> **Theorem: (Comparison Test)**
> Assume that $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \geq a$, if $0 \leq f(x) \leq g(x)$ for every $x \geq a$, and if $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have
> $$\int_a^\infty f\,d\alpha \leq \int_a^\infty g\,d\alpha$$

Proof

Let $I_1(b) = \int_a^b f\,d\alpha$ and $I_2(b) = \int_a^b g\,d\alpha$, $b \geq a$

$\because$ $0 \leq f(x) \leq g(x)$ for every $x \geq a$

$\therefore$ $I_1(b) \leq I_2(b)$ .... (*i*)

$\because$ $\int_a^\infty g\,d\alpha$ converges $\therefore$ $\exists$ a constant $M > 0$ such that

$\int_a^\infty g\,d\alpha \leq M$, $b \geq a$ ....(*ii*)

From (*i*) and (*ii*) we have $I_1(b) \leq M$, $b \geq a$.

$\Rightarrow$ $\lim_{b\to+\infty} I_1(b)$ exists and is finite.

$\Rightarrow$ $\int_a^\infty f\,d\alpha$ converges.

Also $\lim_{b\to+\infty} I_1(b) \leq \lim_{b\to+\infty} I_2(b) \leq M$

$\Rightarrow$ $\int_a^\infty f\,d\alpha \leq \int_a^\infty g\,d\alpha$.

> ***Theorem (Limit Comparison Test)***
>
> Assume that $\alpha \uparrow$ on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \geq a$, where $f(x) \geq 0$ and $g(x) \geq 0$ if $x \geq a$. If
>
> $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$
>
> then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \geq a$, we can find some $N > 0$ such that

$$\left|\frac{f(x)}{g(x)}-1\right|<\varepsilon \qquad \forall\ x \geq N \quad \text{for every} \quad \varepsilon > 0.$$

$$\Rightarrow\ 1-\varepsilon<\frac{f(x)}{g(x)}<1+\varepsilon$$

Let $\varepsilon=\frac{1}{2}$, then we have

$$\frac{1}{2}<\frac{f(x)}{g(x)}<\frac{3}{2}$$

$\Rightarrow\ g(x)<2f(x)$ ....*(i)* and $2f(x)<3g(x)$ ....*(ii)*

From (i) $\int_a^\infty g\, d\alpha < 2\int_a^\infty f\, d\alpha$

$\Rightarrow \int_a^\infty g\, d\alpha$ converges if $\int_a^\infty f\, d\alpha$ converges and $\int_a^\infty f\, d\alpha$ diverges if $\int_a^\infty f\, d\alpha$ diverges.

From (ii) $2\int_a^\infty f\, d\alpha < 3\int_a^\infty g\, d\alpha$

$\Rightarrow \int_a^\infty f\, d\alpha$ converges if $\int_a^\infty g\, d\alpha$ converges and $\int_a^\infty g\, d\alpha$ diverges if $\int_a^\infty f\, d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^\infty f\, d\alpha$ and $\int_a^\infty g\, d\alpha$ converge or diverge together.

➢ ***Note***

The above theorem also holds if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g\, d\alpha$ implies convergence of $\int_a^\infty f\, d\alpha$.

➢ ***Example***

For every real p, the integral $\int_1^\infty e^{-x} x^p\, dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2} dx$.

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{-x} x^p}{1/x^2}$ where $f(x) = e^{-x} x^p$ and $g(x) = \frac{1}{x^2}$.

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-x} x^{p+2} = \lim_{x\to\infty} \frac{x^{p+2}}{e^x} = 0$$

and $\because \int_1^\infty \frac{1}{x^2} dx$ is convergent

$\therefore$ the given integral $\int_1^\infty e^{-x} x^p\, dx$ is also convergent.

> **Theorem**
>
> Let f be a positive decreasing function defined on $[a,+\infty)$ such that $f(x)\to 0$ as $x\to+\infty$. Let α be bounded on $[a,+\infty)$ and assume that $f\in R(\alpha;a,b)$ for every $b\geq a$. Then the integral $\int_a^\infty f\,d\alpha$ is convergent.

Proof

Integration by parts gives

$$\int_a^b f\,d\alpha = \left|f(x)\cdot\alpha(x)\right|_a^b - \int_a^b \alpha(x)\,df$$

$$= f(b)\cdot\alpha(b) - f(a)\cdot\alpha(a) + \int_a^b \alpha\,d(-f)$$

It is obvious that $f(b)\alpha(b)\to 0$ as $b\to+\infty$

($\because$ α is bounded and $f(x)\to 0$ as $x\to+\infty$)

and $f(a)\alpha(a)$ is finite.

$\therefore$ the convergence of $\int_a^b f\,d\alpha$ depends upon the convergence of $\int_a^b \alpha\,d(-f)$.

Actually, this integral converges absolutely. To see this, suppose $|\alpha(x)|\leq M$ for all $x\geq a$ ($\because$ $\alpha(x)$ is given to be bounded)

$$\Rightarrow \int_a^b |\alpha(x)|\,d(-f) \leq \int_a^b M\,d(-f)$$

But $\int_a^b M\,d(-f) = M\left|-f\right|_a^b = M\,f(a) - M\,f(b)\to M\,f(a)$ as $b\to\infty$.

$$\Rightarrow \int_a^\infty M\,d(-f) \text{ is convergent.}$$

$\because$ $-f$ is an increasing function.

$$\therefore \int_a^\infty |\alpha|\,d(-f) \text{ is convergent.} \quad \text{(Comparison Test)}$$

$$\Rightarrow \int_a^\infty f\,d\alpha \text{ is convergent.}$$

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➢ *Questions*

Examine the convergence of

(*i*) $\int_1^\infty \frac{x}{(1+x)^3}dx$ (*ii*) $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ (*iii*) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution

(*i*) Let $f(x)=\frac{x}{(1+x)^3}$ and take $g(x)=\frac{x}{x^3}=\frac{1}{x^2}$

As $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^3}{(1+x)^3}=1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ have identical behaviour for convergence at ∞.

$\because \int_1^\infty \frac{1}{x^2}dx$ is convergent $\therefore \int_1^\infty \frac{x}{(1+x)^3}dx$ is convergent.

(*ii*) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent.

(*iii*) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$ and $\int_1^\infty \frac{1}{x^{5/6}}dx$ is convergent $\therefore \int_1^\infty f(x)dx$ is convergent.