

IMPROPER INTEGRAL OF THE SECOND KIND

➤ **Definition**

Let f be defined on the half open interval $(a, b]$ and assume that $f \in R(\alpha; x, b)$ for every $x \in (a, b]$. Define a function I on $(a, b]$ as follows:

$$I(x) = \int_x^b f d\alpha \quad \text{if } x \in (a, b] \dots\dots\dots (i)$$

The function I so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a^+}^b f(t) d\alpha(t)$ or $\int_{a^+}^b f d\alpha$.

The integral $\int_{a^+}^b f d\alpha$ is said to converge if the limit

$$\lim_{x \rightarrow a^+} I(x) \dots\dots\dots (ii) \text{ exists (finite).}$$

Otherwise, $\int_{a^+}^b f d\alpha$ is said to diverge. If the limit in (ii) exists and equals A , the

number A is called the value of the integral and we write $\int_{a^+}^b f d\alpha = A$.

Similarly, if f is defined on $[a, b)$ and $f \in R(\alpha; a, x) \quad \forall x \in [a, b)$ then

$I(x) = \int_a^x f \, d\alpha$ if $x \in [a, b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f \, d\alpha$ and is convergent if $\lim_{x \rightarrow b-} I(x)$ exists (finite).

> Example

$f(x) = x^{-p}$ is defined on $(0, b]$ and $f \in R(x, b)$ for every $x \in (0, b]$.

$$\begin{aligned} I(x) &= \int_x^b x^{-p} \, dx \quad \text{if } x \in (0, b] \\ &= \int_{0+}^b x^{-p} \, dx = \lim_{\epsilon \rightarrow 0} \int_{0+\epsilon}^b x^{-p} \, dx \\ &= \lim_{\epsilon \rightarrow 0} \left[\frac{x^{1-p}}{1-p} \right]_{\epsilon}^b = \lim_{\epsilon \rightarrow 0} \frac{b^{1-p} - \epsilon^{1-p}}{1-p} \quad , \quad (p \neq 1) \\ &= \begin{cases} \text{finite} & , p < 1 \\ \text{infinite} & , p > 1 \end{cases} \end{aligned}$$

When $p = 1$, we get $\int_{\epsilon}^b \frac{1}{x} \, dx = \log b - \log \epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$.

$\Rightarrow \int_{0+}^b x^{-1} \, dx$ also diverges.

Hence the integral converges when $p < 1$ and diverges when $p \geq 1$.

> Note

If the two integrals $\int_{a+}^c f \, d\alpha$ and $\int_c^{b-} f \, d\alpha$ both converge, we write

$$\int_{a+}^{b-} f \, d\alpha = \int_{a+}^c f \, d\alpha + \int_c^{b-} f \, d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^b f \, d\alpha + \int_b^{\infty} f \, d\alpha \quad \text{which can be written as } \int_{a+}^{\infty} f \, d\alpha .$$

> Example

Consider $\int_{0+}^{\infty} e^{-x} x^{p-1} \, dx \quad , \quad (p > 0)$

This integral must be interpreted as a sum as

$$\begin{aligned} \int_{0+}^{\infty} e^{-x} x^{p-1} \, dx &= \int_{0+}^1 e^{-x} x^{p-1} \, dx + \int_1^{\infty} e^{-x} x^{p-1} \, dx \\ &= I_1 + I_2 \dots \dots \dots (i) \end{aligned}$$

I_2 , the second integral, converges for every real p as proved earlier.

To test I_1 , put $t = \frac{1}{x} \Rightarrow dx = -\frac{1}{t^2} dt$

$$\Rightarrow I_1 = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 e^{-x} x^{p-1} dx = \lim_{\epsilon \rightarrow 0} \int_{\frac{1}{\epsilon}}^1 e^{-\frac{1}{t}} t^{1-p} \left(-\frac{1}{t^2} dt \right) = \lim_{\epsilon \rightarrow 0} \int_1^{\frac{1}{\epsilon}} e^{-\frac{1}{t}} t^{-p-1} dt$$

Take $f(t) = e^{-\frac{1}{t}} t^{-p-1}$ and $g(t) = t^{-p-1}$

Then $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = \lim_{t \rightarrow \infty} \frac{e^{-\frac{1}{t}} t^{-p-1}}{t^{-p-1}} = 1$ and since $\int_1^{\infty} t^{-p-1} dt$ converges when $p > 0$

$\therefore \int_1^{\infty} e^{-\frac{1}{t}} t^{-p-1} dt$ converges when $p > 0$

Thus $\int_{0^+}^{\infty} e^{-x} x^{p-1} dx$ converges when $p > 0$.

When $p > 0$, the value of the sum in (i) is denoted by $\Gamma(p)$. The function so defined is called the Gamma function.

> Note

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

> A Useful Comparison Integral

$$\int_a^b \frac{dx}{(x-a)^n}$$

We have, if $n \neq 1$,

$$\begin{aligned} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n} &= \left| \frac{1}{(1-n)(x-a)^{n-1}} \right|_{a+\epsilon}^b \\ &= \frac{1}{(1-n)} \left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\epsilon^{n-1}} \right) \end{aligned}$$

Which tends to $\frac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n < 1$ or $n > 1$, as $\epsilon \rightarrow 0$.

Again, if $n = 1$,

$$\int_{a+\epsilon}^b \frac{dx}{x-a} = \log(b-a) - \log \epsilon \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Hence the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges iff $n < 1$.

> Question

Examine the convergence of

$$(i) \int_0^1 \frac{dx}{x^{1/3}(1+x^2)} \quad (ii) \int_0^1 \frac{dx}{x^2(1+x)^2} \quad (iii) \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

Solution

$$(i) \int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$$

Here '0' is the only point of infinite discontinuity of the integrand.

We have

$$f(x) = \frac{1}{x^{1/3}(1+x^2)}$$

$$\text{Take } g(x) = \frac{1}{x^{1/3}}$$

$$\text{Then } \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{1}{1+x^2} = 1$$

$\Rightarrow \int_0^1 f(x) dx$ and $\int_0^1 g(x) dx$ have identical behaviours.

$\therefore \int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ also converges.

$$(ii) \int_0^1 \frac{dx}{x^2(1+x)^2}$$

Here '0' is the only point of infinite discontinuity of the given integrand.

We have

$$f(x) = \frac{1}{x^2(1+x)^2}$$

$$\text{Take } g(x) = \frac{1}{x^2}$$

$$\text{Then } \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{1}{(1+x)^2} = 1$$

$\Rightarrow \int_0^1 f(x) dx$ and $\int_0^1 g(x) dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x) dx$ does not converge. Hence the given integral also does not converge.

$$(iii) \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.

We have

$$f(x) = \frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $\frac{1}{2}$, and examine the convergence of

the improper integrals $\int_0^{1/2} f(x) dx$ and $\int_{1/2}^1 f(x) dx$.

To examine the convergence of $\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}} dx$, we take $g(x) = \frac{1}{x^{1/2}}$

Then

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{1}{(1-x)^{1/3}} = 1$$

$\therefore \int_0^{1/2} \frac{1}{x^{1/2}} dx$ converges $\therefore \int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}} dx$ is convergent.

To examine the convergence of $\int_{1/2}^1 \frac{1}{x^{1/2}(1-x)^{1/3}} dx$, we take $g(x) = \frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1} \frac{1}{x^{1/2}} = 1$$

$\therefore \int_{1/2}^1 \frac{1}{(1-x)^{1/3}} dx$ converges $\therefore \int_{1/2}^1 \frac{1}{x^{1/2}(1-x)^{1/3}} dx$ is convergent.

Hence $\int_0^1 f(x) dx$ converges.

> Question

Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m \geq 1$ and $n \geq 1$.

The number '0' is a point of infinite discontinuity if $m < 1$ and the number '1' is a point of infinite discontinuity if $n < 1$.

Let $m < 1$ and $n < 1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of

the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1} dx$ and $\int_{1/2}^1 x^{m-1}(1-x)^{n-1} dx$ at '0' and '1'

respectively.

Convergence at 0:

We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take } g(x) = \frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)} \rightarrow 1$ as $x \rightarrow 0$

As $\int_0^{1/2} \frac{1}{x^{1-m}} dx$ is convergent at 0 iff $1-m < 1$ i.e. $m > 0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1} dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x) = \frac{1}{(1-x)^{1-n}}$

Then $\frac{f(x)}{g(x)} \rightarrow 1$ as $x \rightarrow 1$

As $\int_{\frac{1}{2}}^1 \frac{1}{(1-x)^{1-n}} dx$ is convergent, iff $1-n < 1$ i.e. $n > 0$.

We deduce that the integral $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1} dx$ converges iff $n > 0$.

Thus $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists for positive values of m, n only.

It is a function which depends upon m & n and is defined for all positive values of m & n . It is called Beta function.