained from the Identity by the same transformation.

Elementary matrix operations play an important role in many matrix algebra applications, such as finding the inverse of a matrix and solving simultaneous linear equations.