Lecture # 6 Monday 03-02-2020

Chapter # 3

Volterra Integral Equation

Volterra Integral Equation of the second kind is defined as

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt \quad \text{---} (1)$$

a variety of analytic and numerical method such as successive method, approximation method, suppline method, adomian decomposition and the variational iteration method to handle the volterra integral equation

Method (1) The Adomian Decomposition Method (ADM):

The adomian decomposition method (ADM) was introduced and developed by the "George Adomian". This method consists of decomposition infinite element of components

The decomposition series is defined as

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots\dots$$

From (1) Eq $\Rightarrow$

$$u_0(x) + u_1(x) + u_2(x) + \dots\dots = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt$$

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)\, u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = \lambda \int_0^x K(x,t)\, u_0(t)\, dt$$

Put $n=1$

$$u_2(x) = \lambda \int_0^x K(x,t)\, u_1(t)\, dt$$

Put $n=2$

$$u_3(x) = \lambda \int_0^x K(x,t)\, u_2(t)\, dt$$

⋮

Example: Solve the following Volterra Integral Equation

$$u(x) = 1 - \int_0^x u(t)\, dt$$

Sol: By using ADM

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots \quad \text{--- (1)}$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = -\int_0^x u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = -\int_0^x u_0(t)\, dt$$

$$u_1(x) = -\int_0^x 1\, dt \Rightarrow -|t|_0^x$$

$$\boxed{u_1(x) = -x}$$

Put $n=1$

$$u_2(x) = -\int_0^x u_1(t)\, dt$$

$$u_2(x) = -\int_0^x -t\, dt \Rightarrow \int_0^x t\, dt$$

$$u_2(x) = \left|\frac{t^2}{2}\right|_0^x \Rightarrow \frac{x^2}{2}$$

$$\boxed{u_2(x) = \frac{x^2}{2}}$$

Put $n=2$

$$u_3(x) = -\int_0^x u_2(t)\, dt$$

$$u_3(x) = -\int_0^x \frac{t^2}{2}\, dt \Rightarrow \left|-\frac{t^3}{6}\right|_0^x$$

$$\boxed{u_3(x) = -\frac{x^3}{3!}}$$

Put $n=3$

$$u_4(x) = -\int_0^x u_3(t)\,dt$$

$$u_4(x) = -\int_0^x -\frac{t^3}{3!} \Rightarrow \left|\frac{t^4}{4!}\right|_0^x$$

$$\boxed{u_4(x) = \frac{x^4}{4!}}$$

⋮

Now by ① Equation

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$\boxed{u(x) = e^{-x}}$. Ans.

Example: Solve the following Volterra Integral Equation

$$u(x) = 1 + \int_0^x (t-x)\,u(t)\,dt$$

Sol: By using ADM

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad ①$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = \int_0^x (t-x)\,u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = \int_0^x (t-x)\,u_0(t)\,dt$$

$$u_1(x) = \int_0^x (t-x)\,dt \Rightarrow \left|\frac{t^2}{2}\right|_0^x - x\left|t\right|_0^x$$

$$= \frac{x^2}{2} - x^2$$

$$\boxed{u_1(x) = -\frac{1}{2}x^2}$$

Put $n=1$

$$u_2(x) = \int_0^x (t-x)\,u_1(t)\,dt$$

$$u_2(x) = \int_0^x (t-x)\,-\frac{1}{2}t^2\,dt \Rightarrow \int_0^x \left(-\frac{1}{2}t^3 + \frac{1}{2}xt^2\right)dt$$

$$u_2(x) = -\frac{1}{8}\left|t^4\right|_0^x + \frac{x}{6}\left|t^3\right|_0^x$$

$$u_2(x) = -\frac{x^4}{8} + \frac{x^4}{6} \Rightarrow \frac{-6x^4+8x^4}{48} \Rightarrow \frac{x^4}{24}$$

$$\boxed{u_2(x) = \frac{x^4}{4!}}$$

Put $n=2$

$$u_3(x) = \int_0^x (t-x)\,u_2(t)\,dt$$

$$u_3(x) = \int_0^x (t-x)\,\frac{t^4}{4!}\,dt$$

$$u_3(x) = \int_0^x \left(\frac{t^5}{4!} - \frac{xt^4}{4!}\right)dt \Rightarrow \left|\frac{t^6}{4!\,6} - \frac{x\,t^5}{4!\,5}\right|_0^x$$

$$u_3(x) = \frac{x^6}{4!\,6} - \frac{x^6}{4!\,5} \Rightarrow \frac{5x^6 - 6x^6}{4!\,5\cdot 6} \Rightarrow -\frac{x^6}{6!}$$

$$\boxed{u_3(x) = -\frac{x^6}{6!}}$$

Put $n=3$

$$u_4(x) = \int_0^x (t-x)\,u_3(t)\,dt$$

$$u_4(x) = \int_0^x (t-x)\left(-\frac{t^6}{6!}\right)dt \Rightarrow -\int_0^x \left(\frac{t^7}{6!} - \frac{xt^6}{6!}\right)dt$$

$$u_4(x) = -\left|\frac{t^8}{6!\,8} - x\frac{t^7}{6!\,7}\right|_0^x$$

$$u_4(x) = -\left[\frac{x^8}{6!\,8} - \frac{x^8}{6!\,7}\right] \Rightarrow -\left[\frac{7x^8 - 8x^8}{6!\,7\cdot 8}\right] \Rightarrow \frac{+x^8}{8\cdot 7\cdot 6!}$$

$$\boxed{u_4(x) = +\frac{x^8}{8!}}$$

From ① Eq →

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots$

$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$

$\boxed{u(x) = \cos x}$ Ans.

Example: Solve the following Volterra Integral Equation

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\, u(t)\, dt.$$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots \quad \text{—①}$$

$$u_0(x) = f(x) \qquad \because \left[f(x) = 1 - x - \frac{1}{2}x^2\right]$$

$$\boxed{u_0(x) = 1 - x - \frac{1}{2}x^2}$$

$$u_{n+1}(x) = -\int_0^x (t-x)\, u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = -\int_0^x (t-x)\, u_0(t)\, dt$$

$$u_1(x) = -\int_0^x (t-x)\left(1 - t - \frac{1}{2}t^2\right) dt$$

$$u_1(x) = -\int_0^x \left(t - t^2 - \frac{1}{2}t^3 - x + xt + \frac{1}{2}xt^2\right) dt$$

$$u_1(x) = -\left|\frac{t^2}{2} - \frac{t^3}{3} - \frac{1}{8}t^4 - xt + \frac{xt^2}{2} + \frac{xt^3}{6}\right|_0^x$$

$$u_1(x) = -\left(\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - x^2 + \frac{x^3}{2} + \frac{x^4}{6}\right)$$

$$u_1(x) = -\left(-\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

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$$u_1(x) = \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}$$

Put $n=1$

$$u_2(x) = -\int_0^x (t-x)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (t-x)\left(\frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!}\right) dt$$

$$u_2(x) = -\int_0^x \left(\frac{t^3}{2} - \frac{t^4}{6} - \frac{t^5}{24} - \frac{xt^2}{2} + \frac{xt^3}{6} + \frac{xt^4}{24}\right) dt$$

$$u_2(x) = -\left| \frac{t^4}{8} - \frac{t^5}{30} - \frac{t^6}{144} - \frac{xt^3}{6} + \frac{xt^4}{24} + \frac{xt^5}{120} \right|_0^x$$

$$u_2(x) = -\left(\frac{x^4}{8} - \frac{x^5}{30} - \frac{x^6}{144} - \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}\right)$$

$$u_2(x) = -\left[\frac{6x^4 - 8x^4}{48} - \frac{24x^5 + 30x^5}{720} - \frac{120x^6 - 144x^6}{17280}\right]$$

$$u_2(x) = -\left[-\frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}\right] \Rightarrow \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!}$$

$$u_2(x) = \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!}$$

Put $n=2$

$$u_3(x) = -\int_0^x (t-x)\, u_2(t)\, dt$$

$$u_3(x) = -\int_0^x (t-x)\left(\frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^6}{6!}\right) dt$$

$$u_3(x) = -\int_0^x \left(\frac{t^5}{24} - \frac{t^6}{120} - \frac{t^7}{720} - \frac{xt^4}{24} + \frac{xt^5}{120} + \frac{xt^6}{720}\right) dt$$

$$u_3(x) = -\left| \frac{t^6}{144} - \frac{t^7}{840} - \frac{t^8}{5760} - \frac{xt^5}{120} + \frac{xt^6}{720} + \frac{xt^7}{5040} \right|_0^x$$

$u_3(x) = -\left(\frac{x^6}{144} - \frac{x^7}{840} - \frac{x^8}{5760} - \frac{x^6}{120} + \frac{x^7}{720} + \frac{x^8}{5040}\right)$

$u_3(x) = -\left[\frac{120x^6 - 144x^6}{17280} - \frac{720x^7 + 840x^7}{604800} - \frac{5040x^8 + 5760x^8}{29030400}\right]$

$u_3(x) = -\left[-\frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320}\right]$

$$\boxed{u_3(x) = \frac{x^6}{6!} - \frac{x^7}{7!} - \frac{x^8}{8!}}$$

⋮

From ① Eq ⇒

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$

$u(x) = 1 - x - \frac{x^2}{2!} + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^6}{6!} - \frac{x^7}{7!} - \frac{x^8}{8!} + \cdots$

$u(x) = 1 - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)$

$\boxed{u(x) = 1 - \sinh x}$ Ans.

Example: Solve the Volterra Integral Equation

$$u(x) = 5x^3 - x^5 + \int_0^x t\, u(t)\, dt$$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad ①$$

$u_0(x) = f(x)$ ∵ $[f(x) = 5x^3 - x^5]$

$$\boxed{u_0(x) = 5x^3 - x^5}$$

$$u_{n+1}(x) = \int_0^x t\, u_n(t)\, dt$$

Put $n=0$

$u_1(x) = \int_0^x t(5t^3 - t^5)\,dt$

$u_1(x) = \int_0^x (5t^4 - t^6)\,dt$

$u_1(x) = \left| \frac{5t^5}{5} - \frac{t^7}{7} \right|_0^x$

$$\boxed{u_1(x) = x^5 - \frac{x^7}{7}}$$

Put $n=1$

$u_2(x) = \int_0^x t\left(t^5 - \frac{t^7}{7}\right)dt$

$u_2(x) = \int_0^x \left(t^6 - \frac{t^8}{7}\right)dt$

$u_2(x) = \left| \frac{t^7}{7} - \frac{t^9}{63} \right|_0^x$

$$\boxed{u_2(x) = \frac{x^7}{7} - \frac{x^9}{63}}$$

Put $n=2$

$u_3(x) = \int_0^x t\left(\frac{t^7}{7} - \frac{t^9}{63}\right)dt$

$u_3(x) = \int_0^x \left(\frac{t^8}{7} - \frac{t^{10}}{63}\right)dt$

$u_3(x) = \left| \frac{t^9}{63} - \frac{t^{11}}{693} \right|_0^x$

$$\boxed{u_3(x) = \frac{x^9}{63} - \frac{x^{11}}{693}}$$

⋮

From ① Eq $\Rightarrow$

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots$

$u(x) = 5x^3 - x^5 + x^5 - \frac{1}{7}x^7 + \frac{1}{7}x^7 - \frac{x^9}{63} + \frac{x^9}{63} - \frac{x^{11}}{693} + \dots$

$\boxed{u(x) = 5x^3}$ Ans.

(Exercise 3.21): In Exercise (1-26), Solve the following volterra integral equation by using Adomian Decomposition Method (ADM):

(1) $u(x) = 6x - 3x^2 + \int_0^x u(t)\,dt$.

Sol: By using Adomian Decomposition Method (ADM)

$u(x) = \sum_{n=0}^{\infty} u_n(x)$

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots$ —①

$u_0(x) = f(x)$ $[f(x) = 6x - 3x^2]$

$\boxed{u_0(x) = 6x - 3x^2}$

$u_{n+1}(x) = \int_0^x u_n(t)\,dt$

Put $n=0$

$u_1(x) = \int_0^x u_0(t)\,dt$

$u_1(x) = \int_0^x (6t - 3t^2)\,dt \Rightarrow \left| \frac{6t^2}{2} - \frac{3t^3}{3} \right|_0^x$

$\boxed{u_1(x) = 3x^2 - x^3}$

Put $n=1$

$u_2(x) = \int_0^x u_1(t)\,dt$

$u_2(x) = \int_0^x (3t^2 - t^3)\,dt \Rightarrow \left| \frac{3t^3}{3} - \frac{t^4}{4} \right|_0^x$

$\boxed{u_2(x) = x^3 - \frac{x^4}{4}}$

Put $n=2$

$u_3(x) = \int_0^x u_2(t)\,dt$

$u_3(x) = \int_0^x \left(t^3 - \frac{t^4}{4}\right) dt \Rightarrow \left| \frac{t^4}{4} - \frac{t^5}{20} \right|_0^x$

$$\boxed{u_3(x) = \frac{x^4}{4} - \frac{x^5}{20}}$$

Put $n = 3$

$u_4(x) = \int_0^x u_3(t)\,dt$

$u_4(x) = \int_0^x \left(\frac{t^4}{4} - \frac{t^5}{20}\right) dt \Rightarrow \left| \frac{t^5}{20} - \frac{t^6}{120} \right|_0^x$

$$\boxed{u_4(x) = \frac{x^5}{20} - \frac{x^6}{120}}$$

From ① Eq $\Rightarrow$

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$

$u(x) = 6x - 3x^2 + 3x^2 - x^3 + x^3 - \frac{x^4}{4} + \frac{x^4}{4} - \frac{x^5}{20} + \frac{x^5}{20} - \frac{x^6}{120} + \cdots$

$$\boxed{u(x) = 6x}$$ Ans.

(2) $u(x) = 6x - x^3 + \int_0^x (x-t)\,u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$u(x) = \sum_{n=0}^{\infty} u_n(x)$

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$ ①

$u_0(x) = f(x)$ $\quad \because [f(x) = 6x - x^3]$

$$\boxed{u_0(x) = 6x - x^3}$$

$u_{n+1}(x) = \int_0^x (x-t)\,u_n(t)\,dt$

Put $n=0$

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$u_1(x) = \int_0^x (x-t)\, u_0(t)\, dt$

$u_1(x) = \int_0^x (x-t)(6t - t^3)\, dt \Rightarrow \int_0^x (6xt - xt^3 - 6t^2 + t^4)\, dt$

$u_1(x) = \left| \frac{6xt^2}{2} - \frac{xt^4}{4} - \frac{6t^3}{3} + \frac{t^5}{5} \right|_0^x \Rightarrow 3x^3 - \frac{x^5}{4} - 2x^3 + \frac{x^5}{5}$

$$\boxed{u_1(x) = x^3 - \frac{x^5}{20}}$$

Put $n=1$

$u_2(x) = \int_0^x (x-t)\, u_1(t)\, dt$

$u_2(x) = \int_0^x (x-t)\left(t^3 - \frac{t^5}{20}\right) dt \Rightarrow \int_0^x \left(xt^3 - \frac{xt^5}{20} - t^4 + \frac{t^6}{20}\right) dt$

$u_2(x) = \left| \frac{xt^4}{4} - \frac{xt^6}{120} - \frac{t^5}{5} + \frac{t^7}{140} \right|_0^x \Rightarrow \left(\frac{x^5}{4} - \frac{x^7}{120} - \frac{x^5}{5} + \frac{x^7}{140}\right)$

$u_2(x) = \frac{x^5}{20} - \frac{140x^7 + 120x^7}{16800}$

$$\boxed{u_2(x) = \frac{x^5}{20} - \frac{x^7}{840}}$$

Put $n=2$

$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt$

$u_3(x) = \int_0^x (x-t)\left(\frac{t^5}{20} - \frac{t^7}{840}\right) dt \Rightarrow \int_0^x \left(\frac{xt^5}{20} - \frac{xt^7}{840} - \frac{t^6}{20} + \frac{t^8}{840}\right) dt$

$u_3(x) = \left| \frac{xt^6}{120} - \frac{xt^8}{6720} - \frac{t^7}{140} + \frac{t^9}{7560} \right|_0^x \Rightarrow \left(\frac{x^7}{120} - \frac{x^9}{6720} - \frac{x^7}{140} + \frac{x^9}{7560}\right)$

$$\boxed{u_3(x) = \frac{x^7}{840} - \frac{x^9}{60480}}$$

Put $n=3$

$$u_4(x) = \int_0^x (x-t)\, u_3(t)\,dt$$

$$u_4(x) = \int_0^x (x-t)\left(\frac{t^7}{840} - \frac{t^9}{60480}\right)dt \Rightarrow \int_0^x \left(\frac{xt^7}{840} - \frac{xt^9}{60480} - \frac{t^8}{840} + \frac{t^{10}}{60480}\right)dt$$

$$u_4(x) = \left| \frac{xt^8}{6720} - \frac{xt^{10}}{604800} - \frac{t^9}{7560} + \frac{t^{11}}{665280} \right|_0^x$$

$$u_4(x) = \left(\frac{x^9}{6720} - \frac{x^{11}}{604800} - \frac{x^9}{7560} + \frac{x^{11}}{665280}\right)$$

$$\boxed{u_4(x) = \frac{x^9}{60480} - \frac{x^{11}}{6652800}}$$

⋮

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$$

$$u(x) = 6x - x^2 + x^2 - \frac{x^5}{20} + \frac{x^5}{20} - \frac{x^7}{840} + \frac{x^7}{840} - \frac{x^9}{60480} + \frac{x^9}{60480} - \frac{x^{11}}{6652800} + \cdots$$

$$\boxed{u(x) = 6x}$$ Ans.

(3) $u(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad — ①$$

$$u_0(x) = f(x)$$

$$\because \left[f(x) = 1 - \frac{1}{2}x^2\right]$$

$$\boxed{u_0(x) = 1 - \frac{1}{2}x^2}$$

$$u_{n+1}(x) = \int_0^x u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = \int_0^x u_0(t)\,dt$$

$$u_1(x) = \int_0^x \left(1 - \frac{1}{2}t^2\right)dt$$

$$u_1(x) = \left| t - \frac{1}{6}t^3 \right|_0^x$$

$$\boxed{u_1(x) = x - \frac{x^3}{6}}$$

Put $n=1$

$$u_2(x) = \int_0^x u_1(t)\,dt$$

$$u_2(x) = \int_0^x \left(t - \frac{t^3}{6}\right)dt$$

$$u_2(x) = \left| \frac{t^2}{2} - \frac{t^4}{24} \right|_0^x$$

$$\boxed{u_2(x) = \frac{x^2}{2} - \frac{x^4}{24}}$$

Put $n=2$

$$u_3(x) = \int_0^x u_2(t)\,dt$$

$$u_3(x) = \int_0^x \left(\frac{t^2}{2} - \frac{t^4}{24}\right)dt$$

$$u_3(x) = \left| \frac{t^3}{6} - \frac{t^5}{120} \right|_0^x$$

$$\boxed{u_3(x) = \frac{x^3}{6} - \frac{x^5}{120}}$$

Put $n=3$

$$u_4(x) = \int_0^x u_3(t)\,dt$$

$$u_4(x) = \int_0^x \left(\frac{t^3}{6} - \frac{t^5}{120}\right)dt$$

$$u_4(x) = \left|\frac{t^4}{24} - \frac{t^6}{720}\right|_0^x$$

$$\boxed{u_4(x) = \frac{x^4}{24} - \frac{x^6}{720}}$$

⋮

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 - \frac{1}{2}x^2 + x - \frac{x^3}{6} + \frac{x^4}{2} - \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$\boxed{u(x) = 1 + x}$$ Ans.

(4) $u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad ①$$

$$u_0(x) = f(x) \qquad \left[f(x) = x - \frac{2}{3}x^3\right]$$

$$\boxed{u_0(x) = x - \frac{2}{3}x^3}$$

$$u_{n+1}(x) = -2\int_0^x u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = -2\int_0^x \left(t - \frac{2}{3}t^3\right)dt$$

$$u_1(x) \Rightarrow -2\left|\frac{t^2}{2} - \frac{2}{12}t^4\right|_0^x \Rightarrow -2\left|\frac{t^2}{2} - \frac{1}{6}t^4\right|_0^x$$

$$u_1(x) \Rightarrow -2\left[\frac{x^2}{2} - \frac{1}{6}x^4\right] \Rightarrow -x^2 + \frac{1}{3}x^4$$

$$\boxed{u_1(x) = -x^2 + \frac{1}{3}x^4}$$

Put $n=1$

$$u_2(x) = -2\int_0^x u_1(t)\,dt$$

$$u_2(x) = -2\int_0^x \left(-t^2 + \frac{1}{3}t^4\right)dt \Rightarrow -2\left|-\frac{t^3}{3} + \frac{1}{15}t^5\right|_0^x$$

$$u_2(x) = -2\left[-\frac{x^3}{3} + \frac{1}{15}x^5\right]$$

$$\boxed{u_2(x) = \frac{2x^3}{3} - \frac{2}{15}x^5}$$

Put $n=2$

$$u_3(x) = -2\int_0^x u_2(t)\,dt$$

$$u_3(x) = -2\int_0^x \left(\frac{2}{3}t^3 - \frac{2}{15}t^5\right)dt \Rightarrow -2\left|\frac{2}{12}t^4 - \frac{2}{90}t^6\right|_0^x$$

$$u_3(x) = -2\left[\frac{1}{6}x^4 - \frac{1}{45}x^6\right]$$

$$\boxed{u_3(x) \Rightarrow -\frac{1}{3}x^4 + \frac{2}{45}x^6}$$

Put n=3

$$u_4(x) = -2\int_0^x u_3(t)\,dt$$

$$u_4(x) = -2\int_0^x \left(-\frac{1}{3}t^4 + \frac{2}{45}t^6\right)dt \Rightarrow -2\left|-\frac{1}{15}t^5 + \frac{2}{315}t^7\right|_0^x$$

$$u_4(x) = -2\left[-\frac{1}{15}x^5 + \frac{2}{315}x^7\right]$$

$$\boxed{u_4(x) = \frac{2}{15}x^5 - \frac{4}{315}x^7}$$

⋮

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$$

$$u(x) = x - \frac{2}{3}x^3 - x^2 + \frac{1}{3}x^4 + \frac{2}{3}x^3 - \frac{2}{15}x^5 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \frac{2}{15}x^5 - \frac{4}{315}x^7 \cdots$$

$$\boxed{u(x) = x - x^2}$$ Ans.

---

(5) $u(x) = 1 + x + \int_0^x (x-t)\,u(t)\,dt$.

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad —①$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1+x]$$

$$\boxed{u_0(x) = 1+x}$$

$$u_{n+1}(x) = \int_0^x (x-t)\,u_n(t)\,dt$$

Put n=0

$$u_1(x) = \int_0^x (x-t)\,u_0(t)\,dt$$

$$u_1(x) = \int_0^x (x-t)(1+t)\,dt$$

$$u_1(x) = \int_0^x (x + xt - t - t^2)\,dt$$

$$u_1(x) = \left| xt + \frac{xt^2}{2} - \frac{t^2}{2} - \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = x^2 + \frac{x^3}{2} - \frac{x^2}{2} - \frac{x^3}{3}$$

$$\boxed{u_1(x) = \frac{x^2}{2} + \frac{x^3}{6}}$$

Put $n=1$

$$u_2(x) = \int_0^x (x-t)\,u_1(t)\,dt$$

$$u_2(x) = \int_0^x (x-t)\left(\frac{t^2}{2} + \frac{t^3}{6}\right) dt$$

$$u_2(x) = \int_0^x \left[\frac{xt^2}{2} + \frac{xt^3}{6} - \frac{t^3}{2} - \frac{t^4}{6}\right] dt$$

$$u_2(x) = \left| \frac{xt^3}{6} + \frac{xt^4}{24} - \frac{t^4}{8} - \frac{t^5}{30} \right|_0^x$$

$$u_2(x) = \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^4}{8} - \frac{x^5}{30}$$

$$\boxed{u_2(x) = \frac{x^4}{24} + \frac{x^5}{720}}$$

Put $n=2$

$$u_3(x) = \int_0^x (x-t)\,u_2(t)\,dt$$

$$u_3(x) = \int_0^x (x-t)\left(\frac{t^4}{24} + \frac{t^5}{720}\right) dt$$

$$u_3(x) = \int_0^x \left(\frac{xt^4}{24} + \frac{xt^5}{720} - \frac{t^5}{24} - \frac{t^6}{720}\right) dt$$

$$u_3(x) = \left| \frac{xt^5}{120} + \frac{xt^6}{4320} - \frac{t^6}{144} - \frac{t^7}{5040} \right|_0^x$$

$$u_3(x) = \left[\frac{x^6}{120} + \frac{x^7}{4320} - \frac{x^6}{144} - \frac{x^7}{5040}\right]$$

$$u_3(x) = \frac{24x^6}{17280} + \frac{720x^7}{21772800}$$

$$\boxed{u_3(x) = \frac{x^6}{720} + \frac{x^7}{30240}}$$

Put $n = 3$

$$u_4(x) = \int_0^x (x-t)\, u_3(t)\, dt$$

$$u_4(x) = \int_0^x (x-t)\left(\frac{t^6}{720} + \frac{t^7}{30240}\right) dt$$

$$u_4(x) = \int_0^x \left(\frac{xt^6}{720} + \frac{xt^7}{30240} - \frac{t^7}{720} - \frac{t^8}{30240}\right) dt$$

$$u_4(x) = \left| \frac{xt^7}{5040} + \frac{xt^8}{241920} - \frac{t^8}{5760} - \frac{t^9}{272160} \right|_0^x$$

$$u_4(x) = \left[\frac{x^8}{5040} + \frac{x^9}{241920} - \frac{x^8}{5760} - \frac{x^9}{272160}\right]$$

$$u_4(x) = \left[\frac{720x^8}{29030400} + \frac{30240x^9}{65840947200}\right]$$

$$\boxed{u_4(x) = \frac{x^8}{40320} + \frac{x^9}{2177280}}$$

From ① Eq/ ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

$$\boxed{u(x) = e^x} \text{. Ans.}$$

(6) $u(x) = 1 - x + \int_0^x (x-t)\,u(t)\,dt$.

Sol. By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad \text{—①}$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1 - x]$$

$$\boxed{u_0(x) = 1 - x}$$

$$u_{n+1}(x) = \int_0^x (x-t)\,u_n(t)\,dt$$

Put $n = 0$

$$u_1(x) = \int_0^x (x-t)(1-t)\,dt$$

$$u_1(x) = \int_0^x (x - xt - t + t^2)\,dt$$

$$u_1(x) = \left| xt - \frac{xt^2}{2} - \frac{t^2}{2} + \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = x^2 - \frac{x^3}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\boxed{u_1(x) = \frac{x^2}{2} - \frac{x^3}{6}}$$

For $n = 1$

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$$u_2(x) = \int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = \int_0^x (x-t)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) dt \Rightarrow \int_0^x \left(\frac{xt^2}{2} - \frac{xt^3}{6} - \frac{t^3}{2} + \frac{t^4}{6}\right) dt$$

$$u_2(x) = \left| \frac{xt^3}{6} - \frac{xt^4}{24} - \frac{t^4}{8} + \frac{t^5}{30} \right|_0^x$$

$$u_2(x) = \left(\frac{x^4}{6} - \frac{x^5}{24} - \frac{x^4}{8} + \frac{x^5}{30}\right)$$

$$u_2(x) \quad \frac{2x^4}{48} - \frac{6x^5}{720}$$

$$\boxed{u_2(x) = \frac{x^4}{24} - \frac{x^5}{120}}$$

Put $n=2$

$$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = \int_0^x (x-t)\left(\frac{t^4}{24} - \frac{t^5}{120}\right) dt \Rightarrow \int_0^x \left(\frac{xt^4}{24} - \frac{xt^5}{120} - \frac{t^5}{24} + \frac{t^6}{120}\right) dt$$

$$u_3(x) = \left| \frac{xt^5}{120} - \frac{xt^6}{720} - \frac{t^6}{144} + \frac{t^7}{840} \right|_0^x$$

$$u_3(x) = \left(\frac{x^6}{120} - \frac{x^7}{720} - \frac{x^6}{144} + \frac{x^7}{840}\right)$$

$$u_3(x) = \frac{24x^6}{17280} - \frac{120x^7}{604800}$$

$$\boxed{u_3(x) = \frac{x^6}{720} - \frac{x^7}{5040}}$$

Put $n=3$

$$u_4(x) = \int_0^x (x-t)\, u_3(t)\, dt$$

$$u_4(x) = \int_0^x (x-t)\left(\frac{t^6}{720} - \frac{t^7}{5040}\right)dt \Rightarrow \int_0^x \left(\frac{xt^6}{720} - \frac{xt^7}{5040} - \frac{t^7}{720} + \frac{t^8}{5040}\right)dt$$

$$u_4(x) = \left| \frac{xt^7}{5040} - \frac{xt^8}{40320} - \frac{t^8}{5760} + \frac{t^9}{45360} \right|_0^x$$

$$u_4(x) = \left(\frac{x^8}{5040} - \frac{x^9}{40320} - \frac{x^8}{5760} + \frac{x^9}{45360}\right)$$

$$= \left(\frac{720\, x^8}{29030400} - \frac{5040\, x^9}{1828915200}\right)$$

$$\boxed{u_4(x) = \frac{x^8}{40320} - \frac{x^9}{362880}}$$

From ① Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} - \frac{x^9}{9!} + \cdots$$

$$\boxed{u(x) = e^{-x}}$$ Ans.

(7) $u(x) = 1 + x - \int_0^x (x-t)\, u(t)\, dt$

Sol: $u(x) = 1 + x - \int_0^x (x-t)\, u(t)\, dt$

By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

(42)

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots \quad -(1)$$

$$u_0(x) = f(x) \qquad\qquad : [f(x) = 1+x]$$

$$\boxed{u_0(x) = 1+x}$$

$$u_n(x) = -\int_0^x (x-t)\, u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = -\int_0^x (x-t)\, u_0(t)\, dt$$

$$u_1(x) = -\int_0^x (x-t)(1+t)\, dt \Rightarrow -\int_0^x (x + xt - t - t^2)\, dt$$

$$u_1(x) = -\left| xt + \frac{xt^2}{2} - \frac{t^2}{2} - \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = -\left[x^2 + \frac{x^3}{2} - \frac{x^2}{2} - \frac{x^3}{3} \right]$$

$$u_1(x) = -\left(\frac{x^2}{2} + \frac{x^3}{6} \right)$$

$$\boxed{u_1(x) = -\frac{x^2}{2} - \frac{x^3}{6}}$$

Put $n=1$

$$u_2(x) = -\int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (x-t)\left(-\frac{t^2}{2} - \frac{t^3}{6}\right) dt$$

$$u_2(x) = -\int_0^x \left(-\frac{xt^2}{2} - \frac{xt^3}{6} + \frac{t^3}{2} + \frac{t^4}{6}\right) dt$$

$$u_2(x) = -\left| -\frac{xt^3}{6} - \frac{xt^4}{24} + \frac{t^4}{8} + \frac{t^5}{30} \right|_0^x$$

(42)

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots \quad -①$

$u_0(x) = f(x) \qquad \because [f(x) = 1+x]$

$\boxed{u_0(x) = 1+x}$

$u_{n+1}(x) = -\int_0^x (x-t)\,u_n(t)\,dt$

Put $n=0$

$u_1(x) = -\int_0^x (x-t)\,u_0(t)\,dt$

$u_1(x) = -\int_0^x (x-t)(1+t)\,dt \Rightarrow -\int_0^x (x + xt - t - t^2)\,dt$

$u_1(x) = -\left| xt + \frac{xt^2}{2} - \frac{t^2}{2} - \frac{t^3}{3} \right|_0^x$

$u_1(x) = -\left[x^2 + \frac{x^3}{2} - \frac{x^2}{2} - \frac{x^3}{3} \right]$

$u_1(x) = -\left(\frac{x^2}{2} + \frac{x^3}{6} \right)$

$\boxed{u_1(x) = -\frac{x^2}{2} - \frac{x^3}{6}}$

Put $n=1$

$u_2(x) = -\int_0^x (x-t)\,u_1(t)\,dt$

$u_2(x) = -\int_0^x (x-t)\left(-\frac{t^2}{2} - \frac{t^3}{6}\right)dt$

$u_2(x) = -\int_0^x \left(-\frac{xt^2}{2} - \frac{xt^3}{6} + \frac{t^3}{2} + \frac{t^4}{6}\right)dt$

$u_2(x) = -\left| -\frac{xt^3}{6} - \frac{xt^4}{24} + \frac{t^4}{8} + \frac{t^5}{30} \right|_0^x$

$$u_2(x) = -\left[-\frac{x^4}{6} - \frac{x^5}{24} + \frac{x^4}{8} + \frac{x^5}{30}\right]$$

$$= -\left[-\frac{2x^4}{48} - \frac{6x^5}{720}\right]$$

$$\boxed{u_2(x) = \frac{x^4}{24} + \frac{x^5}{120}}$$

For $n=2$

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{24} + \frac{t^5}{120}\right) dt$$

$$u_3(x) = -\int_0^x \left(\frac{xt^4}{24} + \frac{xt^5}{120} - \frac{t^5}{24} - \frac{t^6}{120}\right) dt$$

$$u_3(x) = -\left|\frac{xt^5}{120} + \frac{xt^6}{720} - \frac{t^6}{144} - \frac{t^7}{840}\right|_0^x$$

$$u_3(x) = -\left[\frac{x^6}{120} + \frac{x^7}{720} - \frac{x^6}{144} - \frac{x^7}{840}\right]$$

$$u_3(x) = -\left[\frac{24x^6}{17280} + \frac{120x^7}{604800}\right]$$

$$\boxed{u_3(x) = -\frac{x^6}{720} - \frac{x^7}{5040}}$$

From ① Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots$$

$$u(x) = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$u(x) = \cos x + \sin x$$ Ans.

(B) $u(x) = 1 - \int_0^x (x-t)\, u(t)\, dt$

Sol. By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad \text{--- (1)}$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$u_0(x) = 1$$

$$u_{n+1}(x) = -\int_0^x (x-t)\, u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = -\int_0^x (x-t)\, u_0(t)\, dt$$

$$u_1(x) = -\int_0^x (x-t)(1)\, dt$$

$$u_1(x) = -\int_0^x (x-t)\, dt$$

$$u_1(x) = -\left| xt - \frac{t^2}{2} \right|_0^x$$

$$u_1(x) = -\left[x^2 - \frac{x^2}{2} \right]$$

$$u_1(x) = -\frac{x^2}{2}$$

Put $n=1$

$$u_2(x) = \int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (x-t)\left(-\frac{t^2}{2}\right)dt$$

$$u_2(x) = -\int_0^x \left(-\frac{xt^2}{2} + \frac{t^3}{2}\right)dt$$

$$u_2(x) = -\left| -\frac{xt^3}{6} + \frac{t^4}{8} \right|_0^x$$

$$u_2(x) = -\left[-\frac{x^4}{6} + \frac{x^4}{8} \right]$$

$$\boxed{u_2(x) = \frac{x^4}{24}}$$

Put $n=2$

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\,dt$$

$$u_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{24}\right)dt$$

$$u_3(x) = -\int_0^x \left(\frac{xt^4}{24} - \frac{t^5}{24}\right)dt$$

$$u_3(x) = -\left| \frac{xt^5}{120} - \frac{t^6}{144} \right|_0^x$$

$$u_3(x) = -\left[\frac{x^6}{120} - \frac{x^6}{144} \right]$$

$$\boxed{u_3(x) = -\frac{x^6}{720}}$$

From ① Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots$$

$$u_2(x) = -\int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (x-t)\left(-\frac{t^3}{6}\right) dt$$

$$u_2(x) = -\int_0^x \left(-\frac{xt^3}{6} + \frac{t^4}{6}\right) dt$$

$$u_2(x) = -\left| -\frac{xt^4}{24} + \frac{t^5}{30} \right|_0^x$$

$$u_2(x) = -\left[-\frac{x^5}{24} + \frac{x^5}{30} \right]$$

$$\boxed{u_2(x) = \frac{x^5}{120}}$$

Put $n=2$

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = -\int_0^x (x-t)\left(\frac{t^5}{120}\right) dt$$

$$u_3(x) = -\int_0^x \left(\frac{xt^5}{120} - \frac{t^6}{120}\right) dt$$

$$u_3(x) = -\left| \frac{xt^6}{720} - \frac{t^7}{840} \right|_0^x$$

$$u_3(x) = -\left[\frac{x^7}{720} - \frac{x^7}{840} \right]$$

$$\boxed{u_3(x) = -\frac{x^7}{5040}}$$

From ① Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\boxed{u(x) = \sin x}$$ Ans.

(10) $u(x) = 1 + \int_0^x u(t)\,dt$

Sol: By using Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad ①$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = \int_0^x u_n(t)$$

Put n=0

$$u_1(x) = \int_0^x u_0(t)\,dt$$

$$u_1(x) = \int_0^x 1\,dt$$

$$u_1(x) = \left| t \right|_0^x$$

$$\boxed{u_1(x) = x}$$

Put n=1

$$u_2(x) = \int_0^x u_1(t)\,dt$$

$$u_2(x) = \int_0^x t\,dt$$

$$u_2(x) = \left| \frac{t^2}{2} \right|_0^x$$

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$$\boxed{u_2(x) = \frac{x^2}{2}}$$

Put $n = 2$

$$u_3(x) = \int_0^x u_2(t)\,dt$$

$$u_3(x) = \int_0^x \left(\frac{t^2}{2}\right)dt$$

$$u_3(x) = \left|\frac{t^3}{6}\right|_0^x$$

$$\boxed{u_3(x) = \frac{x^3}{6}}$$

From ① Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{u(x) = e^x}$$ Ans.

(ii) $u(x) = 1 + 2\int_0^x t\, u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad —①$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = 2\int_0^x t\, u_n(t)\,dt$$

Scanned with CamScanner

Put n=0

$$u_1(x) = 2\int_0^x t\, u_0(t)\, dt$$

$$u_1(x) = 2\int_0^x t\,(1)\, dt$$

$$u_1(x) = 2\left|\frac{t^2}{2}\right|_0^x \Rightarrow 2\left(\frac{x^2}{2}\right)$$

$$\boxed{u_1(x) = x^2}$$

Put n=1

$$u_2(x) = 2\int_0^x t\; u_1(t)\, dt$$

$$u_2(x) = 2\int_0^x t\,(t^2)\, dt \Rightarrow 2\int_0^t t^3\, dt$$

$$u_2(x) = 2\left(\frac{t^4}{4}\right|_0^x \Rightarrow 2\left(\frac{x^4}{4}\right)$$

$$\boxed{u_2(x) = \frac{x^4}{2}}$$

Put n=2

$$u_3(x) = 2\int_0^x t\; u_2(t)\, dt$$

$$u_3(x) = 2\int_0^x t\left(\frac{t^4}{2}\right) dt$$

$$u_3(x) = 2\int_0^x \left(\frac{t^5}{2}\right) dt$$

$$u_3(x) = 2\left|\frac{t^6}{12}\right|_0^x$$

$$u_3(x) = 2\left(\frac{x^6}{12}\right)$$

$$\boxed{u_3(x) = \frac{x^6}{6}}$$

$\vdots$

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots$$

$$u(x) = 1 + x^2 + \frac{x^4}{4!} + \frac{x^6}{3!} + \cdots$$

$$\boxed{u(x) = e^{x^2}}$$ Ans

(12) $u(x) = 1 - x^2 - \int_0^x (x-t)\,u(t)\,dt$

Sol. By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad \text{—①}$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1 - x^2]$$

$$\boxed{u_0(x) = 1 - x^2}$$

$$u_{n+1}(x) = -\int_0^x (x-t)\,u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = -\int_0^x (x-t)\,u_0(t)\,dt$$

$$u_1(x) = -\int_0^x (x-t)(1-t^2)\,dt$$

(112)

$$u_1(x) = -\int_0^x (x - xt^2 - t + t^3)\, dt$$

$$u_1(x) = -\left[xt - \frac{xt^3}{3} - \frac{t^2}{2} + \frac{t^4}{4} \right]_0^x$$

$$u_1(x) = -\left[x^2 - \frac{x^4}{3} - \frac{x^2}{2} + \frac{x^4}{4} \right]$$

$$u_1(x) = -\left[\frac{x^2}{2} - \frac{x^4}{12} \right]$$

$$\boxed{u_1(x) = -\frac{x^2}{2} + \frac{x^4}{12}}$$

Put $n=1$

$$u_2(x) = -\int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (x-t)\left(-\frac{t^2}{2} + \frac{t^4}{12}\right) dt$$

$$u_2(x) = -\int_0^x \left(-\frac{xt^2}{2} + \frac{xt^4}{12} + \frac{t^3}{2} - \frac{t^5}{12}\right) dt$$

$$u_2(x) = -\left| -\frac{xt^3}{6} + \frac{xt^5}{60} + \frac{t^4}{8} - \frac{t^6}{72} \right|_0^x$$

$$u_2(x) = -\left[-\frac{x^4}{6} + \frac{x^6}{60} + \frac{x^4}{8} - \frac{x^6}{72} \right]$$

$$u_2(x) = -\left[\frac{-x^4}{24} + \frac{x^6}{360} \right]$$

$$\boxed{u_2(x) = \frac{x^4}{24} - \frac{x^6}{360}}$$

Put $n=2$

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{24} - \frac{t^6}{360}\right) dt$$

$$u_3(x) = -\int_0^x \left(\frac{xt^4}{24} - \frac{xt^6}{360} - \frac{t^5}{24} + \frac{t^7}{360}\right) dt$$

$$u_3(x) = -\left| \frac{xt^5}{120} - \frac{xt^7}{2520} - \frac{t^6}{144} + \frac{t^8}{2880} \right|_0^x$$

$$u_3(x) = -\left[\frac{x^6}{120} - \frac{x^8}{2520} - \frac{x^6}{144} + \frac{x^8}{2880}\right]$$

$$u_3(x) = -\left[\frac{x^6}{720} - \frac{x^8}{20160}\right]$$

$$\boxed{u_3(x) = -\frac{x^6}{720} + \frac{x^8}{20160}}$$

⋮

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 - x^2 - \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^4}{24} - \frac{x^6}{360} - \frac{x^6}{720} + \frac{x^8}{20160} + \cdots$$

$$u(x) = 1 - \frac{3x^2}{2} + \frac{3x^4}{24} - \frac{3x^6}{720} + \cdots$$

$$u(x) = 3 - \frac{3x^2}{2!} + \frac{3x^4}{4!} - 3\frac{x^6}{6!} + \cdots - 2$$

$$u(x) = 3\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] - 2$$

$\boxed{u(x) = 3\cos x - 2}$ Ans.

(13) $u(x) = 1 + \int_0^x (x-t)\,u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad \text{—(1)}$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = \int_0^x (x-t)\,u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = \int_0^x (x-t)\,u_0(t)\,dt$$

$$u_1(x) = \int_0^x (x-t)\,dt$$

$$u_1(x) = \left| xt - \frac{t^2}{2} \right|_0^x$$

$$u_1(x) = \left[x^2 - \frac{x^2}{2} \right]$$

$$\boxed{u_1(x) = \frac{x^2}{2}}$$

Put $n=1$

$$u_2(x) = \int_0^x (x-t)\,u_1(t)\,dt$$

$$u_2(x) = \int_0^x (x-t)\left(\frac{t^2}{2}\right)dt$$

$$u_2(x) = \int_0^x \left(\frac{xt^2}{2} - \frac{t^3}{2}\right)dt$$

$$u_2(x) = \left| \frac{xt^3}{6} - \frac{t^4}{8} \right|_0^x$$

$$u_2(x) = \left[\frac{x^4}{6} - \frac{x^4}{8} \right]$$

$$\boxed{u_2(x) = \frac{x^4}{24}}$$

Put $n = 2$

$$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = \int_0^x (x-t) \left(\frac{t^4}{24} \right) dt$$

$$u_3(x) = \int_0^x \left[\frac{xt^4}{24} - \frac{t^5}{24} \right] dt$$

$$u_3(x) = \left| \frac{xt^5}{120} - \frac{t^6}{144} \right|_0^x$$

$$u_3(x) = \frac{x^6}{120} - \frac{x^6}{144}$$

$$\boxed{u_3(x) = \frac{x^6}{720}}$$

⋮

From ① Eq/ ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots$$

$$u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\boxed{u(x) = \cosh x}$$. Ans.

(14) $u(x) = x + \int_0^x (x-t)u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad — (1)$$

$$u_0(x) = f(x) \qquad \because [f(x) = x]$$

$$\boxed{u_0(x) = x}$$

$$u_{n+1}(x) = \int_0^x (x-t)u_n(t)\,dt$$

Put $n=0$

$$u_1(x) = \int_0^x (x-t)\,u_0(t)\,dt$$

$$u_1(x) = \int_0^x (x-t)(t)\,dt$$

$$u_1(x) = \int_0^x (xt - t^2)\,dt$$

$$u_1(x) = \left| \frac{xt^2}{2} - \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = \left[\frac{x^3}{2} - \frac{x^3}{3}\right]$$

$$\boxed{u_1(x) = \frac{x^3}{6}}$$

Put $n=1$

$$u_2(x) = \int_0^x (x-t)\,u_1(t)\,dt$$

$$u_2(x) = \int_0^x (x-t)\left(\frac{t^3}{6}\right)dt$$

$$u_2(x) = \int_0^x \left(\frac{xt^3}{6} - \frac{t^4}{6} \right) dt$$

$$u_2(x) = \left| \frac{xt^4}{24} - \frac{t^5}{30} \right|_0^x$$

$$u_2(x) = \left[\frac{x^5}{24} - \frac{x^5}{30} \right]$$

$$\boxed{u_2(x) = \frac{x^5}{120}}$$

Put $n=2$

$$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = \int_0^x (x-t) \left(\frac{t^5}{120} \right) dt$$

$$u_3(x) = \int_0^x \left(\frac{xt^5}{120} - \frac{t^6}{120} \right) dt$$

$$u_3(x) = \left| \frac{xt^6}{720} - \frac{t^7}{840} \right|_0^x$$

$$u_3(x) = \left[\frac{x^7}{720} - \frac{x^7}{840} \right]$$

$$\boxed{u_3(x) = \frac{x^7}{5040}}$$

⋮

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

(118)

$\boxed{u(x) = \sinh x}$ Ans.

(15) $u(x) = 1 - \int_0^x u(t)\,dt$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad (1)$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = -\int_0^x u(t)\,dt$$

Put $n=0$

$$u_1(x) = -\int_0^x u_0(t)\,dt$$

$$u_1(x) = -\int_0^x 1\,dt \Rightarrow -|t|_0^x$$

$$\boxed{u_1(x) = -x}$$

Put $n=1$

$$u_2(x) = -\int_0^x u_1(t)\,dt$$

$$u_2(x) = -\int_0^x (-t)\,dt \Rightarrow -\left| -\frac{t^2}{2} \right|_0^x$$

$$\boxed{u_2(x) = \frac{x^2}{2}}$$

Put $n=2$

$$u_3(x) = -\int_0^x u_2(t)\,dt$$

$$u_3(x) = -\int_0^x \left(\frac{t^2}{2}\right) dt$$

$$u_3(x) = -\left|\frac{t^3}{6}\right|_0^x$$

$$\boxed{u_3(x) = -\frac{x^3}{6}}$$

$\vdots$

From ① Eq ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\boxed{u(x) = e^{-x}}$$ Ans.

(16) $u(x) = 1 - 2\int_0^x t\ u(t)\, dt.$

Sol: By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad — ①$$

$$u_0(x) = f(x) \qquad \because [f(x) = 1]$$

$$\boxed{u_0(x) = 1}$$

$$u_{n+1}(x) = -2\int_0^x t\ u_n(t)\, dt$$

Put n=0

$$u_1(x) = -2\int_0^x t\ u_0(t)\, dt$$

$$u_1(x) = -2\int_0^x t\, dt$$

$$u_1(x) = -2\left|\frac{t^2}{2}\right|_0^x$$

$$u_1(x) = -2\left(\frac{x^2}{2}\right)$$

$$u_1(x) = -x^2$$

Put $n=1$

$$u_2(x) = -2\int_0^x t\, u_1(t)\,dt$$

$$u_2(x) = -2\int_0^x t\,(-t^2)\,dt \Rightarrow -2\int_0^x -t^3\,dt$$

$$u_2(x) = -2\left| -\frac{t^4}{4}\right|_0^x$$

$$u_2(x) = -2\left(-\frac{x^4}{4}\right)$$

$$u_2(x) = \frac{x^4}{2}$$

Put $n=2$

$$u_3(x) = -2\int_0^x t\, u_2(t)\,dt$$

$$u_3(x) = -2\int_0^x t\left(\frac{t^4}{2}\right)dt \Rightarrow -2\int_0^x \frac{t^5}{2}\,dt$$

$$u_3(x) = -2\left|\frac{t^6}{12}\right|_0^x$$

$$u_3(x) = -2\left(\frac{x^6}{12}\right)$$

$$u_3(x) = \frac{-x^6}{6}$$

From (1) Eq $\Rightarrow$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\boxed{u(x) = e^{-x^2}}$$ Ans.

(17) $u(x) = -2 + 3x - x^2 - \int_0^x (x-t)\, u(t)\, dt$

Sol. By Adomian Decomposition Method (ADM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad (1)$$

$$u_0(x) = f(x) \qquad [f(x) = -2 + 3x - x^2]$$

$$\boxed{u_0(x) = -2 + 3x - x^2}$$

$$u_{n+1}(x) = -\int_0^x (x-t)\, u_n(t)\, dt$$

Put $n=0$

$$u_1(x) = -\int_0^x (x-t)\, u_0(t)\, dt$$

$$u(x) = -\int_0^x (x-t)(-2+3t-t^2)\, dt$$

$$u_1(x) = -\int_0^x (-2x + 3xt - xt^2 + 2t - 3t^2 + t^3)\, dt$$

$$u_1(x) = -\left| -2xt + \frac{3}{2}xt^2 - \frac{xt^3}{3} + \frac{2t^2}{2} - \frac{3t^3}{3} + \frac{t^4}{4} \right|_0^x$$

$$u_1(x) = -\left[-2x^2 + \frac{3}{2}x^3 - \frac{x^4}{3} + x^2 - x^3 + \frac{x^4}{4} \right]$$

$$u_1(x) = -\left[-x^2 + \frac{x^3}{2} - \frac{x^4}{12} \right]$$

(121)

$$u_1(x) = x^2 - \frac{x^3}{2} + \frac{x^4}{12}$$

Put n=1

$$u_2(x) = -\int_0^x (x-t)\, u_1(t)\, dt$$

$$u_2(x) = -\int_0^x (x-t)\left(t^2 - \frac{t^3}{2} + \frac{t^4}{12}\right) dt$$

$$u_2(x) = -\int_0^x \left(xt^2 - \frac{xt^3}{2} + \frac{xt^4}{12} - t^3 + \frac{t^4}{2} - \frac{t^5}{12}\right) dt$$

$$u_2(x) = -\left| \frac{xt^3}{3} - \frac{xt^4}{8} + \frac{xt^5}{60} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{72} \right|_0^x$$

$$u_2(x) = -\left[\frac{x^4}{3} - \frac{x^5}{8} + \frac{x^6}{60} - \frac{1}{4}x^4 + \frac{x^5}{5} - \frac{x^6}{72}\right]$$

$$u_2(x) = -\left[\frac{x^4}{12} - \frac{x^5}{40} + \frac{x^6}{360}\right]$$

$$u_2(x) = -\frac{x^4}{12} + \frac{x^5}{40} - \frac{x^6}{360}$$

Put n=2

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\, dt$$

$$u_3(x) = -\int_0^x (x-t)\left(-\frac{t^4}{12} + \frac{t^5}{40} - \frac{t^6}{360}\right) dt$$

$$u_3(x) = -\int_0^x \left(-\frac{xt^4}{12} + \frac{xt^5}{40} - \frac{xt^6}{360} + \frac{t^5}{12} - \frac{t^6}{40} + \frac{t^7}{360}\right) dt$$

$$u_3(x) = -\left| -\frac{xt^5}{60} + \frac{xt^6}{240} - \frac{xt^7}{2520} + \frac{t^6}{72} - \frac{t^7}{280} + \frac{t^8}{2880} \right|_0^x$$

$$u_3(x) = -\left[-\frac{x^6}{180} + \frac{x^7}{840} - \frac{x^8}{2520} + \frac{x^6}{720} - \frac{x^7}{1680} + \frac{x^8}{2880}\right]$$

$$u_3(x) = -\left[-\frac{x^6}{360} + \frac{x^7}{1680} - \frac{x^8}{20160}\right]$$

$$\boxed{u_3(x) = \left[\frac{x^6}{360} - \frac{x^7}{1680} + \frac{x^8}{20160}\right]}$$

⋮

From Eq (1) ⇒

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = -2 + 3x - x^4 + x^4 - \frac{x^3}{2} + \frac{x^4}{12} - \frac{x^4}{12} + \frac{x^5}{40} - \frac{x^6}{360} + \frac{x^6}{360} - \frac{x^7}{1680} + \frac{x^8}{20160} + \cdots$$

$$u(x) = -2 + 3\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right]$$

$$u(x) = -2 + 3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$\boxed{u(x) = -2 + 3\sin x}$$ Ans.