> ***Definition***
>
> The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

> ***Definition***
>
> The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a \le x \le b$. Such points are called singularities of $f(x)$.

Notations

We shall denote the set of all functions f such that $f \in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha \uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

> ***Definition***
>
> Assume that $f \in R(\alpha;a,b)$ for every $b \ge a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:
>
> $$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b \ge a \;\ldots\ldots\ldots\ldots (i)$$
>
> The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.
>
> The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit
>
> $$\lim_{b\to\infty} I(b) \;\ldots\ldots\ldots\ldots (ii)$$
>
> exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.
>
> If the limit in *(ii)* exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

Example

Consider $\int_1^b x^{-p}\,dx$.

$\int_1^b x^{-p}\,dx=\dfrac{\left(1-b^{1-p}\right)}{p-1}$ if $p \ne 1$, the integral $\int_1^\infty x^{-p}\,dx$ diverges if $p<1$. When $p>1$, it converges and has the value $\dfrac{1}{p-1}$.

If $p=1$, we get $\int_1^b x^{-1}\,dx=\log b \to \infty$ as $b \to \infty$. $\Rightarrow \int_1^\infty x^{-1}\,dx$ diverges.

➢ Example

Consider $\int_0^b \sin 2\pi x\, dx$

$$\because \int_0^b \sin 2\pi x\, dx = \frac{(1-\cos 2\pi b)}{2\pi} \not\to \infty \quad \text{as } b \to \infty,$$

$\therefore$ the integral $\int_0^\infty \sin 2\pi x\, dx$ diverges.

➢ Note

If $\int_{-\infty}^{a} f\, d\alpha$ and $\int_{a}^{\infty} f\, d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\, d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\, d\alpha = \int_{-\infty}^{a} f\, d\alpha + \int_{a}^{\infty} f\, d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\, d\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty} \int_{-b}^{b} f\, d\alpha$.

➢ Theorem

Assume that $\alpha \uparrow$ on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^{\infty} f\, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f\, d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

We have $I(b) = \int_a^b f(x)\, d\alpha(x), \quad b \ge a$

$\Rightarrow I \uparrow$ on $[a,+\infty)$

Then $\lim_{b\to+\infty} I(b) = \sup\{I(b) \mid b \ge a\} = M > 0$ and the theorem follows

$\Rightarrow \int_a^b f\, d\alpha \le M$ for every $b \ge a$ whenever the integral converges.

> ***Question***

Show that $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ is convergent.

Solution

We have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim_{a\to\infty}\left[\int_{-a}^{0} \frac{1}{1+x^2}dx + \int_{0}^{a} \frac{1}{1+x^2}dx\right]$$

$$= \lim_{a\to\infty}\left[\int_{0}^{a} \frac{1}{1+x^2}dx + \int_{0}^{a} \frac{1}{1+x^2}dx\right] = 2\lim_{a\to\infty}\left[\int_{0}^{a} \frac{1}{1+x^2}dx\right]$$

$$= 2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a = 2\left(\frac{\pi}{2}\right) = \pi$$

therefore the integral is convergent.

> ***Question***

Show that $\int_{0}^{\infty} \frac{\tan^{-1}x}{1+x^2}dx$ is convergent.