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Converting Boundary Value Problem to Fredholm Integral Equation

Type 1 We first consider a boundary value problem

$$y''(x) + g(x)\, y(x) = h(x) \quad -(1)$$

with boundary conditions $y(0) = \alpha$, $y(1) = \beta$

Set $y''(x) = u(x)$ —(2)

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt$$

Again integrate

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = y(0) + x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = \alpha$

$$y(x) = \alpha + x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad -(3)$$

put $x = 1$

So $$y(1) = \alpha + y'(0) + \int_0^1 (1-t)\,u(t)\,dt$$

where $y(1) = \beta$

$$\beta = \alpha + y'(0) + \int_0^1 (1-t)\,u(t)\,dt$$

$$(\beta - \alpha) = y'(0) + \int_0^1 (1-t)\,u(t)\,dt$$

$$(\beta - \alpha) - \int_0^1 (1-t)\,u(t)\,dt = y'(0)$$

Put the value of $y'(0)$ in ③ Equation

$$y(x) = \alpha + x\left[(\beta - x) - \int_0^1 (1-t)u(t)\,dt\right] + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = \alpha + (\beta - \alpha)x - x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt$$

from ① Eq $\Rightarrow$

$$y''(x) + g(x)\left[\alpha + (\beta-\alpha)x - x\int_0^x (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt\right] = h(x)$$

$$y''(x) = h(x) - g(x)\left[\alpha + (\beta-x)x - x\int_0^x (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt\right]$$

$$y''(x) = h(x) - g(x)\alpha - (\beta-x)x\,g(x) + x\,g(x)\int_0^x (1-t)u(t)\,dt - g(x)\int_0^x (x-t)u(t)\,dt$$

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Example: Convert the following Boundary Value Problem to Fredholm Integral Equation

$$y''(x) + y(x) = 0 \quad ; \quad y(0) = 0 \, , \, y'(1) = 0$$

Sol:

$$y''(x) + y(x) = 0 \quad \text{— (A)}$$

Set $y''(x) = u(x)$

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Equation

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\!\!\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 0$

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{— (2)}$$

from (1) Eq $\Rightarrow$

put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1) = 0$

$$0 = y'(0) + \int_0^1 u(t)\,dt$$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in (2) Equation

$$y(x) = x\left[-\int_0^1 u(t)\,dt\right] + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = -\int_0^1 x\,u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = -\int_x^1 \cancel{x\,u(t)}\,dt - \int_x^1 x\,u(t)\,dt + \int_0^1 \cancel{x\,u(t)}\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

Put in Equation (A)

$$u(x) - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt = 0$$

$$u(x) = \int_0^x t\,u(t)\,dt + \int_x^1 x\,u(t)\,dt$$

where $u(x) = \int_0^1 K(x,t)\,u(t)\,dt$

$$K(x,t) = \begin{cases} t & 0 \le t \le x \\ x & x \le t \le 1 \end{cases}$$. Ans.

Example #2: Convert the following Boundary Value Problem to Fredholm Integral Equation

$$y''(x) + 2\,y(x) = 4 \quad ; \quad y(0) = 0 \;,\; y'(1) = 1$$

Sol: $y''(x) + 2\,y(x) = 4$ — (A)

Set

$$y''(x) = u(x)$$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Equation

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$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)u(t)\,dt$$

put $y(0)=0$

$$y(x) = xy'(0) + \int_0^x (x-t)u(t)\,dt \quad (2)$$

from (1) Eq ⇒

put $x=1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1)=1$

$$1 = y'(0) + \int_0^1 u(t)\,dt$$

$$y'(0) = 1 - \int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in (2) Eq

$$y(x) = x\left[1 - \int_0^1 u(t)\,dt\right] + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = x - x\int_0^1 u(t)\,dt + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = x - x\int_0^x u(t)\,dt - x\int_x^1 u(t)\,dt + \int_0^x x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = x - \int_0^x \cancel{x\,u(t)}\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)}\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = x - \int_x^1 x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

put in Eq (A)

$$u(x) + 2\left[x - \int_x^1 x\,u(t)\,dt - \int_0^x t\,u(t)\,dt\right] = 4$$

$$u(x) + 2x - 2\int_0^x t\,u(t)\,dt - 2\int_x^1 x\,u(t)\,dt = 4$$

$$u(x) = 4 - 2x + \int_0^x 2t\, u(t)\,dt + \int_x^1 2x\, u(t)\,dt$$

$$u(x) = 4 - 2x + \int_0^1 K(x,t)\, u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 2t & 0 \le t \le x \\ 2x & x \le t \le 1 \end{cases}$$ Ans.

Exercise 2.6 : Convert the following Boundary Value Problems to Fredholm Integral Equation

(i) $y'' + 4y = 0$; $y(0) = 0$, $y'(1) = 0$

Sol.

$$y''(x) + 4\,y(x) = 0 \quad \text{— (A)}$$

Set $\quad y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Equation

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 0$

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{— (2)}$$

from (1) Eq $\Rightarrow$

put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put, $y'(1) = 0$

$$0 = y'(0) + \int_0^1 u(t)\,dt$$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in ② Equation

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = -\int_0^1 x\,u(t)\,dt + \int_0^x x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x \cancel{x\,u(t)\,dt} - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)\,dt} - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in Ⓐ Equation.

$$u(x) + 4\left[-\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = 0$$

$$u(x) - \int_0^x 4t\,u(t)\,dt - \int_x^1 4x\,u(t)\,dt = 0$$

$$u(x) = \int_0^x 4t\,u(t)\,dt + \int_x^1 4x\,u(t)\,dt$$

$$u(x) = \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 4t & 0 \le t \le x \\ 4x & x \le t \le 1 \end{cases}$$ Ans.

(iii) $y'' + xy = 0 \quad ; \quad y(0) = 0 \quad , \quad y'(1) = 0$

Sol. $y''(x) + x\,y(x) = 0$ — (A)

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Eq.

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\!\!\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 0$

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{— (2)}$$

from (1) Eq $\Rightarrow$

put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1) = 0$

$$0 = y'(0) + \int_0^1 u(t)\,dt$$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in (2) Equation

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x \cancel{x}\,u(t)\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x}\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in Eq. (A)

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$$u(x) + x\left[-\int_x^0 t\,u(t)\,dt - \int_1^x x\,u(t)\,dt\right] = 0$$

$$u(x) - \int_0^x xt\,u(t)\,dt - \int_x^1 x^2 u(t)\,dt = 0$$

$$u(x) = \int_0^x xt\,u(t)\,dt + \int_x^1 x^2 u(t)\,dt$$

$$u(x) = \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} xt & 0 \le t \le x \\ x^2 & x \le t \le 1 \end{cases}$$ Ans.

(iii) $y'' + 2y = x \quad ; \quad y(0) = 1 \;, \; y'(1) = 0$

Sol. $y''(x) + 2\,y(x) = x$ — (A)

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad -①$$

Integrate ① Equation

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 1$

$$y(x) - 1 = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = 1 + x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad -②$$

from (B) Equation ⇒ put $x=1$

$$y(1) = y(0) + \int_0^1 u(t)\,dt$$

put $y(1)=0$

$$y(0) = -\int_0^1 u(t)\,dt$$

put the value of $y(0)$ in (a) Equation

$$y(x) = 1 - x\int_0^1 u(t)\,dt + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = 1 - \int_0^1 x\,u(t)\,dt + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = 1 - \int_0^x \cancel{x\,u(t)}\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)}\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = 1 - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in (A) Equation

$$u(x) + 2\left[1 - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = x$$

$$u(x) + 2 - \int_0^x 2t\,u(t)\,dt - \int_x^1 2x\,u(t)\,dt = x$$

$$u(x) = x - 2 + \int_0^x 2t\,u(t)\,dt + \int_x^1 2x\,u(t)\,dt$$

$$u(x) = x - 2 + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 2t & 0 \le t \le x \\ 2x & x \le t \le 1 \end{cases}$$

Ans.

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(iv) $y'' + xy = 4$; $y(0)=0$, $y'(1)=0$.

Sol. $y''(x) + x\,y(x) = 4$ — (A)

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Eq

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)u(t)\,dt$$

put $y(0)=0$

$$y(x) = x\,y'(0) + \int_0^x (x-t)u(t)\,dt \quad \text{— (2)}$$

from (1) Eq $\Rightarrow$

put $x=1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1)=0$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in (2) Equation

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = -\int_0^x \cancel{x\,u(t)}\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)}\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in (A) Eq

$$u(x) + x\left[-\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = 4$$

$$u(x) - \int_0^x xt\,u(t)\,dt - \int_x^1 x^2 u(t)\,dt = 4$$

$$u(x) = 4 + \int_0^x xt\,u(t)\,dt + \int_x^1 x^2 u(t)\,dt$$

$$u(x) = 4 + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} xt & 0 \le t \le x \\ x^2 & x \le t \le 1 \end{cases}$$ Ans.

(iv) $y'' + 3xy = 4$; $y(0) = 0$, $y'(1) = 0$

Sol: $y''(x) + 3x\,y(x) = 4$ — (A)

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Equation

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 0$

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{— (2)}$$

from (1) Eq ⇒

put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1) = 0$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y(0)$ in (2) Equation

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = -\int_0^x x\,u(t)\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = -\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in Eq. (A)

$$u(x) + 3x\left[-\int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = 4$$

$$u(x) - \int_0^x 3xt\,u(t)\,dt - \int_x^1 3x^2\,u(t)\,dt = 4$$

$$u(x) = 4 + \int_0^x 3xt\,u(t)\,dt + \int_x^1 3x^2\,u(t)\,dt$$

$$u(x) = 4 + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 3xt & 0 \le t \le x \\ 3x^2 & x \le t \le 1 \end{cases}$$ Ans.

(vi) $y'' + 4y = x$; $y(0) = 1$, $y'(1) = 0$

Sol: $y''(x) + 4y(x) = x$ — (A)

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

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Integrate ① Eq

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)u(t)\,dt$$

put $y(0) = 1$

$$y(x) - 1 = x\,y'(0) + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = 1 + x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{—②}$$

from ① Eq $\Rightarrow$ put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1) = 0$

$$y'(0) = -\int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in ② Equation

$$y(x) = 1 + x\left[-\int_0^1 u(t)\,dt\right] + \int_0^x (x-t)u(t)\,dt$$

$$y(x) = 1 - \int_0^x \cancel{x\,u(t)\,dt} - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)\,dt} - \int_0^x t\,u(t)\,dt$$

$$y(x) = 1 - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

put in Ⓐ Eq.

$$u(x) + 4\left[1 - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = x$$

$$u(x) + 4 - \int_0^x 4t\,u(t)\,dt - \int_x^1 4x\,u(t)\,dt = x$$

$$u(x) = x - 4 + \int_0^x 4t\,u(t)\,dt + \int_x^1 4x\,u(t)\,dt$$

$$u(x) = x - 4 + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 4t & 0 \le t \le x \\ 4x & x \le t \le 1 \end{cases}$$

Ans.

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(vii) $y'' + 4xy = 2 \quad ; \quad y(0) = 0 \quad , \quad y'(1) = 1$

Sol. $y''(x) + 4x\, y(x) = 2 \quad \text{— (A)}$

Set $y''(x) = u(x)$

Integrate

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y'(0) + \int_0^x u(t)\,dt \quad \text{— (1)}$$

Integrate (1) Eq

$$\int_0^x y'(t)\,dt = \int_0^x y'(0)\,dt + \int_0^x\int_0^x u(t)\,dt$$

$$y(x) - y(0) = y'(0)\int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$

put $y(0) = 0$.

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt \quad \text{— (2)}$$

from (1) Eq $\Rightarrow$ put $x = 1$

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

put $y'(1) = 1$

$$1 = y'(0) + \int_0^1 u(t)\,dt$$

$$y'(0) = 1 - \int_0^1 u(t)\,dt$$

put the value of $y'(0)$ in (2) Eq

$$y(x) = x\left[1 - \int_0^1 u(t)\,dt\right] + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = x - x\int_0^1 u(t)\,dt + \int_0^x x\,u(t)\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = x - \int_0^x \cancel{x\,u(t)}\,dt - \int_x^1 x\,u(t)\,dt + \int_0^x \cancel{x\,u(t)}\,dt - \int_0^x t\,u(t)\,dt$$

$$y(x) = x - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt$$

Put in equation ④

$$u''(x) + 4x\left[x - \int_0^x t\,u(t)\,dt - \int_x^1 x\,u(t)\,dt\right] = 2$$

$$u(x) + 4x^2 - \int_0^x 4xt\,u(t)\,dt - \int_x^1 4x^2 u(t)\,dt = 2$$

$$u(x) = 2 - 4x^2 + \int_0^x 4xt\,u(t)\,dt + \int_x^1 4x^2 u(t)\,dt$$

$$u(x) = 2 - 4x^2 + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 4xt & 0 \le t \le x \\ 4x^2 & x \le t \le 1 \end{cases}$$ Ans.

Converting Fredholm Integral Equation to Boundary Value Problem

Example # 1: Convert the Fredholm Integral Equation to an equivalent Boundary Value Problem

$$u(x) = e^x + \int_0^1 K(x,t)\,u(t)\,dt$$

where

$$K(x,t) = \begin{cases} 9t(1-x) & \text{for } 0 \le t \le x \\ 9x(1-t) & \text{for } x \le t \le 1 \end{cases}$$

Sol: $u(x) = e^x + \int_0^1 K(x,t)\,u(t)\,dt$

$$u(x) = e^x + \int_0^x K(x,t)\,u(t)\,dt + \int_x^1 K(x,t)\,u(t)\,dt$$

$$u(x) = e^x + \int_0^x 9t(1-x)\,u(t)\,dt + \int_x^1 9x(1-t)\,u(t)\,dt \quad \text{—①}$$

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Diff. Eq ① w.r.t 'x'

$$u'(x) = e^x - 9\int_0^x t\,u(t)\,dt + 9\int_x^1 (1-t)\,u(t)\,dt \quad \text{—②}$$

Diff. Eq ② w.r.t 'x'

$$u''(x) = e^x - 9x\,u(x) - 9\left[(1-x)\,u(x)\right]$$

$$= e^x - 9x\,u(x) - 9\left[u(x) - x\,u(x)\right]$$

$$u''(x) = e^x - 9x\,\cancel{u(x)} - 9\,u(x) + 9x\,\cancel{u(x)}$$

$$u''(x) + 9u(x) = e^x$$

For boundary conditions

from Eq ① ⇒

put $x = 0$

$$u(0) = e^0 + 0$$

$$u(0) = 1$$

from Eq ② ⇒

put $x = 1$

$$u(1) = e$$

The required equation is

$$u''(x) + 9\,u(x) = e^x \;;\quad u(0) = 1\,,\quad u(1) = e \quad \text{Ans.}$$

Exercise 2.6: Convert Fredholm Integral Equation to Boundary Value Problem

(ix) $u(x) = e^{2x} + \int_0^1 K(x,t)\,u(t)\,dt$

where $K(x,t) = \begin{cases} 3t(1-x) & \text{for } 0 \le t \le x \\ 3x(1-t) & \text{for } x \le t \le 1 \end{cases}$

Sol. $u(x) = e^{2x} + \int_0^1 K(x,t)\, u(t)\, dt$

$$u(x) = e^{2x} + \int_0^x K(x,t)\, u(t)\, dt + \int_x^1 K(x,t)\, u(t)\, dt$$

$$u(x) = e^{2x} + \int_0^x 3t(1-x)\, u(t)\, dt + \int_x^1 3x(1-t)\, u(t)\, dt \quad -① $$

Diff. Eq ① w.r.t 'x'

$$u'(x) = 2e^{2x} - 3\int_0^x t\, u(t)\, dt + 3\int_x^1 (1-t)\, u(t)\, dt \quad -②$$

Diff. Eq ② w.r.t 'x'

$$u''(x) = 4e^{2x} - 3x\, u(x) - 3\left[(1-x)\, u(x)\right]$$

$$u''(x) = 4e^{2x} - 3x\, u(x) - 3\left[u(x) - x\, u(x)\right]$$

$$u''(x) = 4e^{2x} - \cancel{3x\, u(x)} - 3u(x) + \cancel{3x\, u(x)}$$

$$u''(x) + 3u(x) = 4e^{2x}$$

For boundary conditions

from ① Eq $\Rightarrow$ put $x = 0$

$$u(0) = e^{0}$$

$$u(0) = 1$$

from ① Eq $\Rightarrow$ put $x = 1$

$$u(1) = e^{2(1)} + 0$$

$$u(1) = e^{2}$$

So the required equation is

$$u''(x) + 3u(x) = 4e^{2x} \; ; \quad u(0) = 1, \quad u(1) = e^{2}$$ Ans.

(6)

(x) $u(x) = 3x^2 + \int_0^1 K(x,t)\,u(t)\,dt$

where $K(x,t) = \begin{cases} t(1-x) & \text{for } 0 \le t \le x \\ x(1-t) & \text{for } x \le t \le 1 \end{cases}$

Sol. $u(x) = 3x^2 + \int_0^1 K(x,t)\,u(t)\,dt$

$$u(x) = 3x^2 + \int_0^x K(x,t)u(t)\,dt + \int_x^1 K(x,t)u(t)\,dt$$

$$u(x) = 3x^2 + \int_0^x t(1-x)\,u(t)\,dt + \int_x^1 x(1-t)\,u(t)\,dt \quad \text{— ①}$$

Diff Eq ① w.r.t 'x'

$$u'(x) = 6x + \int_0^x t\,u(t)\,dt + \int_x^1 (1-t)\,u(t)\,dt \quad \text{— ②}$$

Diff Eq ② w.r.t 'x'

$$u''(x) = 6 - x\,u(x) - [(1-x)\,u(x)]$$

$$u''(x) = 6 - x\,u(x) - [u(x) - x\,u(x)]$$

$$u''(x) = 6 - \cancel{x\,u(x)} - u(x) + \cancel{x\,u(x)}$$

$$u''(x) + u(x) = 6$$

For boundary conditions.

from ① Eq $\Rightarrow$ put $x = 0$

$$u(0) = 0$$

from ① Eq $\Rightarrow$ put $x = 1$

$$u(1) = 3$$

So the required equation is

$$u''(x) + u(x) = 6 \; ; \quad u(0) = 0 \; , \quad u(1) = 3. \quad \text{Ans.}$$

(xi) $u(x) = \cos x + \int_0^1 K(x,t)\,u(t)\,dt$

where $K(x,t) = \begin{cases} 6t(1-x) & \text{for } 0 \le t \le x \\ 6x(1-t) & \text{for } x \le t \le 1 \end{cases}$

Sol. $u(x) = \cos x + \int_0^1 K(x,t)\,u(t)\,dt$

$u(x) = \cos x + \int_0^x K(x,t)u(t)\,dt + \int_x^1 K(x,t)u(t)\,dt$

$u(x) = \cos x + \int_0^x 6t(1-x)\,u(t)\,dt + \int_x^1 6x(1-t)u(t)\,dt$ — ①

Diff. Eq ① w.r.t 'x'

$u'(x) = -\sin x - 6\int_0^x t\,u(t)\,dt + 6\int_x^1 (1-t)\,u(t)\,dt$ — ②

Diff. Eq ② w.r.t 'x'

$u''(x) = -\cos x - 6x\,u(x) - 6\left[(1-x)u(x)\right]$

$u''(x) = -\cos x - 6x\,u(x) - 6\left[u(x) - x\,u(x)\right]$

$u''(x) = -\cos x - \cancel{6x\,u(x)} - 6\,u(x) + \cancel{6x\,u(x)}$

$u''(x) + 6\,u(x) = -\cos x$

For boundary conditions

from ① Eq $\Rightarrow$ put $x = 0$

$u(0) = \cos(0) + 0$

$u(0) = 1$

from ① Eq $\Rightarrow$ put $x = 1$

$u(1) = \cos(1) + 0$

$u(1) = 0.54$

So the required equation is

$u''(x) + 6\,u(x) = -\cos x \quad ; \quad u(0) = 1 \; , \; u(1) = 0.54$. Ans.

(xii) $u(x) = \sinh x + \int_0^1 K(x,t)\,u(t)\,dt$

where $K(x,t) = \begin{cases} 4t(1-x) & \text{for } 0 \le t \le x \\ 4x(1-t) & \text{for } x \le t \le 1 \end{cases}$

Sol: $u(x) = \sinh x + \int_0^1 K(x,t)\,u(t)\,dt$

$u(x) = \sinh x + \int_0^x K(x,t)\,u(t)\,dt + \int_x^1 K(x,t)\,u(t)\,dt$

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$$u(x) = \sinh x + \int_0^x 4t(1-x)\,u(t)\,dt + \int_x^1 4x(1-t)\,u(t)\,dt \quad \text{---(1)}$$

Diff. Eq (1) w.r.t 'x'

$$u'(x) = \cosh x - 4\int_0^x t\,u(t)\,dt + 4\int_x^1 (1-t)\,u(t)\,dt \quad \text{---(2)}$$

Diff. Eq (2) w.r.t 'x'

$$u''(x) = \sinh x - 4x\,u(x) - 4\left[(1-x)\,u(x)\right]$$

$$u''(x) = \sinh x - 4x\,u(x) - 4\left[u(x) - x\,u(x)\right]$$

$$u''(x) = \sinh x - \cancel{4x\,u(x)} - 4u(x) + \cancel{4x\,u(x)}$$

$$u''(x) + 4u(x) = \sinh x$$

For boundary conditions

From (1) Eq $\Rightarrow$ put $x=0$

$$u(0) = \sinh(0) \qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{e^0 - e^{-0}}{2} \Rightarrow \frac{1 - \frac{1}{e^0}}{2}$$

$$u(0) = \frac{1-1}{2}$$

$$u(0) = 0$$

From (1) Eq $\Rightarrow$ put $x=1$

$$u(1) = \sinh(1)$$

$$u(1) = \frac{e - e^{-1}}{2} \Rightarrow \frac{e - \frac{1}{e}}{2}$$

$$u(1) = \frac{e^2 - 1}{2e}$$

So the required equation is

$$u''(x) + 4u(x) = \sinh x \;; \quad u(0) = 0 \;, \quad u(1) = \frac{e^2-1}{2e}$$

Ans.