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Lecture #3 (Wednesday) 22-01-2

Convert Volterra integro-differential equation to Initial value problem

Example: Find Initial value problem equivalent to the Volterra integro-differential equation

$$u(x) = e^x + \int_0^x u(t)\,dt.$$

Step 1: $u(x) = e^x + \int_0^x u(t)\,dt \quad -①$

Taking differential of ① Eq

$$u'(x) = e^x + \frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_0^x f(t)\,dt = F(x)\right]$$

$$u'(x) = e^x + u(x)$$

$$u'(x) - u(x) = e^x$$

Step 2:

Now for conditions:

from ① Equation $\Rightarrow$

at $x=0$

$$u(0) = e^0 + \int_0^0 u(t)\,dt \qquad \because \left[\int_0^0 u(t)\,dt = 0\right]$$

$$u(0) = 1$$

So required result is

$$u'(x) - u(x) = e^x \quad ; \quad u(0) = 1$$

The above equation is initial value problem. Ans.

Example: Find Initial value problem to Volterra Integral equation

$$u(x) = x^2 + \int_0^x (x-t)\, u(t)\, dt$$

Sol. $u(x) = x^2 + \int_0^x (x-t)\, u(t)\, dt$ — ①

Taking differential of ① Eq

$$u'(x) = 2x + \frac{d}{dx}\int_0^x (x-t)\, u(t)\, dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt\right]$$

$$u'(x) = 2x + \int_0^x u(t)\, dt \quad \text{— ②}$$

Again taking differential of ② Eq

$$u''(x) = 2 + \frac{d}{dx}\int_0^x u(t)\, dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\, dt = F(x)\right]$$

$$u''(x) = 2 + u(x)$$

$$u''(x) - u(x) = 2$$

Now for initial condition

from ① Eq ⇒

put $x=0$

$u(0) = 0$

from ② Eq ⇒

put $x=0$

$u'(0) = 0$

So the required equation is

$$u''(x) - u(x) = 2 \;; \quad u(0) = u'(0) = 0$$

The above equation is initial value problem. Ans.

Example: $u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

Sol: $u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$ — ①

Taking differential of ① Eq

$$u'(x) = \cos x - \frac{2}{2}\int_0^x (x-t)\,u(t)\,dt$$

$$u'(x) = \cos x - \int_0^x (x-t)\,u(t)\,dt \quad \text{— ②}$$

Taking differential of ② Eq

$$u''(x) = -\sin x - \frac{d}{dx}\int_0^x (x-t)\,u(t)\,dt$$

By Leibniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u''(x) = -\sin x - 1\int_0^x u(t)\,dt \quad \text{— ③}$$

Taking differential of ③ Eq

$$u'''(x) = -\cos x - \frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^b f(t)\,dt = F(x)\right]$$

$$u'''(x) = -\cos x - u(x)$$

$$u'''(x) + u(x) = -\cos x$$

Now For conditions:

From ① Eq ⇒ put $x=0$

$$u(0) = 0$$

From ② Eq ⇒ put $x=0$

$$u'(0) = 1$$

From ③ Eq ⇒ put $x=0$

$$u''(0) = 0$$

So the required equation is

$$u'''(x) + u(x) = -\cos x \; ; \; u(0) = 0,\; u'(0) = 1,\; u''(0) = 1$$

The above equation is Initial value problem Ans.

Exercise 2.5: Convert each of the following Volterra integral equations to Initial value problem

(i) $u(x) = x + 2\int_0^x u(t)\,dt.$

Sol: $u(x) = x + 2\int_0^x u(t)\,dt$ —①

Taking differential of ① Eq

$$u'(x) = 1 + 2\frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

$$u'(x) = 1 + 2u(x)$$

$$u'(x) - 2u(x) = 1$$

Now for conditions:

from ① Eq $\Rightarrow$ put $x=0$

$$u(0) = 0$$

So the required equation is

$$u'(x) - 2u(x) = 1 \quad ; \quad u(0) = 0$$

The above equation Initial value problem Ans.

(ii) $u(x) = 1 + e^{x} - \int_0^x u(t)\,dt$

Sol: $u(x) = 1 + e^{x} - \int_0^x u(t)\,dt$ —①

Taking differential of ① Eq

$$u'(x) = e^{x} - \frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

$$u'(x) = e^{x} - u(x)$$

$$u'(x) + u(x) = e^{x}$$

Now for conditions:

From ① Eq $\Rightarrow$ put $x = 0$

$$u(0) = 1 + e^{0}$$

$$u(0) = 1 + 1$$

$$u(0) = 2$$

So the required equation is

$$u'(x) + u(x) = e^{x} \quad ; \quad u(0) = 2$$

The above equation is Initial value problem. Ans.

(iii) $u(x) = 1 + x^2 + \int_0^x (x-t)\,u(t)\,dt$

Sol: $u(x) = 1 + x^2 + \int_0^x (x-t)\,u(t)\,dt$ —①

Taking differential of ① Eq

$$u'(x) = 2x + \frac{d}{dx}\int_0^x (x-t)\,u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$u'(x) = 2x + \int_0^x u(t)\,dt$ ——②

Taking differential of Eq ②,

$u''(x) = 2 + \frac{d}{dx}\int_0^x u(t)\,dt$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^b f(t)\,dt = F(x)\right]$$

$u''(x) = 2 + u(x)$

$u''(x) - u(x) = 2$

Now for conditions:

From ① Eq ⇒ put $x=0$

$u(0) = 1$

From ② Eq ⇒

put $x=0$

$u'(0) = 0$

So the required equation is

$u''(x) - u(x) = 2 \;;\; u(0) = 1 ,\; u'(0) = 0$

The above equation is initial value problem Ans.

(iv) $u(x) = \sin x - \int_0^x (x-t)\,u(t)\,dt$.

Sol: $u(x) = \sin x - \int_0^x (x-t)\,u(t)\,dt$ ——①

Taking differential of ① Eq

$u'(x) = \cos x - \frac{d}{dx}\int_0^x (x-t)\,u(t)\,dt$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u'(x) = \cos x - \int_0^x u(t)\,dt \quad -(2)$$

Taking differential of (2) Eq

$$u''(x) = -\sin x - \frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

$$u''(x) = -\sin x - u(x)$$

$$u''(x) + u(x) = -\sin x$$

Now for conditions:

From (1) Eq $\Rightarrow$

put $x=0$

$$u(0) = 0$$

From (2) Eq $\Rightarrow$ put $x=0$

$$u'(0) = 1$$

So the required equation is

$$u''(x) + u(x) = -\sin x \quad ; \quad u(0)=0 \;,\; u'(0)=1$$

The above equation is Initial value problem. Ans.

(v) $u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)\,dt$

Sol: $u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)\,dt \quad -(1)$

Taking differential of (1) Eq

$$u'(x) = \sin x + 2\frac{d}{dx}\int_0^x (x-t)^2 u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u'(x) = \sin x + 4\int_0^x (x-t)u(t)\,dt \quad \text{--- (2)}$$

Taking differential of (2) Eq.

$$u''(x) = \cos x + 4\,\frac{d}{dx}\int_0^x (x-t)u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u''(x) = \cos x + 4\int_0^x u(t)\,dt \quad \text{--- (3)}$$

Taking differential of (3) Eq.

$$u'''(x) = -\sin x + 4\,\frac{d}{dx}\int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\because \left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

$$u'''(x) = -\sin x + 4u(x)$$

$$u'''(x) - 4u(x) = -\sin x$$

Now for conditions.

from (1) Eq $\Rightarrow$ put $x=0$

$$u(0) = 1-1 \Rightarrow u(0) = 0$$

from (2) Eq $\Rightarrow$ put $x=0$

$$u'(0) = 0$$

from (3) Eq $\Rightarrow$ put $x=0$

$$u''(0) = 1$$

So required equation is

$$u'''(x) - 4u(x) = -\sin x \;;\quad u''(0) = 1,\; u'(0) = u(0) = 0$$

The above equation is Initial value problem Ans.

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(vi) $u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)\, dt$

Sol. $u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)\, dt$ — ①

Taking differential of ① Eq

$$u'(x) = \cosh x + \frac{d}{dx}\int_0^x (x-t)^2 u(t)\, dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt\right]$$

$$u'(x) = \cosh x + 2\int_0^x (x-t)\, u(t)\, dt \quad \text{— ②}$$

Taking differential of ② Eq

$$u''(x) = \sinh x + 2\frac{d}{dx}\int_0^x (x-t)\, u(t)\, dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt\right]$$

$$u''(x) = \sinh x + 2\int_0^x u(t)\, dt \quad \text{— ③}$$

Taking differential of ③ Eq

$$u'''(x) = \cosh x + 2\frac{d}{dx}\int_0^x u(t)\, dt$$

By Fundamental theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\, dt = f(x)\right]$$

$$u'''(x) = \cosh x + 2u(x)$$

$$u'''(x) - 2u(x) = \cosh x$$

Now for conditions,

from ① Eq ⇒ put $x = 0$

$$u(0) = 2 + \frac{e^0 - e^{-0}}{2} \qquad \left(\sinh x = \frac{e^x - e^{-x}}{2}\right)$$

$$u(0) = 2 + \frac{1-1}{2}$$

$$u(0) = 2$$

from ② Eq $\Rightarrow$ put $x=0$

$$u'(0) = \frac{e^{0}+e^{-0}}{2}$$

$$u'(0) = \frac{1+1}{2} \Rightarrow \frac{2}{2} = 1$$

$$u'(0) = 1$$

$$\left[\cosh x = \frac{e^{x}+e^{-x}}{2}\right]$$

from ③ Eq $\Rightarrow$

$$u''(0) = \frac{e^{0}-e^{-0}}{2} \Rightarrow \frac{1-1}{2}$$

$$u''(0) = 0$$

So the required equation is

$$u'''(x) - 2u(x) = \cosh x \;;\; u''(0)=0,\ u'(0)=1,\ u(0)=2$$

The above equation is Initial value problem. Ans.

(vii) $u(x) = 1 + 2\int_0^x (x-t)^3 u(t)\,dt$.

Sol: $u(x) = 1 + 2\int_0^x (x-t)^3 u(t)\,dt$ —①

Taking differential of ① Eq

$$u'(x) = 2\frac{d}{dx}\int_0^x (x-t)^3 u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u'(x) = 2\times 3\int_0^x (x-t)^2 u(t)\,dt$$

$$u'(x) = 6\int_0^x (x-t)^2 u(t)\,dt \quad —②$$

Taking differential of ② Eq

$$u''(x) = 6\frac{d}{dx}\int_0^x (x-t)^2 u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u''(x) = 12 \int_0^x (x-t)\,u(t)\,dt \quad —③$$

Taking differential of ③ Eq

$$u'''(x) = 12 \frac{d}{dx} \int_0^x (x-t)\,u(t)\,dt$$

By Lebniz Rule

$$\left[\frac{d}{dx} \int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt \right]$$

$$u'''(x) = 12 \int_0^x u(t)\,dt \quad —④$$

Taking differential of ④ Eq

$$u''''(x) = 12 \frac{d}{dx} \int_0^x u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx} \int_a^x f(t)\,dt = F(x) \right]$$

$$u''''(x) = 12\,u(x)$$

$$u''''(x) - 12\,u(x) = 0$$

Now for conditions:

from ① Eq $\Rightarrow$ put $x=0$

$u(0) = 1$

from ② Eq $\Rightarrow$ put $x=0$

$u'(0) = 0$

from ③ Eq $\Rightarrow$ put $x=0$

$u''(0) = 0$

from ④ Eq $\Rightarrow$ put $x=0$

$u'''(0) = 0$

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So the required equation is

$$u''''(x) - 12u(x) = 0 \quad ; \quad u(0) = 1, \quad u'(0) = u''(0) = u'''(0) = 0.$$

The above equation is Initial value problem Ans.

(viii) $u(x) = 1 + e^x + \int_0^x (1+x-t)^3 u(t)\,dt$

Sol: $u(x) = 1 + e^x + \int_0^x (1+x-t)^3 u(t)\,dt$ —①

$$u(x) = 1 + e^x + \int_0^x \left[1 + (x-t)^3 + 3(x-t)^2 + 3(x-t)\right] u(t)\,dt$$

$$u(x) = 1 + e^x + \int_0^x u(t)\,dt + \int_0^x (x-t)^3 u(t)\,dt + 3\int_0^x (x-t)^2 u(t)\,dt + 3\int_0^x (x-t)u(t)\,dt$$

$$u(x) = 1 + e^x + \int_0^x u(t)\,dt + 6\left(\frac{1}{3!}\right)\int_0^x (x-t)^3 u(t)\,dt + 3\left(\frac{2}{2}\right)\int_0^x (x-t)^2 u(t)\,dt$$
$$+ 3\int_0^x (x-t)u(t)\,dt$$

Now differentiate

$$u'(x) = e^x + \frac{d}{dx}\int_0^x u(t)\,dt + 6\,\frac{d}{dx}\cdot\frac{1}{3!}\int_0^x (x-t)^3 u(t)\,dt + 6\,\frac{d}{dx}\cdot\frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$
$$+ 3\,\frac{d}{dx}\int_0^x (x-t)u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

and

By Lebniz Theorem

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

$$u'(x) = e^x + u(x) + 6\left(\frac{1}{2}\right)\int_0^x (x-t)^2 u(t)\,dt + 6\int_0^x (x-t)u(t)\,dt + 3\int_0^x u(t)\,dt \rightarrow ②$$

Now differentiate ②. Equation

$$u''(x) = e^x + u'(x) + 6\,\frac{d}{dx}\cdot\frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt + 6\,\frac{d}{dx}\int_0^x (x-t)u(t)\,dt + 3\frac{d}{dx}\int_0^x u(t)\,dt$$

By Lebniz Theorem

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⑩

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

and

Fundamental Theorem of calculus (part 1)

$$\left[\frac{d}{dx}\int_a^b f(t)\,dt = F(x)\right]$$

$$u''(x) - u'(x) = e^x + 6\int_0^x (x-t)u(t)\,dt + 6\int_0^x u(t)\,dt + 3u(x)$$

$$u''(x) - u'(x) - 3u(x) = e^x + 6\int_0^x (x-t)u(t)\,dt + 6\int_0^x u(t)\,dt \quad — ③$$

Now differentiate ③ Equation

$$u'''(x) - u''(x) - 3u'(x) = e^x + 6\frac{d}{dx}\int_0^x (x-t)u(t)\,dt + 6\frac{d}{dx}\int_0^x u(t)\,dx$$

By Lebniz Theorem

$$\left[\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt\right]$$

and

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^b f(t)\,dt = F(x)\right]$$

$$u'''(x) - u''(x) - 3u'(x) = e^x + 6\int_0^x u(t)\,dt + 6u(x)$$

$$u'''(x) - u''(x) - 3u'(x) - 6u(x) = e^x + 6\int_0^x u(t)\,dt \quad — ④$$

Differentiate ④ Equation

$$u''''(x) - u'''(x) - 3u''(x) - 6u'(x) = e^x + 6\frac{d}{dx}\int u(t)\,dt$$

By Fundamental Theorem of Calculus (part 1)

$$\left[\frac{d}{dx}\int_a^x f(t)\,dt = F(x)\right]$$

$$u''''(x) - u'''(x) - 3u''(x) - 6u'(x) = e^x + 6u(x)$$

$$u''''(x) - u'''(x) - 3u''(x) - 6u'(x) - 6u(x) = e^x$$

Now For conditions

From ① put $x=0$

$$u(0)=2$$

From ② $\Rightarrow$

put $x=0$

$$u'(0)=1+u(0)$$

$$\because u(0)=2$$

$$u'(0)=3$$

From ③ $\Rightarrow$

put $x=0$

$$u''(0)=1+u'(0)+3u(0)$$

$$u''(0)=1+3+3(2)$$

$$u''(0)=10$$

From ④ $\Rightarrow$

put $x=0$

$$u'''(0)=1+u''(0)+3u'(0)+6u(0)$$

$$=1+10+3(3)+6(2)$$

$$u'''(0)=32$$

So the required equation is

$$u^{iv}(x)-u'''(x)-3u''(x)-6u'(x)-6u(x)=e^{x}; \quad u(0)=2,\ u'(0)=3,\ u''(0)=1$$

$$u'''(0)=32.$$

The above equation is Initial value problem.

Ans.