Converting Initial Value Problem to Volterra integral equation

[Initial condition → time dependent
Boundary condition → space dependent]

There are general steps to convert initial value problem to Volterra integral equations:

Consider the second order IVP given by

$$y''(x) + P(x)\,y'(x) + q\,y(x) = g(x) \quad \text{—} ①$$

Subject to initial condition $y(0) = \alpha$, $y'(0) = \beta$

(I) Set the higher order i.e. value from equation is equal to $u(x)$

such that $\quad y''(x) = u(x) \quad$ —②

(II) Integrate this relation until the simple variable form

such that $\quad y''(x) = u(x)$

Integrate

$$y'(x)\Big|_0^x = \int_0^x u(t)\,dt$$

[$u(x)$ is dummy variable
So $u(x) = u(t)$]

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

putting $y'(0) = \beta$

$$y'(x) - \beta = \int_0^x u(t)\,dt$$

$$y'(x) = \beta + \int_0^x u(t)\,dt$$

Again integrate

$$y(x)\Big|_0^x = \beta x + \int_0^x\int_0^x u(t)\,dt$$

$\because$ Multiple to single integral

$$\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)\,u(t)\,dt$$

$$\text{or} \quad \int_0^x\int_0^x\int_0^x u(t)\,dt = \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt$$

So,

$$y(x) - y(0) = \beta x + \int_0^x (x-t)\,u(t)\,dt$$

$$\text{put} \quad y(0) = \alpha$$

$$y(x) - \alpha = \beta x + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = \alpha + \beta x + \int_0^x (x-t)\,u(t)\,dt$$

(III) Putting the value of $y(x)$ in given IVP equation

$$y''(x) + P\left[\beta + \int_0^x u(t)\,dt\right] + Q\left[\alpha + \beta x + \int_0^x (x-t)\,u(t)\,dt\right] = g(x)$$

(IV) Separating the higer order value

$$y''(x) = g(x) - P\beta - P\int_0^x u(t)\,dt - Q\alpha - Q\beta x - Q\int_0^x (x-t)\,u(t)\,dt$$

(V) Now putting higher order value is equal to $u(x)$

from (2) $\Rightarrow$ $y''(x) = u(x)$

$$u(x) = g(x) - P\beta - P\int_0^x u(t)\,dt - Q\alpha + Q\beta x - Q\int_0^x (x-t)\,u(t)\,dt.$$

$$u(x) = g(x) - P\beta - Q\alpha + Q\beta x - \int_0^x P - Q(x-t)\,u(t)\,dt$$

This is required Volteera Integral Equation, where $\lambda = -1$, $f(x) = g(x) - P\beta - Q\alpha + Q\beta x$, $K(x,t) = P - Q(x-t)$. Ans.

Example: Consider the following I.V.P to Volteera Integral Equation:

$y'(x) - 2xy(x) = e^{x^2}$; $y(0) = 1$

Sol: $y'(x) - 2xy(x) = e^{x^2}$ — ①

Set

$$y'(x) = u(x) \quad — ②$$

Integrate

$$y(x) - y(0) = \int_0^x u(t)\,dt$$

$\int_0^x y'(x) = y(x)\Big|_0^x = y(x) - y(0)$

put $y(0) = 1$

$$y(x) - 1 = \int_0^x u(t)\,dt$$

$$y(x) = 1 + \int_0^x u(t)\,dt$$

put the value of $y(x)$ in ①, we get

$$y'(x) - 2x\left[1 + \int_0^x u(t)\,dt\right] = e^{x^2}$$

$$y'(x) - 2x - 2x\int_0^x u(t)\,dt = e^{x^2}$$

$$y'(x) = e^{x^2} + 2x + 2x\int_0^x u(t)\,dt$$

from ② $\Rightarrow$ $y'(t) = u(x)$

$$u(x) = e^{x^2} + 2x + 2x\int_0^x u(t)\,dt$$

Which is required Volterra Integral Equation, where $f(x) = e^{x^2} + 2x$, $\lambda = 2x$, $K(x,t) = 1$, $g(x) = 0$ and $h(x) = x$.

Ans.

Example: $y''(x) - y(x) = \sin x$; $y(0) = y'(0) = 0$

Sol. $y''(x) - y(x) = \sin x$ — ①

Set $y''(x) = u(x)$ — ②

Now integrate

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

put $y'(0) = 0$

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$y'(x) = \int_0^x u(t)\,dt$

Again integrate

$y(x) - y(0) = \int_0^x\int_0^x u(t)\,dt$ ∵ [By multiple to single integral $\int_0^x\int_0^x u(t)dt = \int_0^x (x-t)u(t)dt$

$y(x) - y(0) = \int_0^x (x-t)u(t)\,dt$

put $y(0) = 0$

$y(x) = \int_0^x (x-t)u(t)\,dt$

Put the value of $y(x)$ in ①, we get

$y''(x) - \int_0^x (x-t)u(t)\,dt = \sin x$

$y''(x) = \sin x + \int_0^x (x-t)u(t)\,dt$

② Eq. ⇒

$y''(x) = u(x)$

So, $u(x) = \sin x + \int_0^x (x-t)u(t)\,dt$

Which is required Volterra Integral Equation, where $f(x) = \sin x$, $\lambda = 1$, $K(x,t) = x-t$.

Ans.

(Exercise 2.5): Convert the following I.V.P to Volterra Integral Equation:

(i) $y' - 4y = 0$; $y(0) = 1$

Sol: $y'(x) - 4y(x) = 0$ —①

Set $y'(x) = u(x)$ —②

Integrate

$y(x) - y(0) = \int_0^x u(t)\,dt$

put $y(0) = 1$

$$y(x) - 1 = \int_0^x u(t)\,dt$$

$$y(x) = 1 + \int_0^x u(t)\,dt$$

put the value of $y(x)$ in ① Eq. we get

$$y'(x) - 4\left[1 + \int_0^x u(t)\,dt\right] = 0$$

$$y'(x) - 4 - 4\int_0^x u(t)\,dt = 0$$

$$y'(x) = 4 + 4\int_0^x u(t)\,dt$$

from ② Eq $\Rightarrow$

$$y'(x) = u(x)$$

So $\quad u(x) = 4 + 4\int_0^x u(t)\,dt$

Which is required Volterra Integral Equation, where $f(x) = 4$, $\lambda = 4$, $k(x,t) = 1$. Ans.

(ii) $y' + 4xy = e^{-2x^2}$; $y(0) = 0$

Sol: $y'(x) + 4x\,y(x) = e^{-2x^2}$ —①

Set $\quad y'(x) = u(x)$ —②

Integrate

$$y(x) - y(0) = \int_0^x u(t)\,dt$$

put $y(0) = 0$

$$y(x) = \int_0^x u(t)\,dt$$

put the value of $y(x)$ in ① Eq. we get

$$y'(x) + 4x\left[\int_0^x u(t)\,dt\right] = e^{-2x^2}$$

$$y'(x) = e^{-2x^2} - 4x\int_0^x u(t)\,dt$$

from ② Eq $\Rightarrow$ $\quad y'(x) = u(x)$

So, $\quad u(x) = e^{-2x^2} - 4x\int_0^x u(t)\,dt$

Which is required Volterra integral equation, where $f(x) = e^{-2x^2}$, $\lambda = 4x$, $K(x,t) = 1$. Ans.

(iii) $y'' + 4y = 0$; $y(0) = y'(0) = 0$

Sol. $y''(x) + 4y(x) = 0$ — ①

Set $y''(x) = u(x)$ — ②

Integrate

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

put $y'(0) = 0$

$$y'(x) = \int_0^x u(t)\,dt$$

Again integrate

$$y(x) - y(0) = \int_0^x\int_0^x u(t)\,dt$$

∵ By multiple to single integral $\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)u(t)\,dt$

$$y(x) - y(0) = \int_0^x (x-t)u(t)\,dt$$

put $y(0) = 0$

$$y(x) = \int_0^x (x-t)u(t)\,dt$$

put the value of $y(x)$ in ① Eq, we get

$$y''(x) + 4\int_0^x (x-t)\,u(t)\,dt = 0$$

$$y''(x) = -4\int_0^x (x-t)u(t)\,dt$$

from ② Eq ⇒ $y''(x) = u(x)$

$$u(x) = -4\int_0^x (x-t)\,u(t)\,dt$$

Which is required Volterra Integral Equation where $f(x) = 0$, $\lambda = -4$, $K(x,t) = x - t$.

Ans.

(iv) $y'' - 6y' + 8y = 1 \quad ; \quad y(0) = y'(0) = 1$

Sol. $y''(x) - 6y'(x) + 8y(x) = 1$ —①

Set $y''(x) = u(x)$ —②

Integrate

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

put $y'(0) = 1$

$$y'(x) - 1 = \int_0^x u(t)\,dt$$

$$y'(x) = 1 + \int_0^x u(t)\,dt$$

Again integrate

$$y(x) - y(0) = x + \int_0^x\int_0^x u(t)\,dt$$

∴ By multiple to single integral

$$\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)u(t)\,dt$$

$$y(x) - y(0) = x + \int_0^x (x-t)u(t)\,dt$$

put $y(0) = 1$

$$y(x) - 1 = x + \int_0^x (x-t)\,u(t)\,dt$$

$$y(x) = 1 + x + \int_0^x (x-t)u(t)\,dt$$

Put the value of $y'(x)$ and $y(x)$ in ① Eq, we get

$$y''(x) - 6\left[1 + \int_0^x u(t)dt\right] + 8\left[1 + x + \int_0^x (x-t)\,u(t)\,dt\right] = 1$$

$$y''(x) = 1 + 6 + 6\int_0^x u(t)\,dt - 8 - 8x - 8\int_0^x (x-t)u(t)\,dt$$

$$y''(x) = 7 + 6\int_0^x u(t)\,dt - 8 - 8x - 8\int_0^x (x-t)u(t)\,dt$$

$$y''(x) = -1 + 6\int_0^x u(t)\,dt - 8x - 8\int_0^x (x-t)u(t)\,dt$$

from ② Eq $\Rightarrow$ $y''(x) = u(x)$

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$$u(x) = -1-8x + \int_0^x 6-8(x-t)\,u(t)\,dt$$

Which is required Volterra Integral Equation where $f(x) = -1-8x$, $\lambda = 1$, $K(x,t) = 6-8(x-t)$. Ans.

(v) $y''' - 2y'' + y = x$; $y(0)=1$, $y'(0)=0$, $y''(0)=1$

Sol. $y'''(x) - 2y''(x) + y(x) = x$ — (1)

Set $y'''(x) = u(x)$ — (2)

Integrate

$$y''(x) - y''(0) = \int_0^x u(t)\,dt$$

put $y''(0)=1$

$$y''(x) - 1 = \int_0^x u(t)\,dt$$

$$y''(x) = 1 + \int_0^x u(t)\,dt$$

Again integrate

$$y'(x) - y'(0) = x + \int_0^x\int_0^x u(t)\,dt$$

put $y'(0)=0$

$$y'(x) = x + \int_0^x (x-t)\,u(t)\,dt \qquad \left[\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)\,u(t)\,dt\right.$$

Again integrate

$$y(x) - y(0) = \frac{x^2}{2} + \int_0^x\int_0^x (x-t)\,u(t)\,dt$$

$$y(x) - y(0) = \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt$$

put $y(0)=1$

$$y(x) - 1 = \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt$$

$$y(x) = 1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt$$

put the value of $y(x)$ and $y''(x)$ in (1) Eq. we get

$$y'''(x) - 2\left[1 + \int_0^x u(t)\,dt\right] + 1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt = x$$

$$y'''(x) - 2 - 2\int_0^x u(t)\,dt + 1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt = x$$

$$y''(x) - 1 - 2\int_0^x u(t)\,dt + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt = x$$

$$y''(x) = x + 1 + \frac{x^2}{2} + 2\int_0^x u(t)\,dt - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

$$y''(x) = x + \frac{x^2}{2} + 1 + \int_0^x 2 - \frac{1}{2}(x-t)^2 u(t)\,dt$$

from ② Eq $\Rightarrow$

$$y''(x) = u(x)$$

$$u(x) = x + \frac{x^2}{2} + 1 + \int_0^x 2 - \frac{1}{2}(x-t)^2 u(t)\,dt$$

Which is required Volterra integral Equation where $f(x) = x + \frac{x^2}{2} + 1$, $\lambda = 1$, $K(x,t) = 2 - \frac{1}{2}(x-t)^2$. Ans.

(vi) $y''' - y' = 0$; $y(0) = 2$, $y'(0) = y''(0) = 1$

Sol. $y'''(x) - y'(x) = 0$ — ①

Set $y'''(x) = u(x)$ — ②

Integrate

$$y''(x) - y''(0) = \int_0^x u(t)\,dt$$

u(x) is dummy variable So u(x)=u(t)

$\int_0^x y'''(x)dx \Rightarrow y''(x)\Big|_0^x \Rightarrow y''(x) - y''(0)$

put $y''(0) = 1$

$$y''(x) - 1 = \int_0^x u(t)\,dt$$

$$y''(x) = 1 + \int_0^x u(t)\,dt$$

Again integrate

$$y'(x) - y'(0) = x + \int_0^x\int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = x + \int_0^x (x-t)u(t)\,dt$$

$\because \left[\int_0^x\int_0^x u(t)dt = \int_0^x (x-t)u(t)dt\right]$

put $y'(0) = 1$

$$y'(x) - 1 = x + \int_0^x (x-t)u(t)\,dt$$

$$y'(x) = 1 + x + \int_0^x (x-t)u(t)\,dt$$

Again integrate

$$y(x) - y(0) = x + \frac{x^2}{2} + \int_0^x\int_0^x (x-t)u(t)\,dt$$

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$$y(x) - y(0) = x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

put $y(0) = 2$

$$y(x) - 2 = x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

$$y(x) = 2 + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

putting the value of $y'(x)$ in ① Eq, we get

$$y''(x) - \left[1 + x + \int_0^x (x-t)u(t)\,dt\right] = 0$$

$$y''(x) = 1 + x + \int_0^x (x-t)u(t)\,dt$$

From ② Eq $\Rightarrow$ $y''(x) = u(x)$

$$u(x) = 1 + x + \int_0^x (x-t)u(t)\,dt$$

Which is required Volterra Integral Equation where $f(x) = 1+x$, $\lambda = 1$, $K(x,t) = x - t$ Ans.

(iii) $y''' - y'' = 1$; $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$

Sol: $y'''(x) - y''(x) = 1$ — ①

Set $y'''(x) = u(x)$ — ②

Integrate

$$y''(x) - y''(0) = \int_0^x u(t)\,dt$$

put $y''(0) = 1$

$$y''(x) - 1 = \int_0^x u(t)\,dt$$

$$y''(x) = 1 + \int_0^x u(t)\,dt$$

Again Integrate

$$y'(x) - y'(0) = x + \int_0^x\int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = x + \int_0^x (x-t)\,u(t)\,dt \qquad \because \left[\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)u(t)\,dt\right]$$

put $y'(0)=1$

$$y'(x)-1 = x+\int_0^x (x-t)u(t)\,dt$$

$$y'(x) = 1+x+\int_0^x (x-t)u(t)\,dt$$

putting the value of $y'(x)$ in ① Eq. we get

$$y''''(x)-\left[1+x+\int_0^x (x-t)u(t)\,dt\right]=1$$

$$y''''(x) = 1+1+x+\int_0^x (x-t)u(t)\,dt$$

$$y''''(x) = 2+x+\int_0^x (x-t)u(t)\,dt$$

from ② Eq $\Rightarrow$ $y''''(x)=u(x)$

$$u(x) = 2+x+\int_0^x (x-t)u(t)\,dt$$

Which is required Volterra Integral Equation, where $f(x)=2+x$, $\lambda=1$, $K(x,t)=(x-t)$.

Ans.

(viii) $y''' + y'' + y = x$; $y(0)=y'(0)=1$, $y''(0)=y'''(0)=0$

Sol: $y'''(x)+y''(x)+y(x)=x$ —①

Set $y'''(x)=u(x)$ —②

Integrate

$$y''(x)-y''(0)=\int_0^x u(t)\,dt$$ $\because$ [$u(x)$ is dummy variable So $u(x)=u(t)$]

put $y''(0)=0$

$$y''(x)=\int_0^x u(t)\,dt$$

Again integrate

$$y'(x)-y'(0)=\int_0^x\int_0^x u(t)\,dt$$

$$y'(x)-y'(0)=\int_0^x (x-t)u(t)\,dt$$ $\because \left[\int_0^x\int_0^x u(t)\,dt = \int_0^x (x-t)u(t)\,dt\right]$

put $y'(0)=0$

$$y'(x)=\int_0^x (x-t)u(t)\,dt$$

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Again integrate

$$y'(x) - y'(0) = \int_0^x\int_0^x (x-t)u(t)dt$$

$$y'(x) - y'(0) = \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

put $y'(0)=1$

$$y'(x) - 1 = \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

$$y'(x) = 1 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

Again integrate

$$y(x) - y(0) = x + \frac{1}{2}\int_0^x\int_0^x (x-t)^2 u(t)\,dt$$

$$y(x) - y(0) = x + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

put $y(0)=1$

$$y(x) - 1 = x + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

$$y(x) = 1 + x + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

Putting the values of $y''(x)$ and $y(x)$ in ① Eq. we get

$$y''(x) + \int_0^x (x-t)u(t)dt + 1 + \cancel{x} + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt = \cancel{x}$$

$$y''(x) = -1 - \int_0^x (x-t) - \frac{1}{6}(x-t)^3 u(t)\,dt$$

from ② Eq ⇒

$$y''(x) = u(x)$$

$$u(x) = -1 - \int_0^x (x-t) - \frac{1}{6}(x-t)^3 u(t)\,dt$$

Which is required Volterra Integral Equation where $f(x) = -1$, $\lambda = -1$ and $K(x,t) = (x-t) - \frac{1}{6}(x-t)^3$. Ans.

(ix) $y''' - y'' - y' + y = 0$; $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$.

Sol: $y'''(x) - y''(x) - y'(x) + y(x) = 0$ —①

Set $y'''(x) = u(x)$ —②

Integrate

$y''(x) - y''(0) = \int_0^x u(t)\,dt$

put $y''(0) = 3$

$y''(x) - 3 = \int_0^x u(t)\,dt$

$y''(x) = 3 + \int_0^x u(t)\,dt$

Again integrate

$y'(x) - y'(0) = 3x + \int_0^x\int_0^x u(t)\,dt$ $\quad \left[\int_0^x\int_0^x u(t)dt = \int_0^x (x-t)u(t)dt\right]$

$y'(x) - y'(0) = 3x + \int_0^x (x-t)u(t)\,dt$

put $y'(0) = 2$

$y'(x) - 2 = 3x + \int_0^x (x-t)u(t)\,dt$

$y'(x) = 2 + 3x + \int_0^x (x-t)\,u(t)\,dt$

Again integrate

$y(x) - y(0) = 2x + \frac{3x^2}{2} + \int_0^x\int_0^x (x-t)u(t)\,dt$

$y(x) - y(0) = 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

put $y(0) = 1$

$y(x) - 1 = 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

$y(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

putting the values of $y''(x)$, $y'(x)$ & $y(x)$ in ① Eq, we get

$0 = y'''(x) - \left[3 + \int_0^x u(t)dt\right] - \left[2 + 3x \int_0^x (x-t)u(t)dt\right] + 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt$

$y'''(x) = 3 + \int_0^x u(t)\,dt + 2 + 3x\int_0^x (x-t)u(t)\,dt - 1 - 2x - \frac{3}{2}x^2 - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

$y'''(x) = 4 - 2x - \frac{3}{2}x^2 + \int_0^x 1 + 3x(x-t) - \frac{1}{2}(x-t)^2\, u(t)\,dt$

from ② Eq $\Rightarrow$ $y'''(x) = u(x)$

$u(x) = 4 - 2x - \frac{3}{2}x^2 + \int_0^x 1 + 3x(x-t) - \frac{1}{2}(x-t)^2\, u(t)\,dt$

Which is required Volterra Integral Equation where $f(x) = 4 - 2x - \frac{3}{2}x^2$, $\lambda = 1$, $k(x,t) = 1 + 3x(x-t) - \frac{1}{2}(x-t)^2$.

Ans.