Chapter #2

Basic Concepts of Linear Integral Equation

Integral Equation: An integral equation is an equation in which unknown function $U(x)$ appears under an integral equation is defined as,

$$U(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt \quad \text{---} (1)$$

where

$g(x)$ and $h(x)$ are the limits of integration

λ is constant

$K(x,t)$ is a function of two variables x and t called kernel.

For Example: $y(x) = f(x) + 5 \int_0^1 e^{t+x}\, y(t)\, dt$

where $h(x) = 1$ and $g(x) = 0$

$\lambda = 5$

$K(x,t) = e^{t+x}$

Note.

There is close connection between differential and integral and some problem [scientific problem (research paper)] may be formulated theory and Maxwell Equation.

Classification: Integral Equation are classified according to three different types

(a) Limit of Integration

(b) Placement of unknown function

(c) Nature of unknown function

(a) Limit of Integration:

(i) both fixed (constant) = Fredholm Integral Equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

(ii) one variable = Volteera Integral Equation

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$

(b) Placement of unknown function:

(i) one side integral = First kind Volteera / Fredholm I.E

$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt$ (Fredholm Integral Equation of 1^{st} kin

$u(x) = \lambda \int_a^x K(x,t)\, u(t)\, dt$ (Volteera Integral Equation of 1^{st} kin

(ii) both inside / outside = Second kind Volteera / Fredholm I.E

$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt$ (Fredholm Integral Equation of 2^{nd} kind

$u(x) = f(x) + \lambda \int_a^x k(x,t)\, u(t)\, dt$ (Volteera Integral Equation of 2^{nd} kin

(c) Nature of unknown function:

(i) Identically zero: = homogeneous

$$f(x) = 0$$

So equation becomes,

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt$$ (Fredholm Integral Equation)

$$u(x) = \lambda \int_a^x K(x,t)\, u(t)\, dt$$ (Volterra Integral Equation)

(ii) not identically zero = non-homogeneous

$f(x) \neq 0$

So equation becomes,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$ (Fredholm Integral Equation)

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$ (Volterra Integral Equation)

- Types of Integral Equation.

An integro-differential equations is an equation in which unknown function $u(t)$ appear under integral sign and contain ordinary derivative

$$u'(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt$$

Derivate

Ordinary — [only one independent variable]

Partial — [more than one independent variable]

Singular Integral Equation

Volterra integral equation of 1st and 2nd kind

$$u(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt \quad \text{--- (1)}$$

or

In complex analysis

$\frac{e^z}{z-1}$, put $z=1$

$\frac{e}{1-1} \Rightarrow \frac{e}{0} = \infty$

$\Rightarrow 1$ is the singular point

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$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt \quad \text{—} ②$$

is called singular if the limits of integrations are infinite.

- Fredholm integral equation of 1^{st} and 2^{nd} kind

$$u(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt \quad \text{—} ③$$

or

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)\, u(t)\, dt \quad \text{—} ④$$

is called singular if limit of integrations are infinite.

(Exercise 2.1) Problems

Question # 1: For each of the following integral equation, classify as Fredholm, volteera, Volteera - fredholm, 1^{st} and 2^{nd} kind also classify the equation is singular or not.

(i) $u(t) = 1 + \int_0^x u(t)\, dt$

Sol: This is volteera integral equation, because of the limit of integration is not constant, it is 2^{nd} Kind because the unknown function appear inside/outside the integral, it is not homogeneous because of 1, it is not singular since neither limits of integration are infinite.

$f(x) = 1$
$\lambda = 1$
$K(x,t) = 1$
$g(x) = 0$
$h(x) = x$

Ans.