Lecture # 5 Wednesday 29.01.2020

Solution of Integral Equation

A solution of a differential equation or an integral equation.
There are two types
(i) Exact Solution
(ii) Series Solution

Exact Solution: The solution is called exact if it can be expressed in a closed form such as polynomial, exponential and trigonometric function or etc.
The combination of two or more of the elementary function.

Example: $u(x) = x + e^{x}$
$u(x) = \sin x + e^{2x}$
$u(x) = 1 + \cosh x + \tan x$

Series Solution: Convert problems, some terms cannot obtain exact solution. In this case we determine the solution in series form.

Example: Show that $u(x) = \sin x$ is the solution of the volterra integro differential equation

$$u'(x) = 1 - \int_0^x u(t)\,dt$$

Sol. $u'(x) = 1 - \int_0^x u(t)\,dt \quad \text{--- ①}$

where $u(x) = \sin x$

Diff. w.r.t x

$u'(x) = \cos x$

put the value of $u(x)$ in ① Eq

$\cos x = 1 - \int_0^x \sin t \, dt$

$\cos x = 1 - \left| -\cos t \right|_0^x$

$\cos x = 1 - \left[-\cos x + \cos(0) \right]$

$\cos x = \cancel{1} + \cos x - \cancel{1}$

$\cos x = \cos x$

L.H.S = R.H.S Ans.

Exercise 2.6: In (1-4) show the given function $u(x)$ is a solution of corresponding Fredholm Integral Equation

(1) $u(x) = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x \, u(t) \, dt$

$u(x) = \sin x + \cos x$

Sol: $u(x) = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x \, u(t) \, dt$

put the value of $u(x)$

$\sin x + \cos x = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x \, (\sin t + \cos t) \, dt$

$\sin x + \cos x = \cos x + \frac{1}{2} \sin x \int_0^{\pi/2} (\sin t + \cos t) \, dt$

$\sin x + \cos x = \cos x + \frac{1}{2} \sin x \left[-\cos t \Big|_0^{\pi/2} + \sin t \Big|_0^{\pi/2} \right]$

$\sin x + \cos x = \cos x + \frac{1}{2} \sin x \left\{ -\cos\left(\frac{\pi}{2}\right) + \cos(0) + \sin\left(\frac{\pi}{2}\right) + \sin(0) \right\}$

$\sin x + \cos x = \cos x + \frac{1}{2} \sin x \left[+1 + 1 \right]$

$\sin x + \cos x = \cos x + \frac{1}{\cancel{2}} \sin x \, (\cancel{2})$

$\sin x + \cos x = \sin x + \cos x$.

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans.

(2) $u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t} u(t)\,dt$

$u(x) = e^{2x}$

Sol: $u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t} u(t)\,dt$

put the value of $u(x)$

$$e^{2x} = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t} e^{2t}\,dt$$

$$e^{2x} = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x} \cdot e^{-\frac{5}{3}t} \cdot e^{2t}\,dt$$

$$e^{2x} = e^{2x} \cdot e^{1/3} - \frac{1}{3} e^{2x}\int_0^1 e^{-\frac{5}{3}t+2t}\,dt$$

$$e^{2x} = e^{2x} \cdot e^{1/3} - \frac{1}{3} e^{2x}\int_0^1 e^{1/3 t}\,dt$$

$$e^{2x} = e^{2x} \cdot e^{1/3} - \frac{\cancel{1}}{\cancel{3}} e^{2x}\left|\frac{e^{1/3 t}}{\cancel{1/3}}\right|_0^1$$

$$e^{2x} = e^{2x} \cdot e^{1/3} - e^{2x}\left[e^{1/3} - e^{0}\right]$$

$$e^{2x} = \cancel{e^{2x} \cdot e^{1/3}} - \cancel{e^{2x} \cdot e^{1/3}} + e^{2x} \qquad \because (e^0 = 1)$$

$$e^{2x} = e^{2x}$$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans.

(3) $u(x) = x + \int_{-1}^{1} (x^4 - t^4)\, u(t)\,dt$

$u(x) = x$

Sol: $u(x) = x + \int_{-1}^{1} (x^4 - t^4)\, u(t)\,dt$

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(6)

Put the value of $u(x)$

$$x = x + \int_{-1}^{1} (x^4 - t^4)\, t\, dt$$

$$x = x + \int_{-1}^{1} (x^4 t - t^5)\, dt$$

$$x = x + x^4 \left| \frac{t^2}{2} \right|_{-1}^{1} - \left| \frac{t^6}{6} \right|_{-1}^{1}$$

$$x = x + x^4 \left(\frac{1}{2} - \frac{1}{2} \right) - \left(\frac{1}{6} - \frac{1}{6} \right)$$

$$x = x + \cancel{\frac{x^4}{2}} - \cancel{\frac{x^4}{2}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}}$$

$$x = x$$

LHS = RHS

$\Rightarrow$ This is Exact solution Ans.

(4) $$u(x) = x + (1-x)e^{x} + \int_0^1 x^2 e^{t(x-1)} u(t)\, dt$$

$$u(x) = e^{x}$$

Sol. $$u(x) = x + (1-x)e^{x} + \int_0^1 x^2 e^{t(x-1)} u(t)\, dt$$

put the value of $u(x)$

$$e^x = x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)} e^t\, dt$$

$$e^x = x + (1-x)e^x + \int_0^1 x^2 e^{tx} \cdot \cancel{e^{-t}} \cdot \cancel{e^{t}}\, dt$$

$$e^x = x + (1-x)e^x + x^2 \int_0^1 e^{tx}\, dt$$

$$e^x = x + (1-x)e^x + \cancel{x^2} \left| \frac{e^{tx}}{\cancel{x}} \right|_0^1$$

$$e^x = x + (1-x)e^x + x(e^x - e^0)$$

$$e^x = \cancel{x} + e^x - \cancel{xe^x} + \cancel{xe^x} - \cancel{x}$$

$$e^x = e^x$$

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L.H.S = R.H.S

$\Rightarrow$ This is Exact Equation. Ans.

In (5-8) show that the given function $u(x)$ is a solution of the corresponding Volterra-Integral Equation.

(5) $u(x) = 1 + 2\int_0^x u(t)\,dt$

$u(x) = e^{2x}$

Sol: $u(x) = 1 + 2\int_0^x u(t)\,dt$

put the value of $u(x)$

$e^{2x} = 1 + 2\int_0^x e^{2t}\,dt$

$e^{2x} = 1 + 2\left|\frac{e^{2t}}{2}\right|_0^x$

$e^{2x} = 1 + 2\left[\frac{e^{2x}}{2} - \frac{e^0}{2}\right]$

$e^{2x} = 1 + 2\left[\frac{e^{2x}}{2} - \frac{1}{2}\right]$

$e^{2x} = \cancel{1} + e^{2x} - \cancel{1}$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans.

(6) $u(x) = 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x u(t)\,dt$

$u(x) = 4x + \sin x$

Sol: $u(x) = 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x u(t)\,dt$

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Put the value of $u(x)$

$$4x+\sin x = 4x+\sin x + 2x^2 - \cos x + 1 - \int_0^x (4t+\sin t)\,dt$$

$$4x+\sin x = 4x+\sin x + 2x^2 - \cos x + 1 - \left|\frac{4t^2}{2}\right|_0^x - \left|-\cos t\right|_0^x$$

$$4x+\sin x = 4x+\sin x + 2x^2 - \cos x + 1 - 2x^2 + 0 - [-\cos x + \cos(0)]$$

$$4x+\sin x = 4x+\sin x + \cancel{2x^2} - \cancel{\cos x} + \cancel{1} - \cancel{2x^2} + \cancel{\cos x} - \cancel{1}$$

$$4x+\sin x = 4x+\sin x$$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans.

(7). $u(x) = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)\,u(t)\,dt$

$u(x) = 2\cos x - 1$

Sol: $u(x) = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)\,u(t)\,dt$

Put the value of $u(x)$

$$2\cos x - 1 = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)(2\cos t - 1)\,dt$$

$$2\cos x - 1 = 1 - \frac{1}{2}x^2 - \int_0^x (2x\cos t - x + 2t\cos t + t)\,dt$$

$$2\cos x - 1 = 1 - \frac{1}{2}x^2 - 2x\int_0^x \cos t\,dt + x\int_0^x dt - 2\int_0^x t\cos t\,dt - \int_0^x t\,dt$$

$$2\cos x - 1 = 1 - \frac{1}{2}x^2 + 2x\,|\sin t|_0^x + x\,|t|_0^x - 2\left[t\sin t|_0^x - \int \sin t\,dt\right] - \frac{t^2}{2}$$

$$2\cos x - 1 = 1 - \frac{1}{2}x^2 + 2x[\sin x - \sin(0)] + x[x-0] - 2[x\sin x - t\cos t|_0^x] - \frac{x^2}{2}$$

$$2\cos x - 1 = 1 - \cancel{\frac{1}{2}x^2} + \cancel{2x\sin x} + \cancel{x^2} - \cancel{2x\sin x} + 2\cos x - 2 - \cancel{\frac{x^2}{2}}$$

$$2\cos x - 1 = 2\cos x - 1$$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans:

(8) $u(x) = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x u(t)\,dt$

$u(x) = 2x + \sin x$

Sol: $u(x) = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x u(t)\,dt$

Put the value of $u(x)$

$2x + \sin x = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x (2t + \sin t)\,dt$

$2x + \sin x = 1 + 2x + \sin x + x^2 - \cos x - 2\int_0^x t\,dt - \int_0^x \sin t\,dt$

$2x + \sin x = 1 + 2x + \sin x + x^2 - \cos x - 2\left|\frac{t^2}{2}\right|_0^x - \left|-\cos t\right|_0^x$

$2x + \sin x = 1 + 2x + \sin x + \cancel{x^2} - \cancel{\cos x} - \cancel{x^2} + \cancel{\cos x} - \cos(0)$

$2x + \sin x = \cancel{1} + 2x + \sin x - \cancel{1}$

$2x + \sin x = 2x + \sin x$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution. Ans.

(10) $u'(x) = e^x + (e-1) - \int_0^1 u(t)\,dt$; $u(0) = 1$, $u(x) = e^x$

Sol: $u'(x) = e^x + (e-1) - \int_0^1 u(t)\,dt$

where

$u(x) = e^x$

Diff w.r.t x

$u'(x) = e^x$

Put the value of $u'(x)$

$e^x = e^x + e - 1 - \int_0^1 e^t\,dt$

$$e^x = e^x + e - 1 - |e^t|_0^1$$

$$e^x = e^x + e - 1 - e + e^0$$

$$e^x = e^x + \cancel{e} - \cancel{1} - \cancel{e} + \cancel{1}$$

$$e^x = e^x$$

L.H.S = R.H.S

⇒ This is Exact Solution. Ans.

(11) $u''(x) = 1 - \sin x - \int_0^{\pi/2} t\, u(t)\, dt \; ; \; u(0) = 0, \; u'(0) = 1$

$u(x) = \sin x$.

Sol: $u''(x) = 1 - \sin x - \int_0^{\pi/2} t\, u(t)\, dt$

where

$$u(x) = \sin x$$

Diff w.r.t 'x'

$$u'(x) = \cos x$$

Again diff w.r.t 'x'

$$u''(x) = -\sin x$$

Put the value of $u''(x)$

$$-\sin x = 1 - \sin x - \int_0^{\pi/2} t \sin t\, dt$$

$$-\sin x = 1 - \sin x - \left[t\, |-\cos t|_0^{\pi/2} - \int_0^{\pi/2} -\cos t\, dt \right]$$

$$-\sin x = 1 - \sin x - \int_0^{\pi/2} \cos t\, dt$$

$$-\sin x = 1 - \sin x - |\sin t|_0^{\pi/2}$$

$$-\sin x = 1 - \sin x - \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$-\sin x = \cancel{x} - \sin x - \cancel{x} + 0$

$-\sin x = -\sin x$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution. Ans.

(13) $u'(x) = 2 + x + x^2 - \int_0^x u(t)\,dt \;;\; u(0) = 1,\; u(x) = 1 + 2x$

Sol: $u'(x) = 2 + x + x^2 - \int_0^x u(t)\,dt$

where

$u(x) = 1 + 2x$

Diff w.r.t 'x'

$u'(x) = 2$

Put the value of $u'(x)$

$2 = 2 + x + x^2 - \int_0^x (1 + 2t)\,dt$

$2 = 2 + x + x^2 - \int_0^x 1\,dt - 2\int_0^x t\,dt$

$2 = 2 + x + x^2 - |t|_0^x - 2\left|\frac{t^2}{2}\right|_0^x$

$2 = 2 + \cancel{x} + \cancel{x^2} - \cancel{x} - \cancel{x^2}$

$2 = 2$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution Ans.

(14) $u''(x) = x\cos x - 2\sin x + \int_0^x t\,u(t)\,dt$

$u(0) = 0$, $u'(0) = 1$, $u(x) = \sin x$

Sol: $u''(x) = x\cos x - 2\sin x + \int_0^x t\,u(t)\,dt$

where $u(x) = \sin x$

Diff w.r.t 'x'

$u'(x) = \cos x$

Again diff w.r.t 'x'

$u''(x) = -\sin x$

Put the value of $u''(x)$

$$-\sin x = x\cos x - 2\sin x + \int_0^x t\sin t\,dt$$

$$-\sin x = x\cos x - 2\sin x + \left[t\,|-\cos t|_0^x - \int_0^x -\cos t\,dt \right]$$

$$-\sin x = \cancel{x\cos x} - 2\sin x - \cancel{x\cos x} + \int_0^x \cos t\,dt$$

$$-\sin x = -2\sin x + |+\sin t|_0^x$$

$$-\sin x = -2\sin x + \sin x$$

$$-\sin x = -\sin x$$

L.H.S = R.H.S

$\Rightarrow$ This is Exact Solution. Ans.

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(15) $u''(x) = 1 + \int_0^x (x-t)\,u(t)\,dt \;;\; u(0)=1,\; u'(0)=0$

$u(x) = \cosh x$

Sol. $u''(x) = 1 + \int_0^x (x-t)\,u(t)\,dt$

where

$u(x) = \cosh x$

Diff w.r.t 'x'

$u'(x) = \sinh x$

Again diff w.r.t 'x'

$u''(x) = \cosh x$

Put the value of $u''(x)$

$$\cosh x = 1 + \int_0^x (x-t)\cosh t\,dt$$

$$\cosh x = 1 + \int_0^x (x\cosh t - t\cosh t)\,dt$$

$$\cosh x = 1 + x\int_0^x \cosh t\,dt - \int_0^x t\cosh t\,dt$$

$$\cosh x = 1 + x\,\left|\sinh t\right|_0^x - \left[t\,\left|\sinh t\right|_0^x - \int_0^x \sinh t\,dt\right]$$

$$\cosh x = 1 + \cancel{x\sinh x} - \cancel{x\sinh x} + \left|\cosh t\right|_0^x$$

$$\cosh x = \cancel{1} + \cosh x - \cancel{1}$$

$$\cosh x = \cosh x$$

L.H.S = R.H.S

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⇒ This is Exact Solution Ans