

## Chapter #2

### Basic Concepts of Linear Integral Equation

► Integral Equation: An integral equation is an equation in which unknown function  $U(x)$  appears under an integral equation is defined as,

$$U(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) U(t) dt \quad \text{--- (1)}$$

where

$g(x)$  and  $h(x)$  are the limits of integration

$\lambda$  is constant

$K(x,t)$  is a function of two variables  $x$  and  $t$  called kernel

For Example:  $y(x) = f(x) + 5 \int_0^1 e^{t+x} y(t) dt$

where  $h(x) = 1$  and  $g(x) = 0$

$$\lambda = 5$$

$$K(x,t) = e^{t+x}$$

Note:

There is close connection between differential and integral and some problem [scientific problem (research paper)] may be formulated theory and Maxwell Equation.

► Classification: Integral Equation are classified according to three different types

(a) Limit of Integration

(b) Placement of unknown function

(c) Nature of unknown function(a) Limit of Integration:

(i) both fixed (constant) = Fredholm Integral Equation

$$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt$$

(ii) one variable = Volterra Integral Equation

$$u(x) = f(x) + \lambda \int_a^x k(x,t) u(t) dt$$

(b) Placement of unknown function:

(i) one side integral = First kind Volterra / Fredholm I.E

$$u(x) = \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Fredholm Integral Equation of 1st kind})$$

$$u(x) = \lambda \int_a^x k(x,t) u(t) dt \quad (\text{Volterra Integral Equation of 1st kind})$$

(ii) both inside / outside = Second kind Volterra / Fredholm I.E

$$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Fredholm Integral Equation of 2nd kind})$$

$$u(x) = f(x) + \lambda \int_a^x k(x,t) u(t) dt \quad (\text{Volterra Integral Equation of 2nd kind})$$

(c) Nature of unknown function:

(i) identically zero = homogeneous

$$f(x) = 0$$



So equation becomes,

$$u(x) = \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Fredholm Integral Equation})$$

$$u(x) = \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Volterra Integral Equation})$$

(ii) not identically zero = non-homogeneous  
 $f(x) \neq 0$

So equation becomes,

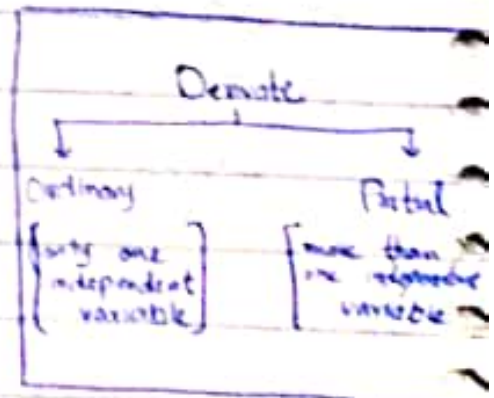
$$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Fredholm Integral Equation})$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt \quad (\text{Volterra Integral Equation})$$

### Types of Integral Equation:

An integro-differential equation is an equation in which unknown function  $u(t)$  appears under integral sign and contain ordinary derivative.

$$u'(x) = f(x) + \lambda \int_a^b k(x,t) u(t) dt$$



### Singular Integral Equation

Volterra integral equation of 1<sup>st</sup> and 2<sup>nd</sup> kind

$$u(x) = \lambda \int_a^b k(x,t) u(t) dt \quad \text{--- (1)}$$

or

In complex analysis  
 $\frac{e^z}{z-1}$  put  $z=1$   
 $\frac{e}{1-1} \Rightarrow \frac{e}{0} \Rightarrow \infty$   
 $\Rightarrow 1$  is the singular point

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) u(t) dt \quad \text{--- (2)}$$

is called singular if the limits of integrations are infinite.

• Fredholm integral equation of 1<sup>st</sup> and 2<sup>nd</sup> kind

$$u(x) = \lambda \int_{g(x)}^{h(x)} K(x,t) u(t) dt \quad \text{--- (3)}$$

or:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) u(t) dt \quad \text{--- (4)}$$

is called singular if limit of integrations are infinite.

(Exercise 2.1)

### Problems

Question # 1: For each of the following integral equation, classify as Fredholm, Volterra, Volterra - Fredholm, 1<sup>st</sup> and 2<sup>nd</sup> kind also classify the equation is singular or not.

(i)  $u(x) = 1 + \int_0^x u(t) dt$

Sol: This is Volterra integral equation, because of

the limit of integration is not constant, it is 2<sup>nd</sup>

kind because the unknown function appear inside/outside

the integral, it is not homogeneous because of 1,

it is not singular since neither limits of integration

are infinite.

$f(x) = 1$
$\lambda = 1$
$K(x,t) = 1$
$g(x) = 0$
$h(x) = x$

Ans.