Kernel of Integral Equation:-

Kernel:-

$K(x,t)$ is the function of two variable x and t is called Kernel.

There are many types of Kernels:

(1) Degenerate Kernels.
or
(2) Separable Kernels.
(3) Difference Kernels.
(4) Symmetric Kernels.
(5) Resolvent Kernels.

(1) **Degenerate Kernels:-** / Separable Kernel.

The Degenerate Kernels is Defined as

$$K(x,t) = \sum_{K=1}^{n} a_K(x)\, b_K(t)$$

For Example:-

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt.$$ → 2nd kind of FIE

$$u(x) = f(x) + \lambda \int_a^b \sum_{K=1}^{n} a_K(x)\, b_K(t)\, u(t)\, dt.$$

$$u(x) = f(x) + \lambda \sum_{K=1}^{n} a_K(x) \int_a^b b_K(t)\, u(t)\, dt$$

$$C_K = \int_a^b b_K(t)\, u(t)\, dt$$

$$u(x) = f(x) + \lambda \sum_{K=1}^{n} a_K(x)\, C_K$$

(2) **Difference Kernel:-**

When the Kernel depends on the difference $x-\xi$ is called difference Kernel.

$$K(x,\xi) = K(x-\xi)$$

(3) Symmetric Kernel :-

Symmetric kernel is defined as

$K(x,t) = k(t,x)$.

$K(x,t) = \sin(x+t) = \sin(t+x) = k(t,x)$

(4) Resolvent Kernel :-

e.g. Volterra
Fredhol

The Resolvent Kernel is defined as

$$\Gamma(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \quad ; \quad D(\lambda) \neq 0$$

where

$$D(x,t;\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} B_n(x,t)$$

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} c_n$$

and

$$B_n(x,t) = c_n K(x,t) - n\int_a^b K(x,s) B_{n-1}(s,t) ds$$

or

$$c_n = \int_a^b B_{n-1}(t,t)\, dt$$

(5) Iterated Kernels

Iterated kernels is defined as

$$u_n(x) = f(x) + \sum_{i=1}^{n} \lambda^i \phi_i(x)$$

where

$$\phi_i(x) = \int_a^b K_i(x,y) f(y)\, dy$$

and

$$K_i(x,y) = \int_a^b K(x,t) K_{i-1}(t,y)\, dt$$

Exp # 01 Solve the Fredholm Integral Equation,

$$u(x) = x + \lambda \int_0^1 (xt^2 + x^2 t) u(t)\,dt$$

Sol :-

By degenerated Kernel

$$K(x,t) = xt^2 + x^2 t$$

$$= \sum_{k=1}^{2} a_k(x) b_k(t)$$

$$= a_1(x) b_1(t) + a_2(x) b_2(t)$$

$a_1(x) = x$; $b_1(t) = t^2$

$a_2(x) = x^2$ $b_2(t) = t$

$$\boxed{u(x) = f(x) + \lambda \sum_{k=1}^{2} a_k(x) c_k}$$

$$f_1 = \int_0^1 b_1(t) f(t)\,dt$$

$$= \int_0^1 t^2 \cdot t\,dt = \int_0^1 t^3 dt$$

$$\boxed{f_1 = 1/4}$$

$$f_2 = \int_0^1 b_2(t) f(t)\,dt$$

$$= \int_0^1 t \cdot t\,dt = \int_0^1 t^2 dt$$

$$\boxed{f_2 = \frac{1}{3}}$$

Then

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix}$$

Now To matrix A.

$a_{11}, a_{12}, a_{21}, a_{22} = ?$

$$a_{11} = \int_0^1 b_1(t) a_1(t)\,dt$$

$$= \int_0^1 t^2 t\,dt = \int_0^1 t^3 dt$$

$$\boxed{a_{11} = \frac{1}{4}}$$

$$a_{12} = \int_0^1 b_1(t) a_2(t)\,dt$$

$$= \int_0^1 t^2 \cdot t^2 dt = \int_0^1 t^4 dt$$

$$\boxed{a_{12} = \frac{1}{5}}$$

$$a_{21} = \int_0^1 b_2(t) a_1(t) dt = \frac{1}{3}$$

$$a_{21} = \int_0^1 b_2(t) a_2(t) dt = \frac{1}{4}$$

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$(I - \lambda A) C = F$$

– OR –

$$\begin{bmatrix} 1 - \frac{\lambda}{4} & -\frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1 - \frac{\lambda}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \end{bmatrix}$$

$$\left(1-\frac{\lambda}{4}\right)c_1 - \frac{\lambda}{5}c_2 = \frac{1}{4} \to (1$$

$$-\frac{\lambda}{3}c_1 + (1-\lambda/4)c_2 = \frac{1}{3} \to (2$$

$$c_1 = \frac{60+\lambda}{240-120\lambda-\lambda^2}$$

$$c_2 = \frac{80}{240-120\lambda-\lambda^2}$$

Hence

$$u(x) = x+\lambda\sum_{k=1}^{2} a_k(x)\,c_k$$

$$\boxed{u(x) = x+\lambda(a_1c_1+a_2c_2) = x+\lambda\left(x\cdot\frac{60+\lambda}{240-120\lambda-\lambda^2} + x^2\cdot\frac{80}{240-120\lambda-\lambda^2}\right)}$$

which is the require Results.

Example:- 02

Find the resolvent Kernel to solve the Volterra integral equation of the second kind.

Sol:-

$$u(x) = f(x)+\lambda\int_0^x e^{x-t}u(t)\,dt$$

The resolvent kernel

$$\Gamma(x,t;\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,t)$$

$$= K_1(x,t)+\lambda K_2(x,t)+\dots \to (1)$$

$$K_1(x,t) \equiv K(x,t) = e^{x-t}$$

we used

$$K(x,\xi) = e^{x-\xi}, \quad K(\xi,t) = e^{\xi-t}$$

$$K_2(x,t) = \int_t^x K(x,\xi)K_1(\xi,t)\,d\xi$$

$$= \int_t^x e^{x-\xi}\cdot e^{\xi-t}\,d\xi$$

$$= \int_t^x e^{x-t}\,d\xi$$

$$\boxed{K_2(x,t) = (x-t)e^{x-t}}$$

$$\boxed{K_3(x,t) = \frac{(x-t)^2}{2}e^{x-t}}$$

$$K_{n+1}(x,t) = \frac{(x-t)^n}{n!}e^{x-t}$$

Exp:- Find resolvent Kernel of Volterra integral equation with Kernel

$$K(x,t) = 1$$

Sol:-

Resolvent Kernel is

$$\Gamma(x,t;\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,t)$$

$$K_1(x,t) = K(x,t) = 1$$

$$K_2(x,t) = \int_t^x K(x,\xi)K_1(\xi,t)\,dt$$

$$= (x-t)$$

$$K_3(x,t) = \int_t^x K(x,\xi)K_2(\xi,t)\,d\xi$$

$$= \int_t^x (1)(\xi-t)\,d\xi$$

$$K_3(x,t) = \frac{(x-t)^2}{2}$$

and so not

$$K_{n+1}(x,t) = \frac{(x-t)^n}{n!}$$

Now

$$\Gamma(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!}$$

$$\Gamma(x,t;\lambda) = e^{\lambda(x-t)}$$

which is the require Result.

Exp:-

Find the resolvent Kernel

$$\phi(x) = x + \int_0^x (t-x)u(t)\,dt$$

$$Q_2: u(x) = x + \int_0^{1/2} u(t)\,dt.$$

Exp:- Find iterated Kernel for $K(x,t) = x + \sin t$

$a = -\pi,\ b = \pi$

Sol:-

$$K(x,t) = x + \sin t$$

$$K_1(x,t) = x + \sin t$$

$$K_2(x,t) = \int_{-\pi}^{\pi} K(x,\xi) K_1(\xi,t)\,d\xi$$

$$= \int_{-\pi}^{\pi} \underset{II}{(x+\sin\xi)}\underset{I}{(\xi+\sin t)}\,d\xi.$$

$$(\xi+\sin t)(\xi+\sin t)(x\xi - \cos\xi) - \left(\frac{x\xi^2}{2} - \sin\xi\right)\Big|_{-\pi}^{\pi}$$

$$K_2(x,t) = 2\pi(1 + x\sin t).$$

so on.

$$Q:- u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\,u(t)\,dt.$$

Exp:- Solve by Separable Kernel:

$$u(x) = \lambda\int_0^{\pi/2} \cos x \sin t\, u(t)\,dt.$$

Sol:-

$K(x,t) = \cos x \sin t$ is Separable.

$$u(x) = \lambda\cos x\int_0^{\pi/2} \sin t\, u(t)\,dt.$$

$$u(x) = \lambda\cos x\, c \longrightarrow (1)$$

where

$$c = \int_0^{\pi/2} \sin t\, u(t)\,dt.$$

$$c = \int_0^{\pi/2} \sin t(\lambda c\cos t)\,dt.$$

$$c = \lambda c\int_0^{\pi/2} \sin t\cos t\,dt.$$

$$= \frac{\lambda c}{2}\int_0^{\pi/2} \sin 2t\,dt$$

$$= \frac{\lambda c}{2}\left(-\frac{\cos 2t}{2}\Big|_0^{\pi/2}\right)$$

$$c = \frac{\lambda c}{2\cdot 2}(+1+1)$$

$$c = \frac{\lambda c}{2} \qquad \boxed{\lambda = 2}$$

Hence

$$\boxed{u(x) = 2C\cos x}$$

which is the require function

Q:- $u(x) = \lambda \int_0^1 2e^{x+t} u(t)\,dt.$

Q:- $u(x) = \lambda \int_0^{\pi} \sin(x+t)\,u(t)\,dt.$

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