> ***Theorem***
>
> Assume $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.
>
> Or: An absolutely convergent integral is convergent.

Proof

If $x \geq a$, $\pm f(x) \leq |f(x)|$

$\Rightarrow |f(x)| - f(x) \geq 0$

$\Rightarrow 0 \leq |f(x)| - f(x) \leq 2|f(x)|$

$\Rightarrow \int_a^\infty (|f| - f)\,d\alpha$ converges.

Subtracting from $\int_a^\infty |f|\,d\alpha$ we find that $\int_a^\infty f\,d\alpha$ converges.

($\because$ Difference of two convergent integrals is convergent)

➢ ***Note***

$\int_a^\infty f\,d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\,d\alpha$ converges. It is said to be convergent conditionally if $\int_a^\infty f\,d\alpha$ converges but $\int_a^\infty |f|\,d\alpha$ diverges.

➢ ***Remark***

Every absolutely convergent integral is convergent.

> ***Question***

Show that $\int_0^{\infty} e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because\ \left|e^{-x}\cos x\right| < e^{-x}$ and $\int_0^{\infty} e^{-x}\,dx = 1$

$\therefore$ the given integral is absolutely convergent. (comparison test)

> ***Question***

Show that $\int_0^{1} \frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent.

Solution

$\because\ e^{-x} < 1$ and $1 + x^2 > 1$

$$\therefore\ \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

$$\text{Also } \int_0^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{\varepsilon \to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \int_0^{1} \frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent. (by comparison test)

Let $\phi(x)$ be bounded and monotonic in $[a,+\infty)$ and let $\phi(x)\to 0$, when $x\to\infty$. Also let $\int_a^X f(x)dx$ be bounded when $X\geq a$.

Then $\int_a^\infty f(x)\phi(x)dx$ is convergent.

➢ ***Example***

Consider $\int_0^\infty \frac{\sin x}{x}dx$

$\because \frac{\sin x}{x}\to 1$ as $x\to 0$.

$\therefore$ 0 is not a point of infinite discontinuity.

Now consider the improper integral $\int_1^\infty \frac{\sin x}{x} dx$.

The factor $\frac{1}{x}$ of the integrand is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\int_1^X \sin x\, dx\right| = \left|-\cos X + \cos(1)\right| \le \left|\cos X\right| + \left|\cos(1)\right| < 2$

So that $\int_1^X \sin x\, dx$ is bounded above for every $X \ge 1$.

$\Rightarrow \int_1^\infty \frac{\sin x}{x} dx$ is convergent. Now since $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral, we see that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

➢ ***Example***

Consider $\int_0^\infty \sin x^2\, dx$.

We write $\sin x^2 = \frac{1}{2x} \cdot 2x \cdot \sin x^2$

Now $\int_1^\infty \sin x^2\, dx = \int_1^\infty \frac{1}{2x} \cdot 2x \cdot \sin x^2\, dx$

$\frac{1}{2x}$ is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\int_1^X 2x \sin x^2\, dx\right| = \left|-\cos X^2 + \cos(1)\right| < 2$

So that $\int_1^X 2x \sin x^2\, dx$ is bounded for $X \ge 1$.

Hence $\int_1^\infty \frac{1}{2x} \cdot 2x \cdot \sin x^2\, dx$ i.e. $\int_1^\infty \sin x^2\, dx$ is convergent.

Since $\int_0^1 \sin x^2\, dx$ is only a proper integral, we see that the given integral is convergent.

➢ ***Example***

Consider $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$, $a > 0$

Here e^{-ax} is monotonic and bounded and $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

Hence $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is convergent.

➢ ***Question***

Show that $\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}} dx$ converges for $\alpha > 0$.

Solution

$\int_0^{\infty} \sin x \, dx$ is bounded because $\left| \int_0^{x} \sin x \, dx \right| \leq 2 \quad \forall \ x > 0$.

Furthermore the function $\frac{1}{(1+x)^{\alpha}}$, $\alpha > 0$ is monotonic on $[0, +\infty)$.

$\Rightarrow$ the integral $\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}} dx$ is convergent.