REAL ANALYSIS

USEFUL FOR

CSIR UGC NET, GATE, IIT-JAM, NBHM, TIFR & other exams with similar syllabus

First Edition

By

Dr A.P. SINGH

Ph.D. (Mathematics), Indian Young Scientist Awardee
M.Sc. (Gold Medalist) P.U., B.Sc. (Gold Medalist) P.U.

INFOSTUDY PUBLICATIONS
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(a) φ is continuous on $\mathbb{R}$ but not differentiable only at $x=1$, 3/2 and 2.

(b) φ is continuous on $\mathbb{R}$ but not differentiable only at $x=1$.

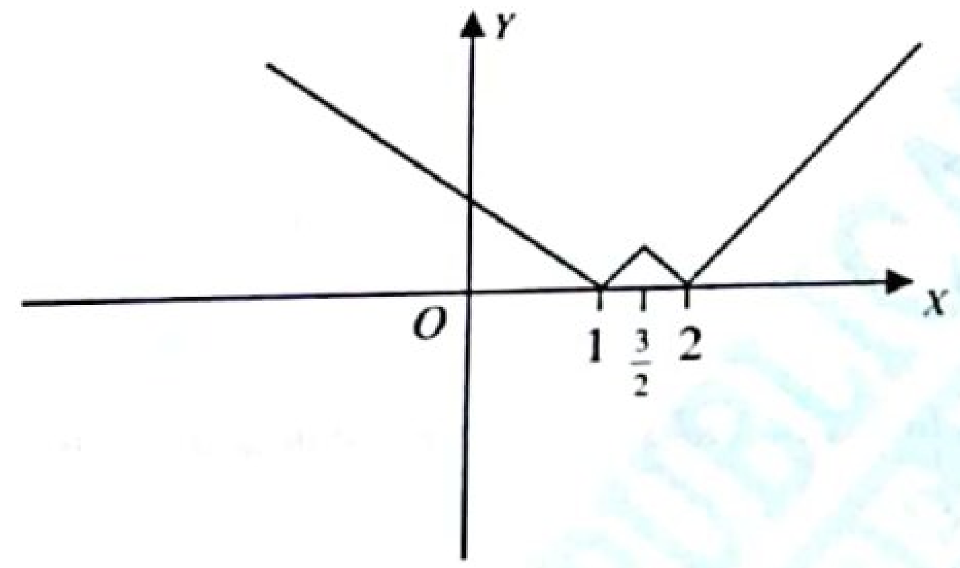
(c) φ is continuous on $\mathbb{R}$ but not differentiable only at $x=1$ and 2.

(d) φ is discontinuous somewhere on $\mathbb{R}$.

Solution: (a)

We have $I=\{1\}\cup\{2\}\subseteq\mathbb{R}$ and $\phi(x)=dist(x,I)$

$=\inf\ \{|x-y|: y\in I\}$



Graph of the function $\phi(x)$ is given below

From, graph it is clear that $\phi(x)$ has corner points at $x = 1, \frac{3}{2}$ and 2.

Thus, $\phi(x)$ is continuous on $\mathbb{R}$ but not differentiable at $x = 1, \frac{3}{2}$ and 2.

[$\because$ A function is not differentiable at its corner points].

Example 13. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases}x^2 & \text{if } x<0,\\ 2x+x^2 & \text{if } x\geq 0\end{cases}$

Then which of the following statements are correct? (CSIR UGC NET DEC-2011)

(a) $f''(x)=2$ for all $x\in\mathbb{R}$ (b) $f''(0)$ does not exist

(c) $f'(x)$ exists for each $x\neq 0$ (d) $f'(0)$ does not exist

Solution: (b,c,d) Now, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{2h+h^2-0}{h}=2$ and $\lim_{h\to 0}\frac{f(0)-f(0-h)}{h}=\frac{0-h^2}{h}=0$

As L.H.D. and R.H.D. are not equal. $\therefore f'(0)$ does not exist

$\Rightarrow f''(0)$ does not exist

Clearly, $f'(x)$ exists $\forall x\neq 0$

Thus, option (b), (c), (d) are correct.

Example 14. Consider the map $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x,y)=(7x+x^4, 3x+4y+y^4)$. Then (CSIR UGC NET JUNE-2012)

(a) f is discontinuous at (0,0).

(b) f is continuous at (0,0) and all directional derivatives exist at (0,0).

(c) f is differentiable at (0,0) but the derivative $Df(0,0)$ is not invertible.

(d) f is differentiable at (0,0) and the derivative $Df(0,0)$ is invertible.

Solution: (b,d) Given, $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x,y) = (7x + x^4, 3x + 4y + y^4)$

Here, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}(3x-2y+x^2, 4x+5y+y^2)$

$= (0,0) = f(0,0)$

Hence, f is continuous at (0,0)

$\therefore$ option (a) is incorrect

Further, let $u = (a_1, a_2)$ be the direction $\&\bar{c}=(0,0)$

Then, $\lim_{h\to 0}\frac{f(\bar{c}+hu)-f(\bar{c})}{h}=\lim_{h\to 0}\frac{f((0,0)+h(a_1,a_2))-f(0,0)}{h}=\lim_{h\to 0}\frac{f(ha_1,ha_2)-f(0,0)}{h}$

$=\lim_{h\to 0}\frac{(7ha_1+h^4a_1^4, 3ha_1+4ha_2+h^4a_2^4)}{h}=\lim_{h\to 0}(7a_1+h^3a_1^4, 3a_1+4a_2+h^3a_2^4$

$= (7a_1, 3a_1 + 4a_2)$ which exists $\forall\ a_1, a_2$

Hence, all directional derivative exists at (0,0)

$\therefore$ option (b) is correct.

Next, we have $f(x,y) = (7x + x^4, 3x + 4y + y^4)$

$\therefore f_1(x,y) = 7x + x^4, f_2(x,y) = 3x + 4y + y^4$

$\therefore Df(x,y)=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}=\begin{vmatrix}7+4x^3 & 0\\ 3 & 4+4y^4\end{vmatrix}$

$\therefore Df(0,0)=\begin{vmatrix}7 & 0\\ 3 & 4\end{vmatrix}=28-0=28\neq 0$

Hence, f is differentiable at (0,0) and $Df(0,0)$ is invertible

Thus, option (c) is incorrect and option (d) is correct.

Example 15. Consider the function $f(x)=\cos(|x-5|)+\sin(|x-3|)+|x+10|^3-(|x|+4)^2$.

At which of the following points is f not differentiable? (CSIR UGC NET JUNE-2012)

(a) $x=5$ (b) $x=3$ (c) $x=-10$ (d) $x=0$

Solution: (b,d) Given $f(x)= \cos(|x-5|)+\sin(|x-3|)+|x+10|^3-(|x|+4)^2$

let, $f_1(x) = \cos(|x-5|)$

$$=\begin{cases}\cos(-(x-5)), & x<5\\ 1, & x=5\\ \cos(x-5), & x>5\end{cases}$$

$$=\begin{cases}\cos(x-5), & x\neq 5\\ 1, & x=5\end{cases}\qquad [\because \cos(-x)=\cos x]$$

$$\therefore L.H.D=\lim_{x\to5^-}\frac{f_1(x)-f_1(5)}{x-5}=\lim_{x\to5^-}\frac{\cos(x-5)-1}{x-5}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to5^-}\{-\sin(x-5)\}=0$$

$$\therefore R.H.D=\lim_{x\to5^+}\frac{f_1(x)-f_1(5)}{x-5}=\lim_{x\to5^+}\frac{\cos(x-5)-1}{x-5}=0$$

$\Rightarrow$ L.H.D = R.H.D

$\therefore$ $f_1(x) = \cos(|x-5|)$ *is differentiable at* $x = 5$ *and hence* $f(x)$ *is differentiable at* $x = 5$

$\therefore$ *option (a) is incorrect*

Next, let $f_2(x) = \sin(|x-3|)$

$$=\begin{cases}\sin(-(x-3)), & x<3\\ 0, & x=3\\ \sin(x-3), & x>3\end{cases}$$

$$\therefore \lim_{x\to3^-}\frac{f_2(x)-f_2(3)}{x-3}=\lim_{x\to3}\frac{\sin(x-3)-0}{x-3}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to3^-}\{-\cos(x-3)\}=-1 \text{ and } \lim_{x\to3^+}\frac{f_2(x)-f_2(3)}{x-3}=\lim_{x\to3}\frac{-\sin(x-3)-0}{x-3}=1$$

Since, L.H.D ≠ R.H.D

$\therefore$ $f_2(x) = \sin(|x-3|$ *is not differentiable at* $x = 3$

Hence $f(x)$ *is not differentiable at* $x = 3$

$\therefore$ *option (b) is correct*

Further, let $f_3(x) = |x + 10|^3$

$$=\begin{cases}-(x+10)^3, & x<-10\\ 0, & x=-10\\ (x+10)^3, & x>-10\end{cases}$$

$$L.H.D.=\lim_{x\to(-10)^-}\frac{f_3(x)-f_3(-10)}{x+10}=\lim_{x\to-10}\frac{-(x+10)^3-0}{x+10}=\lim_{x\to-10}-(x+10)^2=0$$

$$R.H.D.=\lim_{x\to(-10)^+}\frac{f_3(x)-f_3(-10)}{x+10}=\lim_{x\to-10}\frac{(x+10)^3-0}{x+10}=\lim_{x\to-10}(x+10)^2=0$$

L.H.D. = R.H.D

$\therefore f_3(x)$ *is differentiable at* $x = -10$

Hence, $f(x)$ *is differentiable at* $x = -10$

$\therefore$ *option (c) is incorrect*

Next, let $f_4(x) = (|x| + 4)^2$

$$=\begin{cases}(-x+4)^2, & x<0\\ 16, & x=0\\ (x+4)^2, & x>0\end{cases}$$

$$\therefore L.H.D=\lim_{x\to0^-}\frac{f_4(x)-f_4(0)}{x-0}=\lim_{x\to0}\frac{(-x+4)^2-16}{x}\left(\frac{0}{0}\right)$$

$$\lim_{x\to0}\frac{2(-x+4)(-1)}{1}=-8$$

$$R.H.D=\lim_{x\to0^+}\frac{f_4(x)-f_4(0)}{x-0}\lim_{x\to0}\frac{(x+4)^2-16}{x}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{2(x+4)}{1}=8$$

L.H.D. ≠ R.H.D.

$\Rightarrow f_4(x)$ *is not differentiable at* $x = 0$

Thus option (d) is correct

Example 16. *Let f be a twice differentiable function on* $\mathbb{R}$. *Given that* $f''(x) > 0$ *for all* $x\in\mathbb{R}$, *then*

(CSIR UGC NET DEC-2012)

(a) $f(x) = 0$ *has exactly two solutions on* $\mathbb{R}$.

(b) $f(x) = 0$ *has a positive solution if* $f(0) = 0$ *and* $f'(0) = 0$.

(c) $f(x) = 0$ *has no positive solution if* $f(0) = 0$ *and* $f'(0) > 0$.

(d) $f(x) = 0$ *has no positive solution if* $f(0) = 0$ *and* $f'(0) < 0$.

Solution: (c)

Given $f''(x)>0,\ \forall x\in\mathbb{R}$

(i) consider $f(x)=x^2+1=0$ *such that* $f''(x)=2>0\ \forall x\in\mathbb{R}$ *but* $f(x)$ *has no solution in* $\mathbb{R}$

Thus option (a) is incorrect.

(ii) *consider* $f(x) = x^2$

Clearly, $f(0) = 0, f'(0)= 0$ *but* $f(x)$ *has no positive solution*

$\therefore$ *option (b) is incorrect*

(iii) consider $f(x)=x^2-x\ ,\ f'(x)=2x-1$

$f(0)=0,\ f'(0)<0,\ f''(x)>0\ \forall\ x\in\mathbb{R}$

but f(x)=0 has positive solution

Thus option (d) is incorrect.

As all other options ruled out, hence option (c) is correct.

Example 17. Let $f: A \cup E \to \mathbb{R}^2$ be differentiable, where $A = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1\}$ and

$E = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-2)^2 < \frac{1}{2}\}$. Let Df be the derivative of the function f. Which of the following are necessarily correct ? **(CSIR UGC NET DEC-2012)**

(a) If $(Df)(x,y) = 0$ for all $(x,y) \in A \cup E$, then f is constant.

(b) If $(Df)(x,y) = 0$ for all $(x,y) \in A$, then f is constant on A.

(c) If $(Df)(x,y) = 0$ for all $(x,y) \in E$, then f is constant on E.

(d) $(Df)(x,y)=0$ for all $(x,y) \in A \cup E$, then for some $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$, $f(x,y)=(x_0,y_0)$ for all $(x,y) \in A$ and $f(x,y) = (x_1, y_1)$ for all $(x,y) \in E$.

Solution: (b,c,d)

We know that if $Df(x,y) = 0 \quad \forall (x,y) \in S$

Then f is constant, provided S is connected

or

The possibilities of f are either $f(x,y) = \{c \ \forall \ (x,y) \in A \cup E\}$ or $f(x,y) = \begin{cases} c_1, & \forall x \in A \\ c_2, & \forall x \in E \end{cases}$

Thus, options (b), (c), (d) are correct and (a) is incorrect.

Example 18. The function $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$ is differentiable at $x=0$ **(CSIR UGC NET JUNE-2013)**

(a) for no values of a_0, a_1, a_2, a_3

(b) for any value of a_0, a_1, a_2, a_3

(c) only if $a_1=0$

(d) only if both $a_1=0$ and $a_3=0$

Solution: (c)

Option (a) is not correct as take $a_0 = a_1 = a_2 = a_3 = 0$, then

$f(x) = 0 + 0(x) + 0(x)^2 + 0(x)^3 = 0$ is differentiable at $x=0$

Option (b) is not correct as for $a_0 = a_3 = a_2 = 0$ and $a_1=1$

$f(x)=|x|$, which is not differentiable at $x=0$

Option (d) is not correct as for $a_0 = a_1, a_2 = 0, a_3 = 0$ $f(x) = |x|^3$ is differentiable at $x=0$

As options (a), (b), (d) are incorrect.

$\therefore$ option (c) is correct.

Example 19: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function with period $p>0$. Then $g(x) = \int_x^{x+p} f(t)dt$ is a **(CSIR UGC NET JUNE-2013)**

(a) constant function

(b) continuous function

(c) continuous function but not differentiable

(d) neither continuous nor differentiable

Solution: (a,b)

Option (a) is correct as $g(x) = \int_x^{x+p} f(t)dt = \int_0^{x+p} f(t)dt - \int_0^x f(t)dt$

$\Rightarrow g'(x) = f(x+p) - f(x) = 0$

$\Rightarrow g$ is constant

'g' is continuous and differentiable

Thus, options (a) and (b) are correct and (c) and (d) are incorrect.

Example 20: If $f:[0,1] \to (0,1)$ is a continuous mapping, then which of the following is NOT true? **(CSIR UGC NET DEC-2013)**

(a) $F \subseteq [0,1]$ is a closed set implies $f(F)$ is closed in $\mathbb{R}$.

(b) If $f(0)<f(1)$, then $f([0,1])$ must be equal to $[f(0),f(1)]$.

(c) There must exist $x \in (0,1)$ such that $f(x)=x$.

(d) $f([0,1]) \neq (0,1)$

Solution: (b)

For option (b) Since $F \subseteq [0,1]$ is closed.

Also F is bounded, as [0, 1] is bounded

$\Rightarrow$ F is compact.

Every continuous function map compact set to compact set $\Rightarrow f(F)$ is compact set $\Rightarrow f(F)$ is closed in $\mathbb{R}$.

$\Rightarrow$ Option (b) is true

For option (c)

As $f: [0,1] \to (0,1)$ is continuous

$\therefore$ By fixed point theorem, $\exists$ some $x \in (0,1)$ such that $f(x) = x$

Thus, option (c) is true

For option (d),

Since [0, 1] is compact and continuous image of a compact set is compact

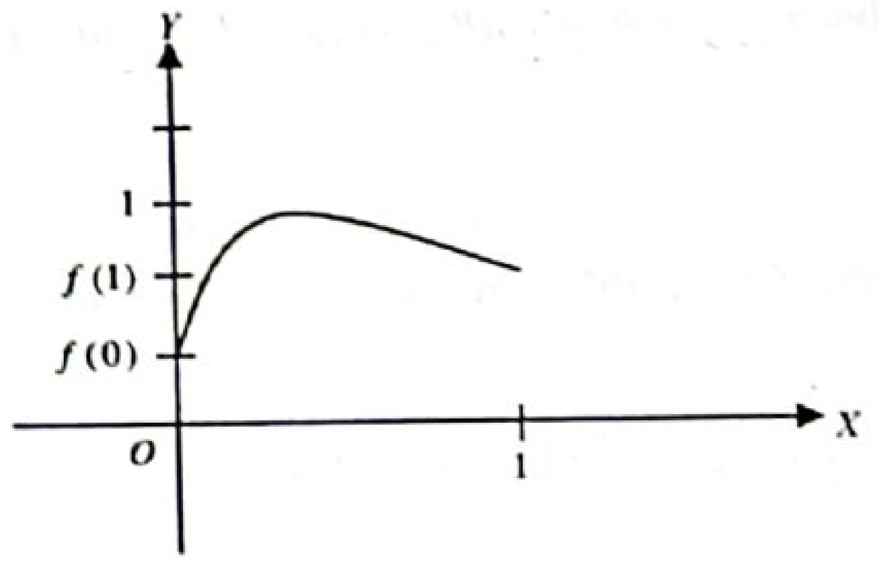
$\therefore f([0,1]) \neq (0,1)$ [$\because$ (0, 1) is not compact)

Thus, option (d) is true

Hence, as options (a), (b), (d) ruled out

$\therefore$ option (b) is correct

Also, we can understand it graphically



From graph it is clear that $f: [0, 1] \to (0, 1)$ is continuous and $f(0) < f(1)$ but $f([0, 1]) \not\subseteq [f(0), f(1)]$

Example 21. For a continuous function $f: \mathbb{R} \to \mathbb{R}$ let $Z(f) = \{x \in \mathbb{R}: f(x) = 0\}$. Then $Z(f)$ is always

(CSIR UGC NET JUNE-2014)

(a) compact (b) open (c) connected (d) closed

Solution: (d)

Given that, $f: \mathbb{R} \to \mathbb{R}$ is a continuous function and $Z(f) = \{x \in \mathbb{R} : f(x) = 0\} = f^{-1}(0)$

Since $f: \mathbb{R} \to \mathbb{R}$ is continuous so for any closed subset E of $\mathbb{R}$, its inverse image $f^{-1}(E)$ is also closed in $\mathbb{R}$.

Here, {0} is closed in $\mathbb{R}$. So $Z(f) = f^{-1}(0)$ is closed in $\mathbb{R}$.

Thus, option (d) is correct.

Further, take $f(x) = x - x^2$

$\therefore Z(f) = \{0, 1\}$, which is neither open nor connected

$\therefore$ options (b) and (c) are incorrect

Option (a) is also incorrect

As, take $f(x) = 0 \ \forall x \in \mathbb{R}$

$\therefore Z(f) = \mathbb{R}$, which is not compact.

Example 22. Let $f: [0, \infty) \to [0, \infty)$ be a continuous function. Which of the following is correct?

(a) There is $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$

(b) If $f(x) \le M$ for all $x \in [0, \infty)$ for some $M > 0$, then there exists $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$

(c) If f has a fixed point, then it must be unique

(d) f does not have a fixed point unless it is differentiable on $(0, \infty)$

Solution: For option (a), Clearly $\nexists$ any $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$

$\therefore$ option (a) is incorrect

For option (c),

Take, $f(x) = x$

Clearly, $f(x)$ is continuous but $f(x)$ have two fixed points 0 and 1.

$\therefore$ option (c) is incorrect

For option (d),

Take, $f(x) = \begin{cases} x, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

Here $f(x)$ is continuous and $x = 1$ is fixed point

But $f(x)$ is not differentiable

Thus option (d) is incorrect

As options (a), (c), (d) ruled out

$\therefore$ option (b) is incorrect.

Example 23. Let $f: \mathbb{R} \to \mathbb{R}$ be a function with continuous derivative such that $f(\sqrt{2}) = 2$ and

$$f(x) = \lim_{t \to 0} \frac{1}{2t} \int_{x-t}^{x+t} s\ f'(s)ds \quad \text{for all } x \in \mathbb{R}.$$

(JAM-2014)

Then $f(3)$ equals

(a) $\sqrt{3}$ (b) $3\sqrt{2}$ (c) $3\sqrt{3}$ (d) 9

Solution: (b)

Given $f(\sqrt{2}) = 2$ and

$$f(x) = \lim_{t \to 0} \frac{1}{2}[(x+t)f'(x+t) - (x-t)f'(x-t)(-1)]$$

[Applying L' Hospital's Rule]

$$\left[\because \frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,t)\,dt = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x} f(x,t)\,dt + f(x, \beta(x))\frac{d\beta(x)}{dx} - f(x, \alpha(x))\frac{d\alpha(x)}{dx}\right]$$

$$\Rightarrow f(x) = \lim_{t \to 0} \frac{1}{2}[x(f'(x+t) + f'(x-t)) + t(f'(x+t) - f'(x-t))]$$

$$\Rightarrow f(x) = \frac{1}{2} \cdot 2xf'(x) \Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x}$$

On integrating, we get $\log f(x) = \log x + \log c \Rightarrow f(x) = cx$

As $f(\sqrt{2}) = 2 \Rightarrow 2 = c\sqrt{2}$

$\Rightarrow c = \sqrt{2} \Rightarrow f(x) = \sqrt{2}x$

Hence, $f(3) = 3\sqrt{2}$.

ASSIGNMENT 3.1

NOTE: CHOOSE THE BEST OPTION

1. If a function f is continuous in a closed interval $[a, b]$, then it attains
(a) supremum only
(b) infimum only
(c) both supremum and infimum
(d) none of these

2. A function f is increasing function on $E \subset \mathbb{R}$ if $x, y \in E$
(a) $x < y \Rightarrow f(x) \le f(y)$
(b) $x < y \Rightarrow f(y) \le f(x)$
(c) $x < y \Rightarrow f(x) = f(y)$
(d) None of these

3. A function f is strictly decreasing on $E \subset \mathbb{R}$, if $x, y \in E$
(a) $x < y \Rightarrow f(x) < f(y)$
(b) $x < y \Rightarrow f(y) < f(x)$
(c) $x < y \Rightarrow f(y) \le f(x)$
(d) $x < y \Rightarrow f(y) \ge f(x)$

4. If f is an increasing function, then
(a) $-f$ is decreasing function
(b) $-f$ is increasing function
(c) f is constant
(d) none of these

5. If f is an increasing function on closed interval $[a, b]$, then for some $c \in (a, b)$ we have
(a) $f(c^-) \ge f(c) \ge f(c^+)$
(b) $f(c^-) \le f(c) \le f(c^+)$
(c) $f(c^-) = f(c) = f(c^+)$
(d) none of these

6. A function $f(x)$ has no jump discontinuity at $x = a$ if
(a) $f(a^+) = f(a^-) = f(a)$
(b) $f(a^+) \ne f(a^-)$
(c) $f(a^+) \ne f(a)$
(d) $f(a^+) = f(a^-) \ne f(a)$

7. A real valued function $f(x)$ has discontinuity of first kind at $x = a$, if
(a) $f(a^+)$ and $f(a^-)$ does not exist
(b) $f(a^+) = f(a^-)$
(c) $f(a^+) \ne f(a^-)$
(d) $f(a^+) = f(a^-) \ne f(a)$

8. Every absolutely continuous function is
(a) continuous
(b) constant
(c) absolutely discontinuous
(d) none of these

9. If f is decreasing function on closed interval, then for some $c \in (a, b)$
(a) $f(c^+)$ and $f(c^-)$ does not exist
(b) $f(c^+) \le f(c^-)$ both exist

(c) $f(c^-) \ge f(c)$
(d) None of these

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points. Then
(a) the total derivative of f exists at all points of $\mathbb{R}^2$
(b) f is continuous on $\mathbb{R}^2$
(c) the function $f(x,y)$ as a function of x for every fixed y and $f(x,y)$ as a function of y for every fixed x are continuous
(d) all directional derivatives of f exists at all points of $\mathbb{R}^2$

11. If $f(x,y)$ is differentiable at (a,b), then at (a,b)
(a) f_x exists but f_y does not exist
(b) (b) f_x does not exist but f_y exists
(c) f_x and f_y both exists
(d) f_x and f_y both do not exist

12. If f_{xy} and f_{yx} both are continuous at (a,b). Then
(a) $f_{xy}(a,b) \ne f_{yx}(a,b)$
(b) $f_{xy}(a,b) = f_{yx}(a,b)$
(c) $f_{xy}(a,b) = 0, f_{yx}(a,b)$
(d) none of these

13. Which of the following statement(s) is/are correct?
(a) If a function $f(x,y)$ is continuous at the point (a,b), it must also be differentiable at (a,b)
(b) If a function $f(x,y)$ is differentiable at (a,b), it must be continuous at (a,b)
(c) If a function $f(x,y)$ possesses both the partial derivatives $f_x(a,b)$ and $f_y(a,b)$, it must be differentiable at (a,b)
(d) If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) \ne f_{yx}(a,b)$

14. Which of the following(s) is / are true?
(a) If $f: X \to \mathbb{R}$, $X \subseteq \mathbb{R}^2$ and $(a,b) \in X$ is such that f_x, f_y are differentiable at (a,b), then $f_x(a,b) = f_y(a,b)$
(b) If a function $f(x,y)$ is differentiable at (a,b), then the partial derivative $f_x(a,b)$ and $f_y(a,b)$ both exists at (a,b)
(c) If a function $f(x,y)$ is discontinuous at (a,b), then both the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ do not exists
(d) If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) \ne f_{yx}(a,b) = 0$

15. Assume f' exists and is monotonic on an open interval (a, b), then on (a, b)
(a) $f(x)$ is bounded
(b) $f(x)$ can have only first kind discontinuity

(c) $f(x)$ is continuous
(d) $f(x)$ is continuous only when length of interval is finite

16. If $f(x)=[x]+x\sin(1/x)$, then in $[-2\pi,2\pi]$
(a) $f(x)$ is discontinuous but limit exists at every point
(b) $f(x)$ is continuous
(c) $f(x)$ is monotonic
(d) limit does not exist at some points of the interval $[-2\pi,2\pi]$

17. If $f(x)=\begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$, then which of the following is correct?
(a) $fof(x)$ is monotonic
(b) $f(x)+f(1-x)$ is constant
(c) $f(x)$ assume every value between 0 & 1
(d) none of these

18. Let $f:\mathbb{R}\to\mathbb{R}$ be a discontinuous function and $D=\{x_0\in\mathbb{R} : f(x)$ has discontinuity of first kind at $x_0\}$, then
(a) D is finite
(b) $\mathbb{R}$ - D is finite
(c) $\mathbb{R}$ - D is countable
(d) D does not contain interval

19. If $f(x)=x|x|$, then choose the correct statement.
(a) $f(x)$ is strictly decreasing function
(b) $f(x)$ is not monotonic
(c) $f(x)$ is differentiable $\forall\ x\in\mathbb{R}$ except at $x=0$
(d) $f(x)$ is differentiable $\forall\ x\in\mathbb{R}$

20. Which of the following is not correct?
(a) The set of points of ordinary (first kind) discontinuities of a function is countable.
(b) The set of points of discontinuity of the function defined by $f(x)=\lim_{n\to\infty}\lim_{m\to\infty}[\cos^{2n}(m!\pi x)]$ is uncountable.
(c) There cannot be a function defined on $\mathbb{R}$ which is continuous on $\mathbb{R}$ but nowhere differentiable.
(d) none of these.

21. The function $f(x)=\frac{|x|}{1+|x|}$ is
(a) increasing in $(0,\infty)$ and decreasing in $(-\infty,0)$
(b) decreasing in $(0,\infty)$ and increasing in $(-\infty,0)$
(c) increasing in $\mathbb{R}$
(d) decreasing in $\mathbb{R}$

22. If $f(x) = x(x-1)(x-2)$ and $g(x) = x(x-2)(x-3)$ in $[0,1/2]$, then the value of c defined in the Cauchy's mean value theorem is

(a) $\frac{(5+\sqrt{13})}{6}$
(b) $\frac{(5-\sqrt{13})}{6}$
(c) $\frac{1}{4}$
(d) none of these

23. The function $f(x)=\frac{1}{x}$, in the interval, $0<x<2$, is
(a) continuous but not uniformly continuous
(b) uniformly continuous, but not continuous
(c) neither continuous nor uniformly continuous
(d) discontinuous everywhere

24. The function $f(x) = |\tan x|$ is
(a) continuous at $x=0$ but not differentiable at $x=0$
(b) differentiable at $x=0$
(c) discontinuous at $x=0$
(d) twice differentiable at $x=0$

25. An example of a function on the real line that is continuous but not uniformly continuous, is
(a) $1/x$
(b) x
(c) $1/x^4$
(d) x^4

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

26. A function f is a monotonic function if
(a) f is a decreasing function
(b) f is an increasing function
(c) f is either increasing or decreasing function
(d) None of these

27. A function $f(x)$ has the discontinuity at $x=a$ if
(a) $f(a^+)=f(a^-)=f(a)$
(b) $f(a^+)=f(a^-)\neq f(a)$
(c) $f(a^+)\neq f(a^-)$
(d) $f(a^+)\neq f(a)$

28. If a function f is decreasing function, then which of the following is false?
(a) $-f$ is increasing function
(b) $-f$ is decreasing function
(c) $-f$ is constant function
(d) None of these

29. A function $f(x)$ has left hand jump discontinuity at $x=a$ if $f(a^+)$ and $f(a^-)$ exist and
(a) $f(a^+)-f(a^-)=0$
(b) $f(a^+)+f(a^-)\neq 0$
(c) $f(a^+)-f(a)=0$
(d) $f(a)-f(a^-)\neq 0$

30. If $f(x,y) = 2x^2 - xy + 2y^2$. Then, at (1,2)
(a) $\frac{\partial f}{\partial x}=2$
(b) $\frac{\partial f}{\partial y}=7$

(c) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ (d) $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial y}$

31. Let $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$. Then

(a) $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exists
(b) $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ does not exists
(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and equals to 0
(d) $\lim_{x\to 0} f(x,y)$ and $\lim_{y\to 0} f(x,y)$ both do not exist

32. Let $f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x^4 + y^2 = 0 \end{cases}$. Then

(a) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists
(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists
(c) the function posses no limit at the origin, but a straight line approach gives the limit zero
(d) none of the above

33. Which of the following (s) is/are true?

(a) $\lim_{(x,y)\to(0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$
(b) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2 + y^4}$ does not exist
(c) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$
(d) $\lim_{(x,y)\to(0,0)} \left(\frac{1}{xy}\right)\sin(x^2y + xy^2) = 0$

34. For what value of k, the function $f(x,y) = \begin{cases} 3xy, & (x,y) \neq (2,3) \\ k, & (x,y) = (2,3) \end{cases}$ is discontinuous?

(a) 6 (b) 9 (c) 16 (d) 18

35. The function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ has

(a) minima at (1,2)
(b) maxima at (-1,-2)
(c) neither maxima nor minima at (1,-2) and (-1,2)
(d) the saddle points are (-1,2) and (1,-2)

36. If f(x) is differentiable function $\forall\ x \in \mathbb{R}$ such that f(1) = 5 and f(x) < 5 $\forall x \neq 1$. Then,

(a) f(x) is monotonically increasing
(b) f(x) is monotonically decreasing
(c) f(x) is bounded
(d) f(x) is not monotonic

37. The function f(x) = |x| + 3 is

(a) continuous as well as differentiable on $\mathbb{R}$
(b) continuous on $\mathbb{R}$ but not differentiable anywhere on $\mathbb{R}$
(c) continuous on $\mathbb{R}$ and differentiable at all but a finitely many points of $\mathbb{R}$
(d) not continuous on $\mathbb{R}$

38. The range of value of 'a' for which the equation $3x^4 - 8x^3 - 6x^2 + 24x + a = 0$ has four real unequal roots is

(a) [- 13, - 8] (b) [- 8, - 5] (c) [- 13, - 8] (d) [- 13, - 8]

39. If f is a differentiable function on the real line such that f(0) = 2, and f(x) < 2 for x ≠ 0, then

(a) $f''(0) > 0$ (b) $f''(0) = 0$ (c) $f''(0) < 0$ (d) none of these

40. The function $f(x) = \begin{cases} x^2, & x > 0 \\ 1-(1-x)^2, & \text{elsewhere} \end{cases}$

(a) not differentiable at 0 but is continuous there
(b) has second derivative at the origin
(c) continuous at the origin but not differentiable
(d) continuous and differentiable at the origin but does not have second derivative there

41. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by f(x,y) = |x| + |y| and g(x,y) = |xy|. Then

(a) f is differentiable at (0, 0), but g is not differentiable at (0, 0)
(b) g is differentiable at (0, 0), but f is not differentiable at (0,0)
(c) Both f and g are differentiable at (0, 0)
(d) Both f and g are continuous at (0, 0).

ASSIGNMENT 3.2

NOTE: CHOOSE THE BEST OPTION

1. If f is strictly increasing on $E \subset \mathbb{R}$, then
   (a) f^{-1} exist (b) f^{-1} does not exist
   (c) f^{-1} is strictly decreasing on $f(E)$ (d) none of these

2. If f is one-to-one, continuous on $[a, b]$, then
   (a) f is strictly monotonic (b) f is monotonic
   (c) f is constant (d) none of these

3. If f' exist and is monotonic on (a, b), then
   (a) f' is continuous on $[a, b]$ (b) f' is discontinuous on $[a, b]$
   (c) f' is constant (d) none of these

4. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$ and $f' > 0$ in (a, b), then
   (a) f is strictly increasing (b) f is strictly decreasing
   (c) f is constant (d) none of these

5. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$, then if $f' = 0$
   (a) f is increasing (b) f is decreasing (c) f is constant (d) none of these

6. If f is strictly increasing on $E \subset \mathbb{R}$, then
   (a) f^{-1} exist and it is strictly decreasing on E (b) f^{-1} exist and it is strictly increasing on E
   (c) f^{-1} does not exist (d) none of these

7. Every point of discontinuity of an increasing function is
   (a) of first kind (b) of second kind (c) removable (d) none of these

8. If f is increasing function on closed interval, then for $c \in (a, b)$
   (a) $f(c^+)$ and $f(c^-)$ exist (b) $f(c^+)$ and $f(c^-)$ does not exist
   (c) $f(c^+) \le f(c^-)$ (d) None of these

9. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x^2]$. The points of discontinuity of f are
   (a) only the integral points (b) all rational numbers
   (c) $\{\pm\sqrt{n}$: n is a non negative integer$\}$ (d) all real numbers

10. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x^2+y^2, & \text{if } x \text{ and } y \text{ are rational} \\ 0, & \text{otherwise} \end{cases}$. Then
   (a) f is not continuous at (0,0)
   (b) f is continuous at (0,0) and f_x, f_y does not exist at every point of $\mathbb{R}^2$
   (c) f is discontinuous at (0,0) and f_x, f_y exists only at (0,0)
   (d) none of the above

11. For $(x, y) \in \mathbb{R}^2$, let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \ne (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$. Then
   (a) f_x and f_y exist at (0,0) and f is continuous at (0,0)
   (b) f_x and f_y exist at (0,0) and f is discontinuous at (0,0)
   (c) f_x and f_y do not exist at (0,0) and f is continuous at (0,0)
   (d) f_x and f_y exist at (0,0) but f is discontinuous at (0,0)

12. Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Then
   (a) f(x,y) is discontinuous at (0,0) (b) f(x,y) is continuous at (0,0)
   (c) $f_x(0,0)=1$ (d) $f_y(0,0)=1$

13. Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Then
   (a) f(x,y) is not defined at the origin (b) $f_x(0,0)=1$
   (c) $f_x(0,0)=0$ (d) $f_y(0,0)$ does not exists

14. Let if possible, $\alpha = \lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$, $\beta = \lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$. Then
   (a) α exists but β does not exist (b) α does not exist but β exists
   (c) α, β do not exist (d) Both α, β exist

15. Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Then
   (a) $f_{xy}(0,0) = f_{yx}(0,0) = 0$ (b) $f_{xy}(0,0) \ne f_{yx}(0,0)$
   (c) $f_{xy}(0,0) = f_{yx}(0,0) = 1$ (d) none of these

16. Let $f(x,y)=\sqrt{|xy|}$, then the value of $f_x(0,0)$ and $f_y(0,0)$ are
(a) 0,0 (b) 0,1 (c) 1,0 (d) 1,1

17. $f(x)=(x+|x|)|x| \forall x \in \mathbb{R}$, then which of the following is incorrect?
(a) f is continuous (b) f is not differentiable for some x
(c) f' is continuous (d) f' is differentiable

18. Let $f(x)=\begin{cases} e^{-1/x^2}\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0 \end{cases}$, then
(a) $\lim_{x\to 0} f(x)$ does not exist
(b) $\lim_{x\to 0} f(x)$ exists but $f(x)$ is discontinuous there
(c) $f(x)$ is continuous at $x = 0$ but not differentiable
(d) $f(x)$ is differentiable

19. Let $f: [0,1] \to \mathbb{R}$ be a discontinuous monotonic function. Then the range set of f is
(a) a bounded set that is not an interval (b) a bounded interval
(c) an unbounded connected set (d) a set that is neither bounded nor connected

20. If f has a finite second order derivative in $[a,b]$ and the line segment joining the points $A(a,f(a))$ and $B(b,f(b))$ intersects the graph of f in a third point P different from A and B, then
(a) $f'(x)$ is monotonic in $[a,b]$ but not strictly monotonic
(b) $f'(x)$ is strictly increasing
(c) $f''(x)$ has no root in (a,b)
(d) $f''(x)$ has at least one root in (a,b)

21. If f is non negative and has a finite third derivative in the open interval $(0,1)$, then $f'''(c)=0$ for some $c\in(0,1)$ if
(a) $f(x)=0$ for at least two values of x in $(0,1)$ (b) $f(x)=0$ for at least one value of x in $(0,1)$
(c) $f'(x)=0$ for at least one value of x in $(0,1)$ (d) none of these

22. If $f(x) = e^{-x}$ and $g(x) = e^{x}$. Then in the Cauchy's mean value theorem the value of c is
(a) geometric mean of a and b (b) $\frac{a+b}{ab}$
(c) $\frac{a+b}{2}$ (d) $\frac{ab}{a+b}$

23. If f is twice differentiable and $|f| < \alpha$, $|f''| < \beta$ in the range $x > a$, then which may be an upper bound by $|f'|$
(a) $\sqrt{\alpha\beta}$ (b) $2\sqrt{\alpha\beta}$
(c) $1/2\sqrt{\alpha\beta}$ (d) $4\sqrt{\alpha\beta}$

24. Which of the following need not to be hold always?
(a) f is uniformly continuous on a bounded interval I, then f is bounded on I
(b) f is continuous and bounded on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$
(c) If $< f_n(x) >$ and $< g_n(x) >$ converges uniformly on I. Then $<f_n(x)> + <g_n(x)>$ is uniformly convergent.
(d) all of the above

25. Select the correct statement
(a) If f and g are uniformly continuous on I, then their product is uniformly continuous.
(b) If f and g are uniformly continuous on I, then f/g is uniformly continuous provided $g(x) \neq 0 \forall x\in I$
(c) If f is differentiable in (a, b) and $f'(x)$ is bounded on (a, b), then $f(x)$ is uniformly continuous.
(d) none of these

26. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If $f(\mathbb{Q}) \subseteq \mathbb{N}$. Then
(a) $f(\mathbb{R}) = \mathbb{N}$ (b) $f(\mathbb{R}) \subseteq \mathbb{N}$ but f need not be constant
(c) f is unbounded (d) f is a constant function

27. If $f(x)$ is real valued function defined on $[0,\infty)$ such that $f(0)=0$ and $f''(x) > 0$ for all x, then the function $h(x)=\frac{f(x)}{x}$ is
(a) increasing in $[0, \infty)$ (b) decreasing in $[0,1]$
(c) increasing in $[0,1]$ and decreasing in $[1, \infty)$ (d) decreasing in $[0,1]$ and increasing in $[1, \infty)$

28. Let $f: [0, 10) \to [0, 10)$ be a continuous mapping. Then
(a) f need not have any fixed point (b) f has atleast 10 fixed points
(c) f has atleast 9 fixed points (d) f has atleast one fixed point

29. The map $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2 |x|^3$ is differentiable at $x = 0$ if and only if
(a) $a_1 = 0$ and $a_2 = 0$ (b) $a_0 = 0$ and $a_1 = 0$
(c) $a_1 = 0$ (d) a_0, a_1, a_2 can take any real value

30. f(x) is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} f'(x) = \alpha$.

Then

(a) α must be 0

(b) α need not be 0, but $|\alpha| < 1$

(c) $\alpha > 1$

(d) $\alpha < -1$

31. Suppose f is real valued function on the interval [-1, 1] such that |f| is continuous. Which of the following is true?

(a) f is continuous

(b) f is not continuous

(c) f need not be continuous

(d) f is bounded

32. Let f : [a, b] ∈ ℝ be a continuous functions and let f(a) < f(b). Then by intermediate value theorem

(a) f([a, b]) = [f(a), f(b)]

(b) f([a, b]) ⊇ [f(a), f(b)]

(c) f([a, b]) ⊆ [f(a), f(b)]

(d) f([a, b]) ≠ [f(a), f(b)]

33. An example of a function on the real line that is continuous but not uniformly continuous

(a) constant function

(b) identity function

(c) sin x

(d) x^2

34. The function $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

(a) differentiable at 0 but not continuous

(b) has second derivative at the origin

(c) continuous at the origin but not differentiable

(d) neither continuous nor differentible at the origin

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

35. The point of inflexion of $f(x) = x^{5/3}$ are not

(a) -1

(b) 0

(c) 1

(d) 5/3

36. Let $f(x,y) = \begin{cases} xy\dfrac{(x^2-y^2)}{(x^2+y^2)}, & x^2+y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases}$ then,

(a) $f_x(x,0) = 0$

(b) $f_y(x,0) = x$

(c) $f_x(0,y) = -y$

(d) $f_y(0,y) = 0$

37. Let $f(x,y) = \begin{cases} y + x\sin\left(\frac{1}{y}\right) & , y \neq 0 \\ 0 & , y = 0 \end{cases}$. Then

(a) $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$ does not exist

(b) $\lim_{y\to0}\lim_{x\to0} f(x,y)$ exists

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists

38. Let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$. Then

(a) $\lim_{x\to0} f(x,0) = f(0,0)$

(b) $\lim_{y\to0} f(0,y) = f(0,0)$

(c) $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

(d) $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$

39. Let $f(x,y) = \begin{cases} 1 , & xy \neq 0 \\ 0 , & xy = 0 \end{cases}$. Then

(a) $\lim_{x\to0}\lim_{y\to0} f(x,y) = 1$

(b) $\lim_{y\to0}\lim_{x\to0} f(x,y) = 1$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

40. Let $f(x,y) = \dfrac{y-x}{y+x} \cdot \dfrac{1+x}{1+y}$. Then

(a) $\lim_{x\to0}\lim_{y\to0} f(x,y) = 1$

(b) $\lim_{y\to0}\lim_{x\to0} f(x,y) = 1$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists

41. Consider $f(x,y) = \begin{cases} x\sin 1/x + y\sin 1/x, & xy \neq 0 \\ x\sin 1/x, & y = 0, x \neq 0 \\ y\sin 1/y, & x = 0, y \neq 0 \\ 0 \quad , & x = y = 0 \end{cases}$. Then,

(a) f(x,y) is continuous at origin

(b) f(x,y) is discontinuous at origin

(c) f(x,y) is differentiable at origin

(d) both partial derivative exits at origin

42. Let f(x) = |sin πx|, x ∈ ℝ, then

(a) f is continuous nowhere

(b) f is continuous everywhere and differentiable nowhere

(c) f is continuous everywhere and differentiable everywhere except of integral values of x

(d) f is differentiable everywhere but not continuous

ASSIGNMENT 3.3

NOTE: CHOOSE THE BEST OPTION

1. Let $f:[0,10]\to[0,10]$ be continuous function, then
   (a) f need not have any fixed point
   (b) f has atleast 10 fixed points
   (c) f has atleast 9 fixed points
   (d) f has atleast one fixed point

2. If $f(x)$ is real valued function defined on $[0,\infty)$ such that $f(0)=0$ and $f''(x)>0$ for all x, then the function $h(x)=\dfrac{f(x)}{x}$ is
   (a) increasing in $[0,\infty)$
   (b) decreasing in $[0,1]$
   (c) increasing in $[0,1)$ and decreasing in $[1,\infty)$
   (d) decreasing in $[0,1]$ and increasing in $[0,\infty)$

3. The continuous function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=(x^2+1)^{2003}$, is
   (a) onto but not one-one
   (b) one-one but not onto
   (c) both one-one and onto
   (d) neither one-one nor onto

4. The function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ has
   (a) discontinuity of first kind at $x=0$
   (b) removable discontinuity at $x=0$
   (c) continuity at $x=0$
   (d) none of these

5. The function $f(x)=\begin{cases}e^{1/x}, & x\neq 0\\ 0, & x=0\end{cases}$ has
   (a) mixed discontinuity at $x=0$
   (b) removable discontinuity
   (c) continuity at $x=0$
   (d) none of these

6. The function $f(x)=\begin{cases}x\sin(1/x), & x\neq 0\\ 1, & x=0\end{cases}$ has
   (a) removal discontinuity at $x=0$
   (b) non-removal discontinuity at $x=0$
   (c) mixed discontinuity at $x=0$
   (d) continuity at $x=0$

7. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function and $f(1)=4$, then the value of $\lim_{x\to 1}\int_4^{f(x)}\dfrac{2t}{x-1}dt$
   (a) $f'(1)$
   (b) $8f'(1)$
   (c) $4f'(1)$
   (d) $2f'(1)$

8. The formula $A_0 f\left(-\frac{1}{2}\right)+A_1 f(0)+A_2 f\left(\frac{1}{2}\right)$ which approximates the integral $\int_{-1}^{1} f(x)\,dx$ is exact for polynomials of degree less than or equal to 2 if
   (a) $A_0=A_2=\frac{4}{3}, A_1=\frac{2}{3}$
   (b) $A_0=A_1=A_2=1$
   (c) $A_0=A_2=\frac{4}{3}, A_1=-\frac{2}{3}$
   (d) none of above

9. Let $f(x,y)=\sqrt{|xy|}$, then
   (a) f_x and f_y do not exist at (0, 0)
   (b) $f_x(0,0)=1$
   (c) $f_y(0,0)=0$
   (d) f_y is differentiable at (0, 0)

10. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$. Then
   (a) f is not continuous at (0,0)
   (b) f is continuous at (0,0) but not differentiable at (0,0)
   (c) f is differentiable only at (0,0)
   (d) f is differentiable everywhere

11. For the function $f(x,y)=2x^4-3x^2y+y^2$ has
   (a) maximum at (0,0)
   (b) minimum at (0,0)
   (c) neither maxima nor minima at (0,0)
   (d) doubtful case at (0,0) always

12. Let $f(x,y)=\sqrt{|xy|}$, then
   (a) $f(x,y)$ is continuous at origin.
   (b) f_x exists at origin but not equal to zero.
   (c) f_y exists at origin but not equal to zero.
   (d) none of these.

13. Let
$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x=y=0\end{cases}$$
   (a) $f(x,y)$ is continuous at origin
   (b) $f(x,y)$ is differentiable at origin
   (b) $f_x(0,0)\neq 0$
   (d) $f_y(0,0)\neq 0$

14. Let $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x=y=0\end{cases}$, then

(a) f(x, y) is continuous at origin
(b) f(x, y)has removable discontinuity of (0, 0)
(b) f(x, y) is not differentiable at (0, 0)
(d) none of these

15. Let $f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & x^2+y^4\neq 0\\ 0, & \text{otherwise}\end{cases}$

Then, the directional derivative at c = (0, 0) in the direction $u=(a_1, a_2)$ is

(a) $f'(c,u)=\frac{a_2}{a_1}, a_1\neq 0$
(b) $f'(c,u)=0, a_1=a_2\neq 0$
(c) $f'(c,u)=\frac{1}{\sqrt{3}}, a_1=a_2=\frac{1}{\sqrt{3}}$
(d) none of these

16. Let

$f(x,y)=\begin{cases}\dfrac{x^2-xy}{x+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Then

(a) $f_x(0,0)=2$ (b) $f_y(0,0)=0$ (c) $f_x(0,0)=0$ (d) $f_y(0,0)=1$

17. The function $f(x,y)=x^2-2xy+y^2+x^3-y^3+x^5$
(a) has maximum value at origin
(b) has minimum valueat origin
(c) has neither maximum nor minimum value at origin.
(d) has maximum value but no minimum value at origin.

18. Let $f(x,y)=\begin{cases}x^3+y^3, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$, then

(a) f(x, y) is discontinuous at (0, 0)
(b) f_x exists at every point including origin
(c) f_y exists at every point excluding origin
(d) none of these

19. Let $f(x,y)=\begin{cases}x^2+2y, & (x,y)\neq(1,2)\\ 0, & (x,y)=(1,2)\end{cases}$, then

(a) f(x, y) is continuous at (1,2)
(b) f(x, y) has removable discontinuity at (1, 2)
(c) $\lim_{(x,y)\to(1,2)} f(x,y)$ does not exists
(d) None of these

20. Let $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$, then

(a) f is continuous at origin
(b) $f_x(0,0)$ does not exists
(c) $f_y(0,0)$ does not exists
(d) none of these

21. Let $f(x,y)=\begin{cases}\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & x=y=0\end{cases}$, then

(a) f is differentiable at (0, 0)
(b) f is not differentiable at (0, 0)
(c) f is continuous at (0, 0)
(d) none of these

22. Let $f(x,y)=x^y$, then
(a) $f_x(a,0)=0$, where a is constant
(b) $f_y(e,0)=0$
(b) $f_{xy}(1,0)=0$
(d) $f_{yx}(1,1)=1$

23. Let $f(x,y)=\begin{cases}|x|+|y|, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$. Then

(a) f (x,y) is continuous at (0, 0)
(b) f (x, y) is discontinuous at (0, 0)
(c) f (x,y) is differentiable at (0, 0)
(d) none of these

24. If $f(x)=\lim_{n\to\infty}\left[\lim_{t\to 0}\dfrac{\sin^2(n!\pi x)}{\sin^2(n!\pi x)+t^2}\right]$. Then

(a) f(x) is nowhere continuous
(b) f(x) has only 2 point of discontinuity
(c) f(x) has only 2 point of continuity
(d) f(x) is continuous everywhere

25. If f is defined by

$f(x)=|x|+|x^2-1|\ \forall\ x\in\mathbb{R}$, then

(a) f is discontinuous on $\mathbb{R}$
(b) f has local minima at $x=\pm 1/2$
(c) f has local maxima at $x=\pm 1/2$
(d) f has no local extrema

26. Let $f:(0,2)\to\mathbb{R}$ be defined by

$f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 2x-1 & \text{if } x \text{ is irrational}\end{cases}$,

Then

(a) f is differentiable exactly at one point
(b) f is differentiable exactly at two points
(c) f is not differentiable at any point in (0,2)
(d) f is differentiable at every point in (0, 2)

27. The equation $x^6 - x - 1 = 0$ has
(a) no positive real roots (b) exactly one positive real root
(c) exactly two positive real roots (d) all positive real roots

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

28. Let $X=(0,1)\cup(2,3)$ is an open subset of $\mathbb{R}$. Let f be a continuous function on X such that the derivative $f'(x)=0$ for all x. Then the range of f has not
(a) uncountable number of points (b) countably infinite number of points
(c) at most 2 points (d) at most 1 point

29. The function $f(x) = \frac{1}{x-a}\sin\frac{1}{x-a}$ has not
(a) discontinuity of first kind at $x = a$ (b) discontinuity of second kind at $x = a$
(c) continuity at $x = a$ (d) removable discontinuity at $x=a$

30. If $f(x)=\tan^{-1}\{\log(e/x^2)/\log(ex^2)\} + \tan^{-1}\left\{\frac{3+2\log_e x}{1-6\log_e x}\right\}$, then $\frac{d^2}{dx^2}\{f(x)\}$ is equal to
(a) -1 (b) 0 (c) 1 (d) 5

31. If $f(x)=\tan^{-1}x,\ x>0$, then by Lagrange's mean value theorem which of the following is not correct?
(a) $\frac{x}{1+x^2} < \tan^{-1}x < x$ (b) $\frac{x}{1+x^2} > \tan^{-1}x > x$
(c) $\frac{x}{1+x^2} > \tan^{-1}x < x$ (d) $\frac{x}{1+x^2} < \tan^{-1}x > x$

32. If $f(x,y)=\begin{cases}\frac{x^2-xy}{x+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$. Then
(a) $f_x(0,0)=1$ (b) $f_y(0,0)=0$
(c) $f_x(0,0)=0$ (d) $f_y(0,0)=1$

33. Let $f(x,y)=\begin{cases}x^3+y^3, & x\neq y\\ 0, & x=y\end{cases}$. Then
(a) f(x,y) is continuous at origin (b) f(x,y) is discontinuous at origin
(c) f_x exists at every point including origin (d) f_y exists at every point including origin

34. Let $f(x,y)=\begin{cases}\frac{x^2y}{x^3+y^3}, & x^3+y^3\neq 0\\ 0, & x=y=0\end{cases}$. Then
(a) f(x,y) is continuous at (0,0) (b) f(x,y) is discontinuous at (0,0)
(c) f(0,y) is continuous at origin (d) f(x,0) is continuous at origin

35. Let $f(x,y)=\begin{cases}\frac{xy^3}{x^2+y^6}; & x\neq 0, y\neq 0\\ 0; & x=y=0\end{cases}$. Then
(a) f(x,y) is continuous at origin (b) f(x,y) is discontinuous at origin
(c) 0f(0,y) is continuous at origin (d) f(x,0) is continuous at origin

36. Let $f(x,y)=\begin{cases}x^2+2y; & (x,y)\neq(1,2)\\ 0; & (x,y)=(1,2)\end{cases}$. Then
(a) f(x,y) is continuous at (1,2) (b) f(x,y) is discontinuous at (1,2)
(c) f(x,y) has removable discontinuity at (1,2) (d) $\lim_{(x,y)\to(1,2)} f(x,y)$ does not exist

37. If $f(x,y)=\begin{cases}xy\tan(y/x), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$. Then, at (0,0)
(a) $xf_x + yf_y = 2f$ (b) $xf_x - yf_y = 2f$
(c) $yf_x + xf_y = 2f$ (d) $yf_x - xf_y = 2f$

38. If $f(x,y)=\begin{cases}\frac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$. Then
(a) f is discontinuous at the origin (b) $f_x(0,0)$ exist
(c) $f_y(0,0)$ exist (d) f is continuous at the origin

39. If $f(x,y)=xy(x^2-y^2)/(x^2+y^2)$, when $(x,y)\neq(0,0)$ and $f(0,0)=0$
(a) $f_{xy}(0,0)=-1$ (b) $f_{yx}(0,0)=1$
(c) $f_{xy}(0,0)=f_{yx}(0,0)$ (d) $f_{xy}(0,0)\neq f_{yx}(0,0)$

40. The function $f(x) = 1 - |1 - x|$ on $\mathbb{R}$ is
(a) not continuous (b) continuous but not differentiable
(c) differentiable but not continuous (d) differentiable at only one point

ANSWERS TO EXERCISE

(PRACTICE SET – I)

Exercise 1: (c)	Exercise 2: (b)	Exercise 3: (b)	Exercise 4: (b)
Exercise 5: (a)	Exercise 6: (c)	Exercise 7: (a,c)	Exercise 8: (b)
Exercise 9: (c)	Exercise 10: (b)		

(PRACTICE SET – II)

Exercise 1: (a)	Exercise 2: (a)	Exercise 3: (b,d)	Exercise 5: (d)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT 3.1

1. (c)	2. (a)	3. (b)	4. (a)	5. (b)	6. (a)	7. (c)
8. (a)	9. (b)	10. (c)	11. (c)	12. (b)	13. (b)	14. (c)
15. (c)	16. (d)	17. (d)	18. (d)	19. (d)	20. (c)	21. (a)
22. (b)	23. (a)	24. (a)	25. (d)			
26. (a,b,c)	27. (b,c,d)	28. (b,c,d)	29. (b,d)	30. (a,b,d)	31. (a,b,c,d)	32. (a,c)
33. (a,b,c,d)	34. (a,b,c)	35. (a,b,c,d)	36. (d)	37. (c)	38. (d)	39. (c)
40. (a)	41. (d)					

ASSIGNMENT 3.2

1. (a)	2. (a)	3. (a)	4. (a)	5. (c)	6. (b)	7. (a)
8. (a)	9. (c)	10. (b)	11. (d)	12. (b)	13. (c)	14. (a)
15. (a)	16. (a)	17. (d)	18. (c)	19. (a)	20. (d)	21. (a)
22. (c)	23. (b)	24. (b)	25. (c)	26. (d)	27. (a)	28. (a)
29. (c)	30. (a)	31. (c)	32. (b)	33. (d)	34. (c)	
35. (a,c,d)	36. (a,b,c,d)	37. (a,b,d)	38. (a,b,d)	39. (a,b,c)	40. (b,c)	41. (a,d)
42. (c)						

ASSIGNMENT 3.3

1. (d)	2. (a)	3. (d)	4. (b)	5. (a)	6. (a)	7. (b)
8. (c)	9. (c)	10. (b)	11. (c)	12. (a)	13. (a)	14. (d)
15. (c)	16. (b)	17. (c)	18. (b)	19. (b)	20. (a)	21. (b)
22. (a)	23. (a)	24. (d)	25. (c)	26. (a)	27. (b)	
28. (a,b,d)	29. (a,c,d)	30. (b)	31. (b,c,d)	32. (a,b)	33. (a,c,d)	34. (b,c,d)
35. (b,c,d)	36. (b,c)	37. (a,b,c,d)	38. (a,b,c)	39. (a,b,d)	40. (b)	

CHAPTER–4
METRIC SPACE

INTRODUCTION

Earlier, we have studied the algebraic properties of real number system i.e. addition, multiplication, division etc. In this section we will study number system with different properties by defining the distance between them. A metric space is a set for which distances between all members of the set are defined. All distances taken together is called a metric on the set. So, in this section we will study the properties of metric spaces.

4.1. *METRIC SPACE*

A set X, whose elements we shall call points, is said to be a metric space if with any two points p and q of X there is associated a real number d(p, q), called the distance from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$

(b) $d(p, q) = d(q, p)$

(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these properties is called a distance function, or a metric.

For examples.

(i) *The most important examples of metric spaces are Euclidean spaces $\mathbb{R}^n$. The distance in $\mathbb{R}^n$ is defined by $d(x, y) = |x - y|$ $(x, y \in \mathbb{R}^n)$*

(ii) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, ..., |x_n - y_n|\}$, *where* $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. *It can be easily verified that 'd' defined above is a metric on $\mathbb{R}^n$.*

(iii) $d(x, y) = |x^n - y^n|$, *is metric on X iff n is odd.*

4.1.1. Discrete Metric: *Let X be any set. Then the discrete metric on set X is given by* $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Important Results:

(1) *If (X, d) is a metric space, then 'd_1' defined by $d_1(x, y) = \min\{1, d(x, y)\}$ is also a metric on X.*

(2) *If d_1 and d_2 are two metrics on a set X. Then any convex combination of d_1 and d_2 is also a metric on X.*

(3) *If (X, d) is a metric space, then $\left(X, \frac{d}{1+d}\right)$ is also a metric space i.e. if d(x, y) is a metric on a set X, then $\frac{d(x,y)}{1+d(x,y)}$ is also a metric on set X and infact this metric is always bounded.*

4.2. *SOME DEFINITIONS*

Let X be a metric space and E be any subset of X.

(1) ***Neighbourhood of a point:*** *Neighbourhood of an element p is a set $N_r(p)$ consisting of all points q such that $d(p, q) < r$. The number r is called the radius of $N_r(p)$.*

(2) ***Interior point:*** *An element p is an interior point of E if there is a neighbourhood N of p such that $p \in N \subset E$.*

(3) ***Open Set:*** *E is open if every element of E is an interior point of E.*

(4) ***Closed Set:*** *E is closed if every limit point of E is a point of E.*

(5) ***Bounded set:*** *E is bounded if there is a real number M and an element $q \in X$ such that $d(p, q) < M$ for all $p \in E$.*

(6) ***Limit point:*** *An element p is a limit point of the set E if every neighbourhood of p contains an element $q \neq p$ such that $q \in E$.*

(7) ***Isolated point:*** *If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E.*

(8) ***Perfect set:*** *E is perfect if E is closed and if every point of E is a limit point of E.*

Note: *If P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.*

(9) ***Dense set:*** *E is dense in X if every element of X either is a limit point of E, or a point of E (or both).*

(10) ***Derived set:*** *Derived set of E is the set of limit points of E. It is denoted by E'.*

(11) ***Closure of a set:*** *Closure of E is the set $\overline{E} = E \cup E'$.*

Results on open and closed sets:

(1) *Every neighbourhood is an open set.*
(2) *A set E is open if and only if its complement is closed.*

Results on closure of a set:

If X is a metric space and $E \subset X$, then

(1) *$\overline{E}$ is closed.*
(2) *$E = \overline{E}$ if and only if E is closed.*
(3) *$\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*
(4) *$\overline{E}$ is the smallest closed subset of X that contains E.*

Results on limit point of a set:

1) ***Weierstrass Theorem:*** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

2) *If p is a limit point of a set E, then every neighbourhood of p contains infinitely many points of E.*

4.3 ***CONVERGENT SEQUENCE***

A sequence $\{p_n\}$ in a metric space X is said to be converge if there is a point $p \in X$ such that for every $\epsilon > 0$ there exist an integer N such that for $n \geq N$ implies that for $d(p_n, p) < \epsilon$. (Here d denotes the distance in X).

We say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ and write $p_n \to p$, or $\lim_{n\to\infty} p_n = p$

If $\{p_n\}$ does not converge, it is said to be divergent

Note: *Convergent sequences are bounded, but bounded sequence in $\mathbb{R}^k$ need not converge. However, there is one important case in which convergence is equivalent to boundedness , this happens for monotonic sequence in $\mathbb{R}$.*

4.4 ***CAUCHY SEQUENCE***

A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$, there is a positive integer N such that $d(p_m, p_n) < \epsilon$ if $n \geq N$ and $m \geq N$.

Results on Cauchy sequence:

(1) *In any metric space X, every convergent sequence is a Cauchy sequence.*
(2) *In $\mathbb{R}^k$, every Cauchy sequence converges.*

4.5 ***DIAMETER OF A SET***

Let E be a non-empty subset of a metric space X, and let S be the set of all real numbers of the form d(p, q), where $p, q \in E$. The sup of S in E is called the diameter of E and is denoted by diam E.

Remark: *If $\overline{E}$ is the closure of a set E in a metric space X, then diam $\overline{E}$ = diam E.*

4.6 ***LIMIT OF A FUNCTION***

Definition 1: *Let X and Y be metric spaces. Suppose $E \subset X$, $f : E \to Y$, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all point $x \in E$ for which $0 < d_X(x, p) < \delta$.*

The symbols d_X and d_Y refer to the metrics in X and Y respectively.

Definition 2: *Let X and Y be metric spaces. Suppose $E \subset X$, $f : E \to Y$, and p is a limit point of E. Then $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n\to\infty} p_n = p$*

Note: If X and Y are replaced by the real line, the complex plane, or by some euclidean space $\mathbb{R}^k$, the distances d_X, d_Y are of course replaced by absolute values or by appropriate norms.

4.6.1. Results on limit of a function:

If $f, g : (E \neq \phi)\ E \to \mathbb{R}^k$, we defining $f + g$ and $f.g$ by:
$(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$ and if λ is a real number, $(\lambda f)(x) = \lambda f(x)$.
Suppose $E(\neq \phi)\ E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$, then

(a) $\lim_{x\to p} (f + g)(x) = A + B$

(b) $\lim_{x\to p} (fg)(x) = AB$

(c) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

4.7 CONTINUOUS FUNCTION

Definition 1: Suppose X and Y are metric spaces, $E \subset X$, $p \in E$ and $f : E \to Y$. Then f is said to be continuous at p iff $\lim_{x\to p} f(x) = f(p)$ i.e. for every $\in > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \in$ for all points $x \in E$ for which $d_X(x, p) < \delta$.

Definition 2: A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Note:

(1) If f is continuous at every point of E, then f is said to be continuous on E. It should be noted that f has to be defined at the point p in order to be continuous at p.

(2) If p is an isolated point of E, then our definition implies that every function f which has E as its domain of definition is continuous at p. For, no matter which $\in > 0$ we can choose $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$ then $d_Y(f(x), f(p)) = 0 < \in$

Results on Continuity:

1) Let f and g be complex continuous functions defined on a metric space X. Then
(a) $f + g$ is continuous on X
(b) fg is continuous on X
(c) f/g are continuous on X where $g(x) \neq 0$, for all $x \in X$.

2) Let $f_1, ..., f_k$ be real functions defined on a metric space X and let $f: X \to \mathbb{R}^k$ defined by $f(x) = (f_1(x), ..., f_k(x))$; $x \in X$, then f is continuous if and only if each of the functions $f_1, ..., f_k$ are continuous.

4.8 DISCONTINUITY

Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x^+)$ and $f(x^-)$ exist. Then f is said to have a discontinuity of the first kind, or a simple discontinuity at x.

Two ways in which a function can have a simple discontinuity is $f(x^+) \neq f(x^-)$ or $f(x^+) = f(x^-) \neq f(x)$. Otherwise the discontinuity is said to be of the second kind.

4.9 UNIFORM CONTINUITY

Let f be a mapping of a metric space X into a metric space Y we say that f is uniformly continuous on X if for every $\in > 0$, there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \in$ for all p and q in X for which $d_X(p,q) < \delta$

4.9.1 Difference between continuity and uniform continuity:

(1) Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.

(2) Evidently, every uniformly continuous function is continuous.

4.10 COMPLETE METRIC SPACE

A metric space in which every Cauchy sequence converges is said to be complete.

Examples.

(1) All compact metric spaces and all Euclidean spaces are complete.

(2) The space $\mathbb{Q}$ of rational numbers, with the standard metric given by $d\,|x, y| = |x-y|$ (i.e., the absolute value of the difference) is not complete.

Result on complete metric:

Every closed subset E of a complete metric space X is complete. (Every Cauchy sequence in E is a Cauchy sequence in X, hence it converges to some $p \in X$ and actually $p \in E$, since E is closed).

4.11. COMPACTNESS

4.11.1. Open cover: Let X be a metric space and $E \subseteq X$. Then open cover of a set E is the collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$

4.11.2. Compact set: A metric space is said to be compact if every open cover of the metric space has a finite sub-cover (sub-cover is also a cover of the set and is a subset of the cover).

4.11.3. Relatively compact: A subset Y of a metric space X is said to be relatively compact if $\overline{Y}$ is compact.

Example 4.11.1. Consider metric space $(\mathbb{R}, d)$ and subset A of $\mathbb{R}$, where $A = [0, \infty)$. Then A is not compact.

Solution: As $A = [0,\infty) \subseteq \bigcup_{n=1}^{\infty}(-1, n)$, is infinite open cover. But there does not exist any finite sub-cover. Thus A is not compact.

Note:

(1) If any set A is unbounded, then A cannot be compact.

(2) If we have metric space $(\mathbb{Q}, d)$ and A be any subset which is closed as well as bounded, then that subset need not be compact.

For example. if $A = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$

Clearly, $A \subseteq \bigcup_{n=1}^{\infty}\left(\sqrt{2} - \frac{1}{n}, \sqrt{3} + \frac{1}{n}\right)$

No finite sub-cover of this infinite cover can cover set A.

Hence, $[\sqrt{2}, \sqrt{3}]$ is not compact.

Results on Compact sets:

1) Compact subsets of a metric space are closed.

2) If F is closed and K is compact, then $F \cap K$ is compact.

3) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub collection of $\{K_\alpha\}$ is non-empty, then $\cap K_\alpha$ is non-empty.

4) If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

5) If E is an infinite subset of a compact set K, then E has a limit point in K.

6) If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:
   (a) E is closed and bounded.
   (b) E is compact
   (c) Every infinite subset of E has a limit point in E.

Note: (b) and (c) are equivalent in any metric space but (a) does not in general imply (b) and (c).

7) If $\{K_n\}$ is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$ and if $\lim_{n\to\infty} \text{diam } K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

8) Let E be a non compact set in $\mathbb{R}$. Then
   (a) there exists a continuous function on E which is not bounded.
   (b) there exists a continuous and bounded function on E which has no maximum. If, in addition E is bounded, then there exists a continuous function on E which is not uniformly continuous.

9) ***Heine-Borel property*** of Real line: Every closed and bounded subset of $\mathbb{R}$ is compact. (Converse of Heine-Borel theorem is true for $\mathbb{R}$. However, it is not true for general metric spaces.). Heine-Borel property can be extended to $\mathbb{R}^n$.

10) Every finite subset of $\mathbb{R}$ is compact.

11) The set of real numbers $\mathbb{R}$ is not compact.

Results on Compact metric space:

1) Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then f(X) is compact i.e. continuous image of a compact set is compact.

2) If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then f(X) is closed and bounded. Thus, f is bounded.

3) Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x;\ x \in X$, is a continuous mapping of Y onto X.

4) Every compact metric space is complete.

5) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

6) Let f be a continuous mapping of a compact metric space X into a metric space Y, then f is uniformly continuous on X.

Example 4.11.2. If (X, d) is a metric space and $A \subseteq X$ and if $d(x, A) = 0\ \forall x \in X$, then
(a) A is closed
(b) A is compact
(c) A is dense in X
(d) None of these

Solution: Here $d(x, A) = 0$

$\Rightarrow$ every $x \in X$ is either member of A or it is a limit point of $A \Rightarrow X = A \cup A'$

$X = \overline{A} \Rightarrow A$ is dense in X. So (c) is correct.

(a) and (b) are incorrect.

As consider, $(\mathbb{R}, d)$ and $\mathbb{Q} \subseteq \mathbb{R}$

$d(x, \mathbb{Q}) = 0\ \forall x \in \mathbb{R}$

but $\mathbb{Q}$ is not closed, not compact.

Example 4.11.3. Consider the subspaces of $\mathbb{R}^2$. Which of the following are compact?
(1) $A = \{(x, y) \mid x^2 + y^2 = 1\}$
(2) $B = \{(x, y) \mid x^2 + y^2 < 1\}$
(3) $C = \{(x, y) \mid 2x^2 + 3y^2 = 1\}$
(4) $D = \{(x, y) \mid 2x^2 - y^2 \leq 1\}$

(5) $E = \{(x, y) \mid x \le y^2\}$

(6) $F = \left\{(x, y) \mid y = \sin\frac{1}{x}, x \neq 0\right\}$

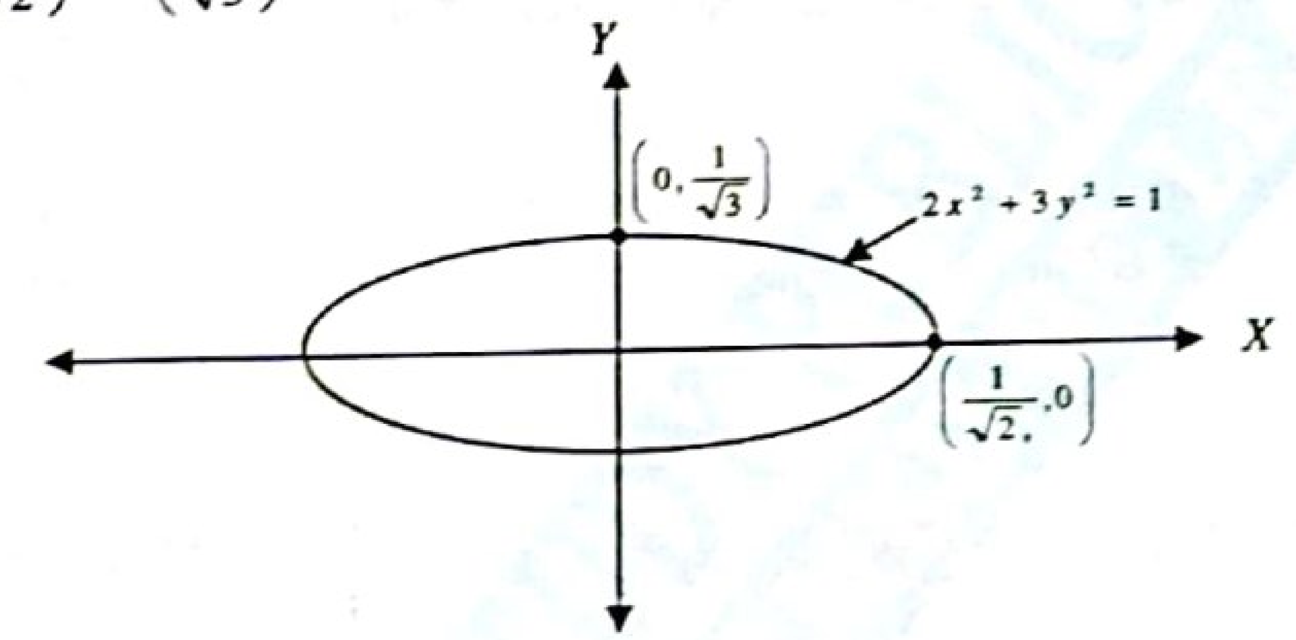
Solution:

(1) $A = \{(x, y) \mid x^2 + y^2 = 1\}$ is compact. [$\because A' = A$ closed]

(2) $B = \{(x, y) \mid x^2 + y^2 < 1\}$ is not compact. [$\because B' = \{(x, y) \mid x^2 + y^2 \le 1\} \not\subseteq B$]

(3) $C = \{(x, y) \mid 2x^2 + 3y^2 = 1\}$ is compact.

$$\because \frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1$$



(4) $D = \{(x, y) \mid 2x^2 - y^2 \le 1\}$ is not compact [$\because$ closed but not bounded]

(5) $E = \{(x, y) \mid x \le y^2\}$ is not compact. [$\because$ closed but not bounded]

(6) $F = \left\{(x, y) \mid y = \sin\frac{1}{x}, x \neq 0\right\}$ is not compact [$\because$ neither closed nor bounded]

Example 4.11.4. *Consider* $(\mathbb{R}, d)$ *and its subsets. Then which of the following are compact?*

(I) $A = \{x \in \mathbf{Q} \mid e < x < 2e\}$

(II) $B = \{x \in \mathbf{R} - \mathbf{Q} \mid 1 < x < 2\}$

(III) $C = \{x \in \mathbf{R} \mid -2 < x^2 \le 3\}$

(IV) $D = \{x \in \mathbf{R} \mid -5 < |x| \le 2\}$

Solution: (I) $A = \{x \in \mathbf{Q} \mid e < x < 2e\}$ is not compact [$\because A' = [e, 2e]$, not closed]

(II) $B = \{x \in \mathbf{R} - \mathbf{Q} \mid 1 < x < 2\}$ is not compact [$\because B' = [1, 2] \not\subseteq B$]

(III) $C = \{x \in \mathbf{R} \mid -2 < x^2 \le 3\}$ is compact. [$\because C = [-\sqrt{3}, \sqrt{3}] = C'$]

(IV) $D = \{x \in \mathbf{R} \mid -5 < |x| \le 2\}$ is compact. [$\because D = [-2, 2] = D'$]

4.12. CONNECTEDNESS

4.12.1. Separated Sets: *Two subsets A and B of a metric space X are said to be separated if both* $A \cap \overline{B}$ *and* $\overline{A} \cap B$ *are empty i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.*

Note:

(1) A necessary condition for two sets to be separated is that they are disjoint i.e. separatedness $\Rightarrow$ disjointness

As, $A \cap B \subseteq \overline{A} \cap B = \phi \Rightarrow A \cap B = \phi$.

(2) Two disjoint sets may or may not be separated sets.

For example. Consider metric space $(\mathbb{R}, d)$ and its two subsets $A = (-\infty, 1)$, $B = [1, \infty)$

$\overline{A} = (-\infty, 1], \overline{A} \cap B = \{1\}$. But $A \cap B = \phi$

4.12.2. Path: *Let A be any non-empty subset of metric space (X, d) and* $x, y \in A$, *then a continuous function* $f: [0, 1] \to A$, *where* $f(0) = x$ *and* $f(1) = y$ *is called a path in A from x to y.*

4.12.3. Path-Connected: *A subset A of X is said to be path-connected if and only if, for all* $x, y \in A$, *there is path in A from x to y.*

4.12.4. Connected Set: *If (X, d) is a metric space and* $A \subseteq X$, *then A is said to be connected if it cannot be written as disjoint union of two non-empty separated sets.*

4.12.5. Disconnected Set: *If a subset A of X is not connected, then it is said to be disconnected.*

Note: (1) Every path-connected set is connected. But converse is not true i.e. Every connected set need not be path-connected.

For example: $S = \left\{\left(x, \sin\frac{1}{x}\right) : x > 0\right\} \cup \{(0,0)\}$ is connected but not path-connected.

(2) Subsets of $\mathbb{R}$ are connected iff path-connected.

Results on connected sets:

1) If f is a continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X, then $f(E)$ is connected.

2) A subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property: if $x \in E, y \in E$, and $x < z < y$, then $z \in E$

3) Any finite subset of $\mathbb{R}$ is connected if it is singleton and disconnected if it has two or more than two elements.

4) Every connected subset of $\mathbb{R}$ is either an interval or a singleton set.

5) $\mathbb{R}$ is a connected set.

6) Subsets of two separated sets are themselves separated.

7) Two closed (open) sets are separated iff they are disjoint.

8) The union of two connected sets, having non – empty intersection, is connected.

9) Continuous image of a connected set is connected.

10) The closure of connected set is connected.

11) $\mathbb{R}^2$ is connected.

12) $\mathbb{Q}\times\mathbb{Q}$ is disconnected.

13) Any discrete space with more than one point is disconnected.

Example 4.12.1. Give example of subsets of $\mathbb{R}$ which are disjoint but not separated.

Solution: Consider Metric Space $(\mathbb{R}, d)$ and its two subsets $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$. These are disjoint but not separated.

$A=\mathbb{Q}$ and $B=\mathbb{R}-\mathbb{Q}$

$\overline{A}=\mathbb{R}\Rightarrow \overline{A}\cap B=B=\mathbb{R}-\mathbb{Q}\neq\phi$.

Example 4.12.2. Consider metric space $(\mathbb{R}, d)$ and its subset as $\mathbb{Q}$. Is it connected?

Solution: Let us take $A=\{x\in\mathbb{Q}\mid x<e\}$ and $B=\{x\in\mathbb{Q}\mid x>e\}$ be two components

Then $\overline{A}=(-\infty,e]$ and $\overline{B}=[e,\infty)$

Thus $\overline{A}\cap B=\phi$ and $A\cap\overline{B}=\phi$

$\Rightarrow A\cup B=\mathbb{Q}$

$\Rightarrow \mathbb{Q}$ is not connected

Example 4.12.3. Is $\mathbb{R}-\mathbb{Q}$ connected ?

Solution: $\mathbb{R}-\mathbb{Q}$ is not connected.

As if we take $A=\{x\in\mathbb{R}-\mathbb{Q}\mid x<2\}$ and $B=\{x\in\mathbb{R}-\mathbb{Q}\mid x>2\}$

A and B are two separated sets and $\mathbb{R}-\mathbb{Q}=A\cup B$

$\Rightarrow \mathbb{R}-\mathbb{Q}$ is not connected.

Thus, some of the disconnected sets of $(\mathbb{R}, d)$ are $\mathbb{Q}$, $\mathbb{R}-\mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z}$.

Example 4.12.4. Consider a metric space $([0, 50), d)$, where d is usual metric over $\mathbb{R}$.
Which of the following are true?

(a) $A=(0, 1)$ is an open subset of (X, d)

(b) $[1, 2)$ is an open subset of (X, d)

(c) $[0, 50)$ is a closed subset of (X, d)

Solution:

(a) True, As $A^0=A$

(b) False, As $1\in[1, 2)$, but 1 is not interior point of A

(c) False, as 50 is limit point of $[0, 50)$ but $50\notin[0, 50)$

Example 4.12.5. If d is usual metric on $\mathbb{R}$. Let $(\mathbb{Q}, d)$ be a metric space. Then which of the statements is false?

(a) $A=\{x\in\mathbb{Q}: 5\le x^2\le 17\}$ is closed set.

(b) $B=\{x\in\mathbb{Q}: 5\le x^2\le 17\}$ is closed set.

(c) $C=\{x\in\mathbb{Q}: 5\le x^2\le 17\}$ is closed set.

(d) $D=\{x\in\mathbb{Q}: 5\le x^2\le 17\}$ is closed set.

Solution: All statements are false as $\sqrt{5}$ and $\sqrt{17}$ are limit points of A, B, C, D but $\sqrt{5}, \sqrt{17}\notin A, B, C, D$.

$\therefore$ None of sets A, B, C, D is closed.

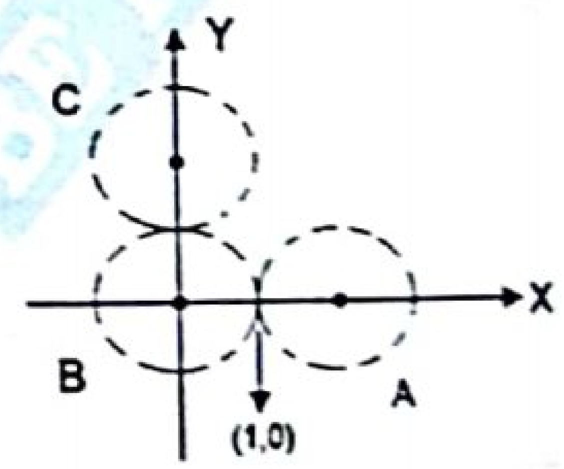
Example 4.12.6. Consider $D(a,b)(r)=D_r(a, b)=\{(x,y):(x-a)^2+(y-b)^2<r^2\}$

Now consider sets

$A=D_1(2, 0)$

$B=D_1(0, 0)$

$C=D_1(0, 2)$



Which of the following is connected?

(a) $A\cup B$ (b) $B\cup C$ (c) $A\cup B\cup C$ (d) $A\cup B\cup\{(1, 0)\}$

Solution: It is clear from diagram, only $A\cup B\cup\{(1, 0)\}$ is connected.

4.13. HOMEOMORPHISM

Let (X,d_1) and (Y,d_2) be two metric spaces. A function $f: X\to Y$ is said to be homeomorphism if

(i) f is both one-one and onto

(ii) f and f^{-1} are both continuous.

4.13.1. Homeomorphic Spaces: *A metric space (X, d_1) is said to homeomorphic to another metric space (Y, d_2), if there exists a homeomorphism of X onto Y.*

Results on Homeomorphisms:

1) *Let (X,d_1) and (Y, d_2) be metric spaces and f be a mapping from X onto Y. Then f is homeomorphism from X to Y iff the following conditions hold:*
(i) *f is one-one and onto*
(ii) *f is continuous*
(iii) *for each open set G in X, f(G) is open in Y.*

2) *The homeomorphism is an equivalence relation in the collection of all metric spaces.*

Examples.

1) *The open interval (–1,1) is homeomorphic to (a, b).*
2) *the mapping $f : \mathbb{R} \to (-1,1)$ defined by $f(x) = x/(1+|x|)$, $\forall x \in \mathbb{R}$, is a homeomorphism of the real line into the open intervals (-1,1) with the usual metric.*
3) *The intervals (a,b) and (c,d) in ($\mathbb{R}$, d) are homeomorphic.*

4.14. ISOMETRY

Let (X,d_1) and (Y,d_2) be two metric spaces. A function $f : X \to Y$ is said to be an isometry if $d_1(x,y) = d_2(f(x), f(y)) \ \forall\ x,y \in X$.

For example:

(i) *Let (X, d) be a metric space. Then Identity mapping of X to itself is an isometry.*
(ii) *The mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f((x,y)) = (x_1,y_1)$ where $x_1 = x\cos\theta - y\sin\theta$, $y_1 = x\sin\theta + y\cos\theta$, θ is constant real number, is an isometry of the plane onto itself.*
(iii) *The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)=3x+4$ is not an isometry.*

Results on isometry:

(1) *Isometry is always one- one function*
(2) *Isometry is uniformly continuous.*

Theorem. *Every isometry is a homeomorphism.*

Proof: *Let f be isometry from metric space (X,d_1) to (Y,d_2). Since each isometry is one- one and onto. So f is one-one, onto. Let a be an arbitrary element of X and take an $\in > 0$, then f is an isometry*
$\Rightarrow d_1(x,a) = d_2(f(x), f(a))\ \forall\ x \in X$
Taking $\delta = \in$, we have
$d_1(x,a) < \delta \Rightarrow d_2(f(x), f(a)) < \in$

Since f is continuous at a and a is an arbitrary point of X, it follows that f is continuous on X.
Now, f is isometry $\Rightarrow f^{-1}$ is isometry.
f^{-1} is continuous, as before.
Thus f is one-one and onto mapping such that f and f^{-1} are both continuous and hence f is a homeomorphism.

Converse is not true : *The mapping $f : \mathbb{R} \to (-1,1)$ defined by $f(x) = \frac{x}{1+|x|}$ is a homeomorphism but is not an isometry.*

PRACTICE SET – I

Exercise 1. Which one of the following subsets of $\mathbb{R}$ (with the usual metric) is NOT complete? **(GATE-2008)**

(a) [1, 2]U[3,4] (b) $[0,\infty)$
(c) [0,1) (d) $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Exercise 2. Let d_1, d_2 and d_3 be metrics on a set X with at least two elements. Which of the following is NOT a metric on X? **(GATE-2014)**

(a) min $\{d_1, 2\}$ (b) max $\{d_2, 2\}$
(c) $\frac{d_3}{1+d_3}$ (d) $\frac{d_1+d_2+d_3}{3}$

Exercise 3. Which of the following statement(s) is (are) TRUE? **(JAM-2016)**

(a) *There exists a connected set in $\mathbb{R}$ which is not compact*
(b) *Arbitrary union of closed intervals in $\mathbb{R}$ need not be compact.*
(c) *Arbitrary union of closed intervals in $\mathbb{R}$ is always closed.*
(d) *Every bounded infinite subset V of $\mathbb{R}$ has a limit point in V itself.*

Exercise 4. Let $f : X \to Y$ be a continuous map between metric spaces. Then f(X) is a complete subset of Y if **(TIFR-2014)**

(a) *the space X is compact* (b) *the space Y is compact*
(c) *the space X is complete* (d) *the space Y is complete*

Exercise 5. Consider the following subsets of $\mathbb{R}^2$: **(CMI-2013)**

$$X_1 = \left\{\left(x, \sin\frac{1}{x}\right) \middle| 0 < x < 1\right\},\ X_2 = [0, 1] \times \{0\},\ \text{and } X_3 = \{(0, 1)\}.\ \text{Then,}$$

(a) $X_1 \cup X_2 \cup X_3$ *is a connected set.*
(b) $X_1 \cup X_2 \cup X_3$ *is a path-connected set.*

(c) $X_1 \cup X_2 \cup X_3$ is not path-connected, but $X_1 \cup X_2$ is path-connected.

(d) $X_1 \cup X_2$ is not path-connected, but every open neighbourhood of a point in this set contains a smaller open neighbourhood which is path-connected.

Exercise 6. Let X be a set with the property that for any two metrices d_1 and d_2 on X, the identity map id : $(X, d_1) \to (X, d_2)$ is continuous. Which of the following are true? **(CMI-2013)**

(a) X must be singleton.
(b) X can be any finite set.
(c) X cannot be infinite.
(d) X may be infinite but not countable.

Exercise 7. Which of the following is/are metrics on $\mathbb{R}$? **(CSIR UGC NET JUNE-2011)**

(a) $d(x,y) = \min(x,y)$ (b) $d(x,y) = |x-y|$
(c) $d(x,y) = |x^2 - y^2|$ (d) $d(x,y) = |x^3 - y^3|$

Exercise 8. Which of the following define a metric on $\mathbb{R}$? **(CSIR UGC NET JUNE-2013)**

(a) $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$
(b) $d(x,y)=|x-2y|+|2y-x|$
(c) $d(x,y)=|x^2-y^2|$
(d) $d(x,y)=|x^3-y^3|$

Exercise 9. Let $f : X \to Y$ be a function from a metric space X to another metric space Y. For any Cauchy sequence $\{x_n\}$ in X, **(CSIR UGC NET JUNE-2014)**
(a) if f is continuous then $\{f(x_n)\}$ is Cauchy sequence in Y.
(b) if $\{f(x_n)\}$ is Cauchy then $\{f(x_n)\}$ is always convergent in Y.
(c) if $\{f(x_n)\}$ is Cauchy in Y, then f is continuous.
(d) $\{x_n\}$ is always convergent in X.

Exercise 10. Let (X, d) be a compact metric space. Which of the following statements are true? **(NBHM-2015)**
(a) X is complete.
(b) X is separable.
(c) If $f : X \to \mathbb{R}$ is a continuous mapping, then it maps Cauchy sequences into Cauchy sequences.

KEY POINTS

- In $\mathbb{R}^k$, every Cauchy sequence converges.
- All compact metric spaces and all Euclidean spaces are complete.
- The space $\mathbb{Q}$ of rational numbers with the metric given by $d(x, y) = |x-y|$ is not complete.
- Let E be a non compact set in $\mathbb{R}$. Then there exists a continuous function on E which is not bounded.
- Let f be a continuous mapping of a compact metric space X into a metric space Y, then f is uniformly continuous on X.
- Every connected subset of $\mathbb{R}$ is either an interval or a singleton set.
- Any discrete space with more than one point is disconnected.
- The intervals (a, b) and (c, d) in ($\mathbb{R}$, d) are homeomorphic.
- Every isometry is a homeomorphism.
- In any metric space X, every convergent sequence is a Cauchy sequence.

SOLVED QUESTIONS FROM PREVIOUS PAPERS

Example 1. Let $E = \{(x,y) \in \mathbb{R}^2 : |x| \le 1, |y| \le 1\}$. Define $f : E \to \mathbb{R}$ by $f(x,y) = \dfrac{x+y}{1+x^2+y^2}$. Then the range of f is a **(GATE-2008)**
(a) connected open set (b) connected closed set
(c) bounded open set (d) closed and unbounded set

Solution: (b) As continuous image of connected set is connected and compact set is compact.

Example 2. The set $X = \mathbb{R}$ with the metric $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$ is **(GATE-2010)**
(a) bounded but not compact (b) bounded but not complete
(c) complete but not bounded (d) compact but not complete

Solution: (a)

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

We know that, $|x-y| < 1 + |x-y|$

$\Rightarrow \frac{|x-y|}{1+|x-y|} < 1 \quad \Rightarrow d(x,y) < 1 \ \forall\, x, y \in \mathbb{R}$

$\Rightarrow d(x,y)$ is bounded.

We know that $(\mathbb{R}, e)$ is a metric space, where $e(x,y) = |x-y| \ \forall\, x, y \in \mathbb{R}$

Consider $d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{e(x,y)}{1+e(x,y)} \ \forall\, x, y \in \mathbb{R}$

Then d and e are equivalent metrics. If possible, let $(\mathbb{R}, d)$ is compact metric space, then every d – open cover of $\mathbb{R}$ has a finite subcover. Consequently, every e – open cover of $\mathbb{R}$ has a finite subcover.

$\Rightarrow (\mathbb{R}, e)$ is compact, which is a contradiction.

Hence, $(\mathbb{R}, d)$ is not compact $\Rightarrow$ (a) is correct.

Example 3. Let d_1 and d_2 denote the usual metric and the discrete metric on $\mathbb{R}$, respectively.

Let $f: (\mathbb{R}, d_1) \longrightarrow (\mathbb{R}, d_2)$ be defined by $f(x) = x,\ x \in \mathbb{R}$. Then **(GATE-2015)**

(a) f is continuous but f^{-1} is NOT continuous

(b) f^{-1} is continuous but f is NOT continuous

(c) both f and f^{-1} are continuous

(d) neither f nor f^{-1} is continuous

Solution: (b) Open sets in $(\mathbb{R}, d_1)$ are $\{(a, b) : a,b \in \mathbb{R}\}$ or any union of open intervals. Open sets in $(\mathbb{R}, d_2)$ are $\{\{x\} : x \in \mathbb{R}\}$ or any union of singletons. $f: (\mathbb{R}, d_1) \longrightarrow (\mathbb{R}, d_2)$, where $f(x) = x$.

Suppose, f is continuous, then as $\{x\}$ is open in $(\mathbb{R}, d_2)$. $f^{-1}(\{x\}) = \{x\}$, is open in $(\mathbb{R}, d_1)$, but this is a contradiction. As, $\{x\}$ is not open in $(\mathbb{R}, d_1)$. Hence, f is not continuous.

Let $f^{-1} : (\mathbb{R}, d_2) \longrightarrow (\mathbb{R}, d_1)$, let U be an open set in $(\mathbb{R}, d_1)$. Then $(f^{-1})^{-1}(U) = U$, which is open in $(\mathbb{R}, d_2)$.

Hence, f^{-1} is continuous.

Example 4. The set $\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ is **(IIT JAM 2014)**

(a) connected but NOT compact in $\mathbb{R}$.

(b) compact but NOT connected in $\mathbb{R}$.

(c) compact and connected in $\mathbb{R}$.

(d) neither compact nor connected in $\mathbb{R}$.

Solution: (a) $S = \left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\} = [0,1)$, which is connected, bounded but not closed as it's derived set is $[0, 1]$ $\therefore$ it is not compact.

Example 5. Let A be the 2×2 matrix $\begin{bmatrix} \sin\frac{\pi}{18} & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \sin\frac{\pi}{18} \end{bmatrix}$. Then the smallest number $n \in \mathbb{N}$ such that $A^n = I$ is

(a) 3 (b) 9 (c) 18 (d) 27 **(TIFR-2015)**

Solution: (b) $A = \begin{bmatrix} \sin\frac{\pi}{18} & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \sin\frac{\pi}{18} \end{bmatrix}$

Clearly $\frac{\pi}{2} - \frac{4\pi}{9} = \frac{9\pi - 8\pi}{18} = \frac{\pi}{18}$

Therefore, $A = \begin{bmatrix} \sin\left(\frac{\pi}{2} - \frac{4\pi}{9}\right) & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \sin\left(\frac{\pi}{2} - \frac{4\pi}{9}\right) \end{bmatrix} \Rightarrow A = \begin{bmatrix} \cos\frac{4\pi}{9} & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \cos\frac{4\pi}{9} \end{bmatrix}$

which is an orthogonal matrix of order 2×2 and $|A| = 1$

$\therefore$ A is a rotation matrix, that rotates each vector by angle $\frac{4\pi}{9}$ i.e. this matrix is equivalent to $e^{\frac{4\pi}{9}i}$

and as $e^{\left(\frac{4\pi}{9}i\right)^9} = 1$. Therefore, $A^9 = I \Rightarrow$ Option (b) is correct.

Example 6. X is a metric space. Y is a closed subset of X such that the distance between any two points in Y is atmost 1. Then **(TIFR-2014)**

(a) Y is compact

(b) any continuous function from $Y \longrightarrow \mathbb{R}$ is bounded

(c) Y is not an open subset of X

(d) none of the above

Solution: (d) Let $X = \mathbb{R}$. Define $d: X \times X \longrightarrow \mathbb{R} \cup \{0\}$ be discrete metric $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$

Let $Y = \mathbb{Q}$, which is closed subset of X {$\because$ X is discrete metric and $d(x,y) = 1 \ \forall\, x,y \in Y; x \neq y$.}

But Y is not compact a $\mathbb{Q} = \cup_{a \in \mathbb{Q}}\{a\}$ is an open cover.

We have each set is open here and it has no finite subcover.

$\Rightarrow$ option (a) is incorrect.

Now, if we take any $f: Y \longrightarrow \mathbb{R}$, $\mathbb{R}$ with usual metric, then f is continuous because under discrete metric, every function is continuous.

But f may not be bounded.

For example, $f(x) = x$ (identity function)

So, option (b) is also incorrect.

$\mathbb{Q}$ is open in $\mathbb{R}$.

[$\because$ under discrete metric, every set is open as well as closed]

$\Rightarrow$ option (c) is incorrect.

So, option (d) is correct.

Example 7. *Let X be a connected subset of real numbers. If every element of X is irrational, then the cardinality of X is* **(CSIR UGC NET DEC-2011)**

(a) infinite (b) countably infinite

(c) 2 (d) 1

Solution: (d) *We know that in $\mathbb{R}$, connected subsets are either intervals or singletons. If X is an interval, then it will contain rationals also, but X contains only irrationals.*

$\therefore$ X is singleton.

Example 8. *Let (X,d) be a metric space and let $A \subseteq X$. For $x \in X$, define $d(x, A) = \inf\{d(x,a) : a \in A\}$. If $d(x,A)=0$ for all $x \in X$, then which of the following assertions must be true?* **(CSIR UGC NET DEC-2011)**

(a) A is compact (b) A is closed (c) A is dense in X (d) A=X

Solution: (c) *Given that (X, d) is a metric space and $A \subseteq X$. Now, for all $x \in X$,*

$d(x, A) = 0 \quad \forall x \in X$

$\Rightarrow$ Either $x \in A$ or x is a limit point of A

$\Rightarrow \bar{A} = X$

$\Rightarrow$ A is dense in X.

$\therefore$ option (c) is correct.

Further, let $(\mathbb{R}, d)$ be the usual metric space i.e. $X = \mathbb{R}$ and $A = \mathbb{Q} \subseteq \mathbb{R}$

Clearly, $d(x, \mathbb{Q}) = 0 \ \forall x \in \mathbb{R}$

But $\mathbb{Q}$ is neither closed nor compact and $\mathbb{Q} \neq \mathbb{R}$ i.e. $A \neq X$.

$\therefore$ Options (a), (b), (d) are incorrect.

Example 9. *Which of the following are metrics on $C=\{f:[0,1] \to \mathbb{R}$ is a continuous function$\}$* **(CSIR UGC NET JUNE-2012)**

(a) $d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$.

(b) $d(f,g) = \inf\{|f(x)-g(x)| : x \in [0,1]\}$.

(c) $d(f,g) = \int_0^1 |f(x)-g(x)|\,dx$.

(d) $d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\} + \int_0^1 |f(x)-g(x)|\,dx$.

Solution: (a,c,d) *We have, $C = \{f:[0,1] \to \mathbb{R}$ is a continuous function$\}$*

For option (a)

$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$

(i) $d(f,g) \geq 0$ and $d(f,g) = 0 \Leftrightarrow f = g$

(ii) $d(f,g) = d(g,f)$ are obvious.

(iii) Let $f, g, h \in C$, then for all $x \in [0,1]$, we have

$|f(x)-g(x)| = |f(x)-h(x)+h(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$

$\therefore \sup(|f(x)-g(x)|) \leq \sup(|f(x)-h(x)|) + \sup(|h(x)-g(x)|)$

$\Rightarrow d(f,g) \leq d(f,h) + d(h,g)$.

Hence, the triangle inequality is proved

$\therefore d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$ is a metric on C.

Hence, option (a) is correct

For option (b)

As for metric $d(f, g) = \inf\{|f(x) - g(x)| : x \in [0, 1]\}$, triangle inequality does not holds. So option (b) is incorrect.

For option (c)

$d(f,g) = \int_0^1 |f(x)-g(x)|, \ \forall f, g \in C$

The conditions

(i) $d(f,g) \geq 0$ and $d(f,g)=0 \Leftrightarrow f = g$

(ii) $d(f,g) = d(g,f)$ are obvious.

Let $f, g, h \in C$, then $\forall x \in [0,1]$

$\int_0^1 |f(x)-g(x)|\,dx = \int_0^1 |f(x)-h(x)+h(x)-g(x)|\,dx \leq \int_0^1 (|f(x)-h(x)| + |h(x)-g(x)|)\,dx$

$= \int_0^1 |f(x)-h(x)|\,dx + \int_0^1 |h(x)-g(x)|\,dx$

$\Rightarrow d(f,g) \leq d(f,h) + d(h,g)$

Hence, the triangle inequality is proved.

Therefore, $d(f,g) = \int_0^1 |f(x)-g(x)|\,dx$ is a metric on C.

Hence, option (c) is correct.

We know sum of two metric spaces is a metric space.

$\therefore$ option (d) is also correct.

Example 10. *Let $D_{(a,b)}(r) = \{(x,y) : (x-a)^2 + (y-b)^2 < r\}$. Which of the following subsets of $\mathbb{R}$ are connected?* **(CSIR UGC NET JUNE-2012)**

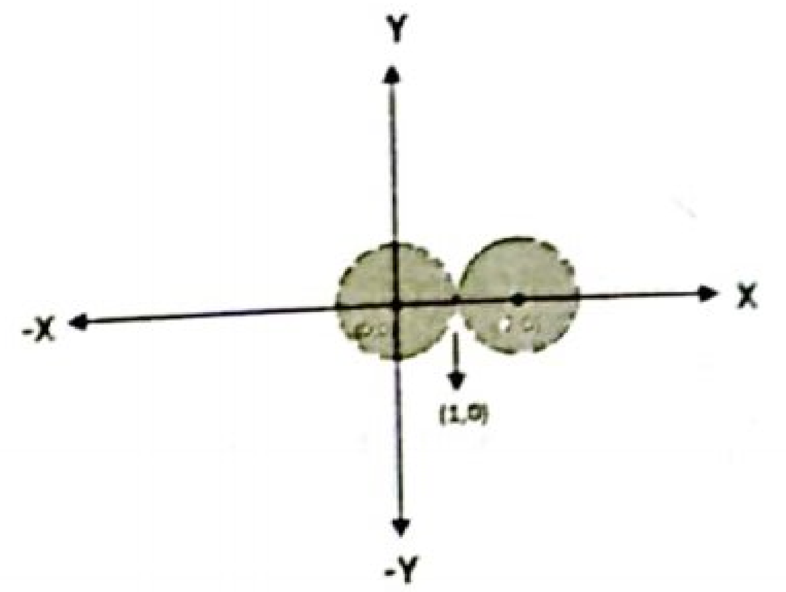
(a) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(2,0)}(1)$ (b) $D_{(0,0)}(1) \cup D_{(2,0)}(1)$

(c) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(0,2)}(1)$ (d) $D_{(0,0)}(1) \cup D_{(0,2)}(1)$

Solution: (a) Given : $D_{(a,b)}(r) = \{(x,y) : (x-a)^2 + (y-b)^2 < r\}$

Here, $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(2,0)}(1) = \{(x,y) : x^2+y^2<1\} \cup \{(1,0)\} \cup \{(x,y) : (x-2)^2+y^2<1\}$

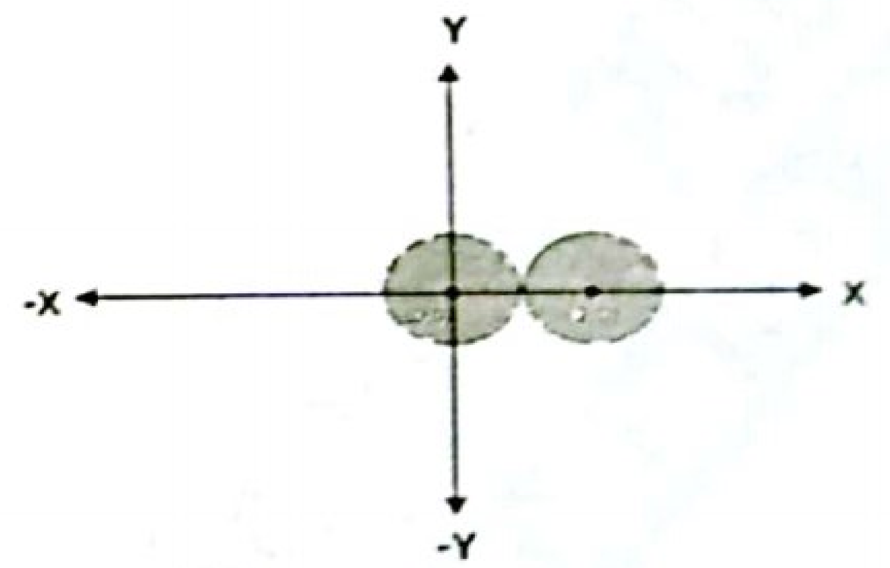
(a)



From figure it is clear that it is a connected set

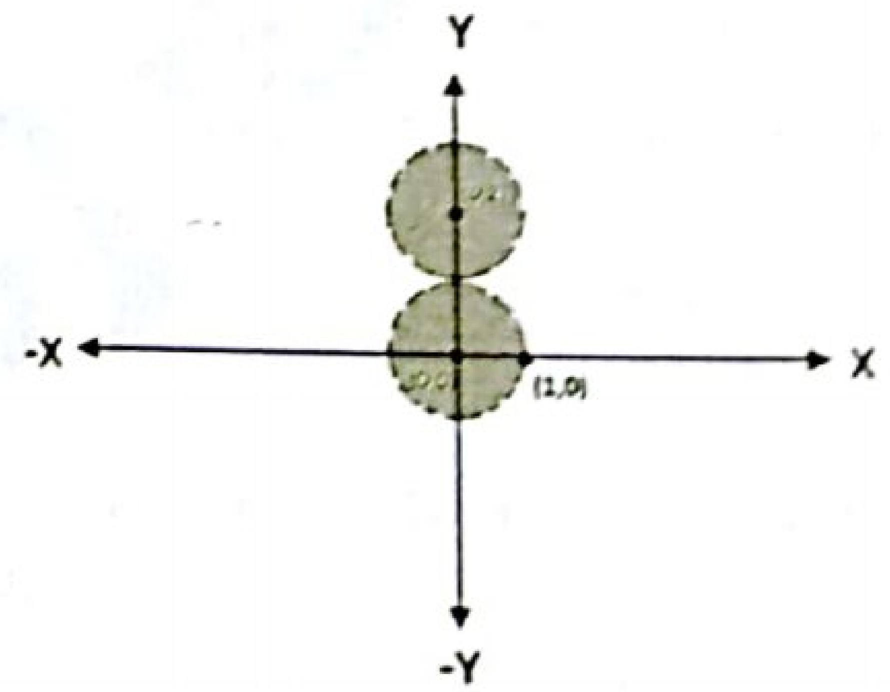
∴ option (a) is correct

(b) $D_{(0,0)}(1) \cup D_{(2,0)}(1) = \{(x,y) : x^2+y^2<1\} \cup \{(x,y) : (x-2)^2+y^2<1\}$



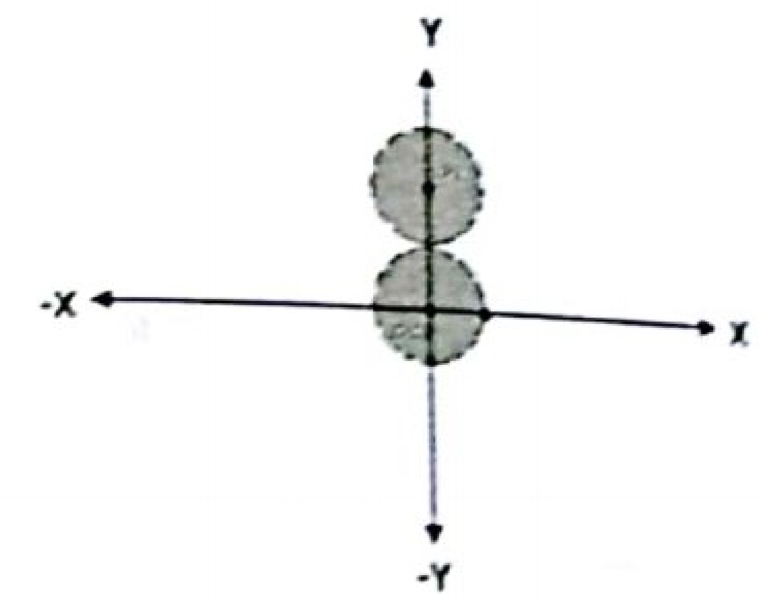
From figure it is clear that, set is disconnected as(1,0) is not in set.

(c) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(0,2)}(1) = \{(x,y) : x^2+y^2<1\} \cup \{(1,0)\} \cup \{(x,y) : x^2+(y-2)^2<1\}$



Clearly, above set is disconnected as (0,1) is not in set.

(d) $D_{(0,0)}(1) \cup D_{(0,2)}(1) = \{(x,y) : x^2+y^2<1\} \cup \{(x,y) : x^2+(y-2)^2<1\}$



Clearly, above set is disconnected as (0,1) is not in set.

Example 11. Let $X = [-1,1] \times [-1,1]$,

$A = \{(x,y) \in X : x^2 + y^2 = 1\}$,

$B = \{(x,y) \in X : |x| + |y| = 1\}$,

$C = \{(x,y) \in X : xy = 0\}$ and

$D = \{(x,y) \in X : x = \pm y\}$. Then **(CSIR UGC NET DEC-2012)**

(a) A is homeomorphic to B.
(b) B is homeomorphic to C.
(c) C is homeomorphic to D.
(d) D is homeomorphic to A.

Solution: (a,c)

We know that a set Y is homeomorphic to Z, if ∃ a function $f : Y \rightarrow Z$ such that

(i) 'f' is continuous,

(ii) 'f' is 1-1 & onto, or

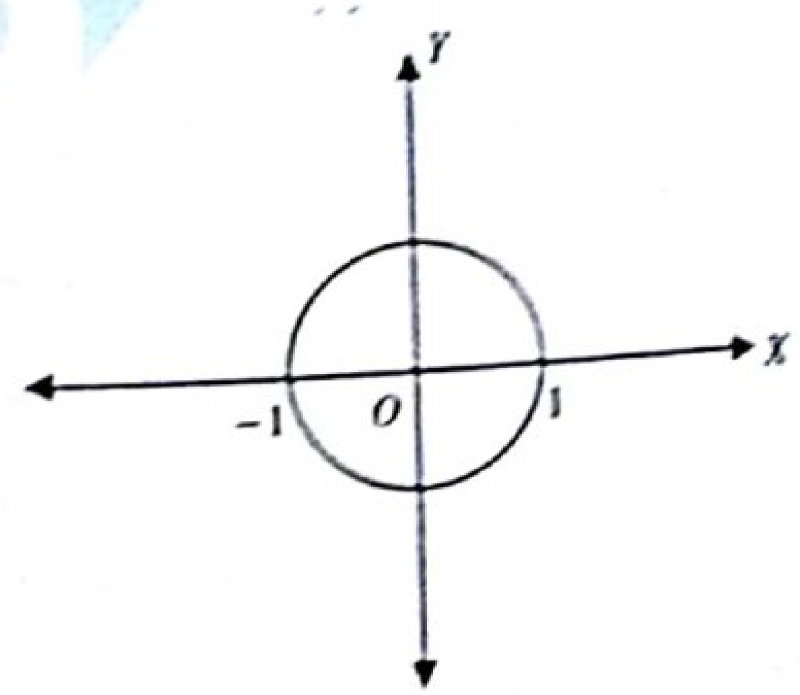
(iii) f^{-1} is continuous for each closed set H in Y, f(H) is closed is Z

Also, we know that in a continuous function, image of connected set is connected.

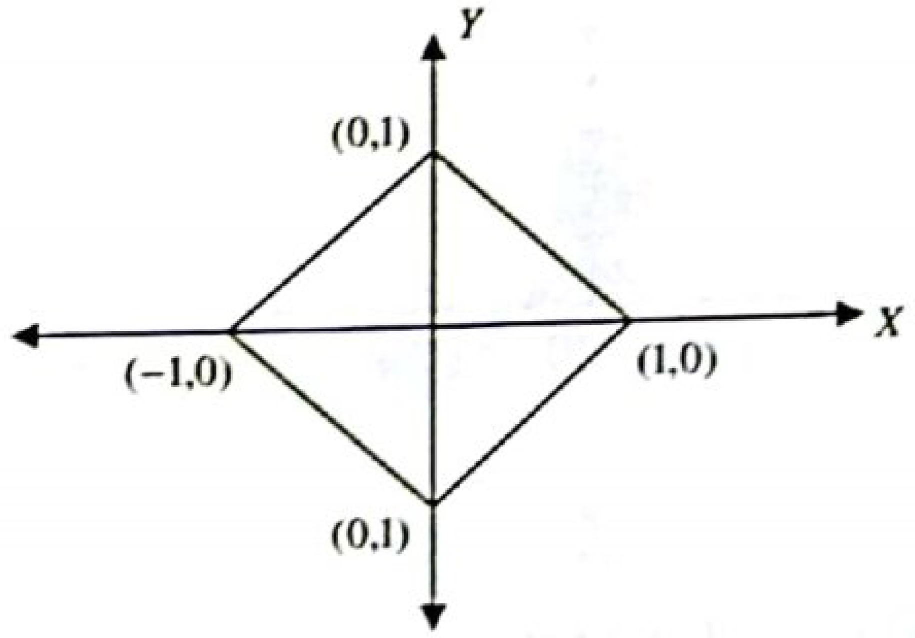
or we can say two sets are homeomorphic if both have same topological properties.

Graphs of sets A, B, C, D are given below:-

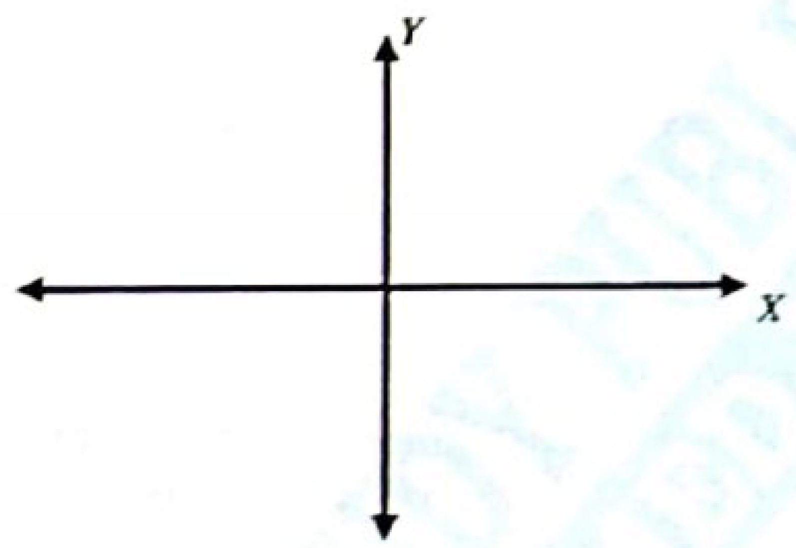
(A)



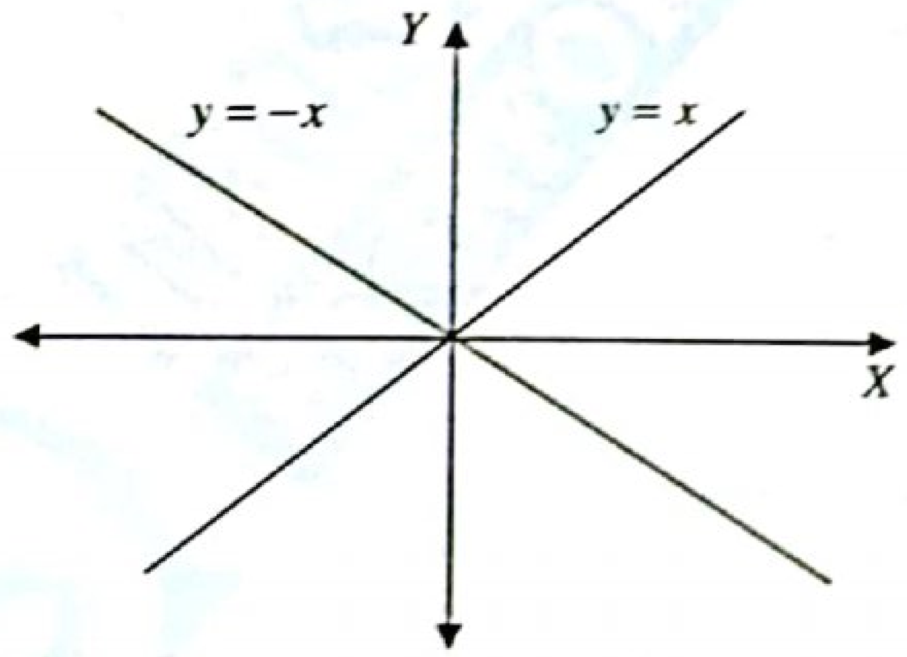
(B)



(C)



(D)



From figure (A) and (B) it is clear that sets A and B are homeomorphic.
Also, set D can be obtained just by rotating set C by angle 45^0
As we know, every rotation is isometry and every isometry is homeomorphic.

$\therefore$ sets C and D are also homeomorphic
Thus option (d) is also correct.
B is not homeomorphic to C, as if (0,0) (one point) is removed from C, then it is divided into four components, whereas B have only one connected component after removal of one point.
Similarly D cannot be homeomorphic to A.
Thus, options (b), (d) are incorrect.

Example 12. Let A and B be two disjoint nonempty subsets of $\mathbb{R}^2$ such that $A \cup B$ is open in $\mathbb{R}^2$. Then, **(CSIR UGC NET DEC-2013)**

(a) if A is open and $A \cup B$ is connected, then B must be closed in $\mathbb{R}^2$.
(b) if A is closed, then B must be open in $\mathbb{R}^2$.
(c) if both A and B are connected, then $A \cup B$ must be disconnected.
(d) if $A \cup B$ is disconnected, then both A and B are open.

Solution: (b,d) option (b) and (d) are correct

Given A, B are two disjoint non-empty subsets of $\mathbb{R}^2$ such that $A \cup B$ is open in $\mathbb{R}^2$
For option (a)
Take,
$A = (0,1) \times (0,1)$ and $B = [1,2) \times (0,1)$.
Then, $A \cup B = (0,2) \times (0,1)$
Clearly, A is open and $A \cup B$ is connected but B is not closed in $\mathbb{R}^2$.
For option (b)
Since A is closed $\Rightarrow A^C$ is open

Also $A \cup B$ is open

$\Rightarrow A^C \cap (A \cup B)$ is open and $A^C \cap (A \cup B) = B - A$
$\therefore B - A$ is open

$\Rightarrow B$ is open [$\because$ A is closed]
$\therefore$ option (b) is correct.
For option (c)
Let $A = (0,1) \times (0,1)$. So A is connected and non empty in $\mathbb{R}^2$
Let $B = (1,2) \times (0,1)$. So B is connected and non empty in $\mathbb{R}^2$
$\therefore A \cup B = (0,2) \times (0,1)$, which is open in $\mathbb{R}^2$, but clearly $A \cup B$ is connected in $\mathbb{R}^2$. So, option (c) is incorrect.

Since $A \cup B$ is disconnected and $A \cup B$ is open $\Rightarrow$ A and B are open

[$\because$ if any of them is closed, then $A \cup B$ cannot be open]

Example 13. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, let $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$. Then Z(f) is always **(CSIR UGC NET JUNE-2014)**

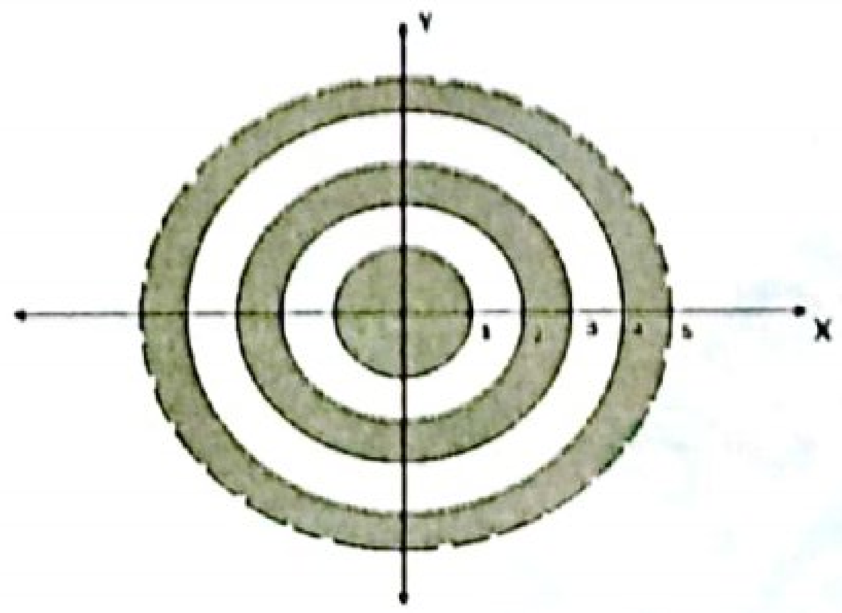
(a) compact (b) open (c) connected (d) closed

Solution: (d) Given that, $f : \mathbb{R} \to \mathbb{R}$, is a continuous function and $Z(f) = \{x \in \mathbb{R} : f(x) = 0\} = f^{-1}(\{0\})$
Since $f : \mathbb{R} \to \mathbb{R}$ is continuous so for any closed subset E of $\mathbb{R}$, its inverse image $f^{-1}(E)$ is also closed in $\mathbb{R}$.
Here {0} is closed in $\mathbb{R}$ so $Z(f) = f^{-1}(\{0\})$ is closed in $\mathbb{R}$.
So option (d) is correct.

Example14. Let $X = \{(x,y)\in \mathbb{R}^2 : x^2+y^2<5\}$ and $K = \{(x,y)\in \mathbb{R}^2 : 1\le x^2+y^2\le 2 \text{ or } 3\le x^2+y^2\le 4\}$. **(CSIR UGC NET JUNE-2014)**
Then,

(a) $X\backslash K$ has three connected components.
(b) $X\backslash K$ has no relatively compact connected component in X.
(c) $X\backslash K$ has two relatively compact connected components in X.
(d) all connected components of $X\backslash K$ are relatively compact in X.

Solution: (a,c) Given $X = \{(x,y)\in \mathbb{R}^2 : x^2+y^2<5\}$ and $K = \{(x,y)\in \mathbb{R}^2 : 1\le x^2+y^2\le 2 \text{ or } 3\le x^2+y^2\le 4\}$
Graph of $X\backslash K$ is given below



From graph it is clear that $X\setminus K$ has three connected components.
Also $X\backslash K$ has two relatively compact connected components in X.

$\therefore$ options (a) and (c) are correct and (b), (d) are incorrect.

Example 15. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function.
Let $G = \{(x,f(x)) : x\in X\}$ be the graph of f. Then **(CSIR UGC NET DEC-2014)**

(a) G is homeomorphic to X.
(b) G is homeomorphic to $\mathbb{R}$.
(c) G is homeomorphic to $X\times\mathbb{R}$.
(d) G is homeomorphic to $\mathbb{R}\times X$.

Solution: (a) Let $X = \{0, 1\}$ and d is metric on X defined as $d(x,y)=\begin{cases}1, & x\ne y\\ 0, & x=y\end{cases}$

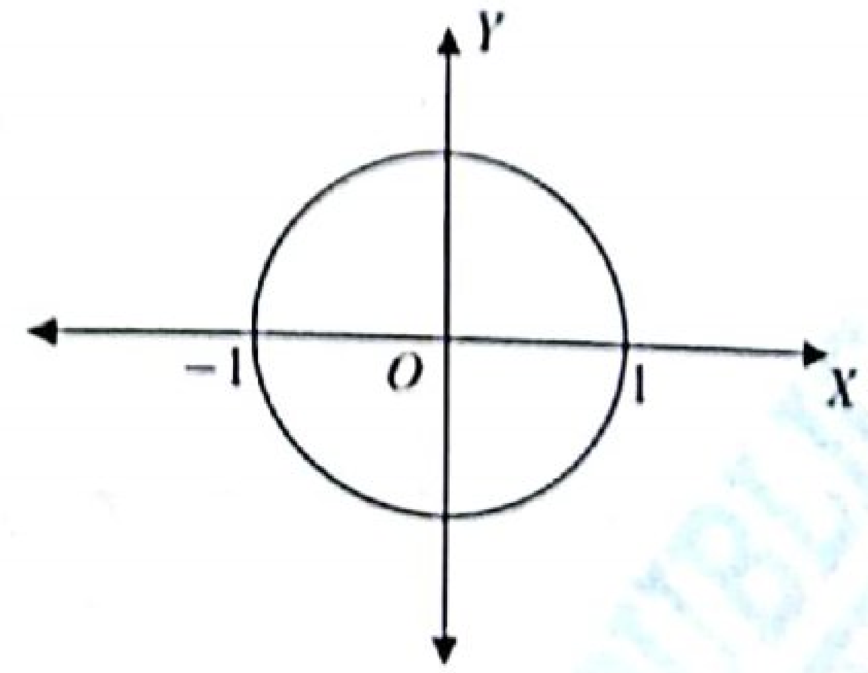
let $f: X\to\mathbb{R}$ is continuous function defined by $f(x) = x$
$\therefore G = \{(0, 0), (1, 1)\}$
Clearly, G is homeomorphic to X
$\therefore$ options (b), (c), (d) are incorrect and (a) is correct.

Example 16. Let $X = \{(a,b)\in\mathbb{R}^2 : a^2+b^2=1\}$ be the unit circle inside $\mathbb{R}^2$. Let $f : X\to\mathbb{R}$ be a continuous function. Then: **(CSIR UGC NET DEC-2014)**

(a) Image (f) is connected.
(b) Image (f) is compact.

(c) the given information is not sufficient to determine whether image (f) is bounded.
(d) f is not injective.

Solution: (a,b,d)
Given $f : X\to\mathbb{R}$ is continuous function, where $X = \{(a, b)\in\mathbb{R}^2 : a^2+b^2=1\}$
Clearly, X is unit circle centered at origin
Thus X is connected and compact



Also, we know that continuous image of compact set is compact and connected set is connected.
$\therefore$ options (a) and (b) are correct.
let $f : X\to\mathbb{R}$ be a continuous defined by $f(x, y) = x$
Here, $f(0, 1) = 0, f(0, -1) = 0$
Thus, 'f' is not one-one
$\therefore$ option (d) is also correct.

ASSIGNMENT 4.1

NOTE: CHOOSE THE BEST OPTION

1. If (X, ρ) is metric space, then for all $x, y, z \in X$
   (a) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ (b) $\rho(x, y) \ge \rho(x, z) + \rho(z, y)$
   (c) $\rho(x, y) \le 0$ (d) none of these

2. If (X, ρ) is metric space, then for all $x, y \in X$
   (a) $\rho(x, y) \le 0$ (b) $\rho(x,y) = 0$ for some $x \ne y$
   (c) $\rho(x,y)=0$ iff $x=y$ (d) none of these

3. If (X, ρ) is metric space, then for all $x,y \in X$
   (a) $\rho(x, y) = 0$ then $x = y$ (b) $\rho(x, y) = 0$ then some $x \ne y$
   (c) $\rho(x, y) < 0$ for some $x \ne y$ (d) $\rho(x, y) < 0$ for some $x = y$

4. The closure of any non-empty set E of a metric space is
   (a) closed set (b) open set (c) null set (d) none of these

5. The complement of non-empty open set of metric space is
   (a) open set (b) closed set (c) null set (d) none of these

6. The complement of non-empty closed set of metric space is
   (a) open set (b) closed set (c) null set (d) none of these

7. A metric space X is compact if
   (a) it is complete (b)it is incomplete (c) it is unbounded (d)none of these

8. A sequentially compact metric space is
   (a) totally bounded (b) unbounded (c) non-compact (d) none of these

9. If every Cauchy sequence $\langle x_n \rangle$ in metric space is convergent, then every sequence in metric space is
   (a) convergent (b) divergent (c) may not convergent (d) none of these

10. The union of any finite collection of non empty closed sets is
    (a) open set (b) closed set (c) null set (d) none of these

11. If every sequence $\langle x_n \rangle$ in metric space X is convergent, then every Cauchy sequence $\langle x_n \rangle$ is also
    (a) convergent (b) divergent (c) constant (d) none of these

12. The set $X=\mathbb{R}$ with the metric $d(x,y)=\dfrac{|x-y|}{1+|x-y|}$ is
    (a) bounded but not compact (b) bounded but not complete
    (c) complete but not bounded (d) compact but not complete

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

13. The union of any collection of open sets of a metric space is not
    (a) open set (b) closed set (c) null set (d) semi open set

14. The union of two non-empty closed sets of a metric space is not
    (a) open set (b) closed set (c) null set (d) none of these

15. A continuous image of compact set is
    (a) compact (b) non-compact (c) unbounded (d) closed

16. A compact subset of a metric space is
    (a) closed (b) open (c) unbounded (d) bounded

17. If A is a closed subset of complete metric space, then
    (a) A is complete metric space (b) A is incomplete metric space
    (c) A is bounded (d) none of these

18. The metric space (X, ρ) is
    (a) open set (b) neither open nor closed
    (c) null set (d) closed set

19. If A and B are respectively open and closed sets in $\mathbb{R}^n$, then
    (a) $A \setminus B$ is open (b) $B \setminus A$ is closed (c) $A \setminus B$ is closed (d) $B \setminus A$ is open

20. If x is an accumulation point of $S \subset \mathbb{R}^n$, then
    (a) every open ball B(x) contains infinitely many points.
    (b) every open ball B(x) contains finitely many points.
    (c) every open ball B(x) does not contain many of its points.
    (d) every open ball B(x) contains x.

21. Which of the following(s) is/are true?
    (a) $S \subset \mathbb{R}^n$ is closed, if it contains all its adherent points.
    (b) $S \subset \mathbb{R}^n$ is closed, if it does not contain all its adherent points.
    (c) $S \subset \mathbb{R}^n$ is closed, if some adherent points does not belong to S.
    (d) $S \subset \mathbb{R}^n$ is closed, if $\mathbb{R}^n \setminus S$ is open.

22. *Which of the following(s) is/are true?*
   (a) If a set $S \subset \mathbb{R}^n$ contains all its adherent points, then S is closed set.
   (b) If a set $S \subset \mathbb{R}^n$ contains all its adherent points, then S is not a closed set.
   (c) If $\mathbb{R}^n \setminus S$ is open, then $S \subset \mathbb{R}^n$ is closed.
   (d) If $\mathbb{R}^n \setminus S$ is open, then $S \subset \mathbb{R}^n$ is open.

23. If $S \subset \mathbb{R}^n$, then
   (a) S is compact $\Rightarrow$ S is closed and bounded.
   (b) S is closed and bounded $\Rightarrow$ S is compact.
   (c) S is closed and bounded $\Leftrightarrow$ S is compact.
   (d) S is closed and bounded, then there is an open covering of S such that a finite sub collection of open covering also covers S.

24. If $\{Q_1, Q_2, ...\}$ is a countable collection of non-empty sets in $\mathbb{R}^n$. Such that
   (i) $Q_{k+1} \subset Q_k$
   (ii) each set Q_k is closed and bounded.
   (a) $\bigcap_{k=1}^{\infty} Q_k$ is open (b) $\bigcap_{k=1}^{\infty} Q_k$ is closed
   (c) $\bigcap_{k=1}^{\infty} Q_k$ is empty (d) $\bigcap_{k=1}^{\infty} Q_k$ is non empty

25. Let X is metric space
   (a) if X is sequentially compact, then X is compact.
   (b) if X is sequentially compact, then X is not compact.
   (c) if X is compact, X is totally bounded.
   (d) if X is compact, X is totally unbounded.

26. Which of the following(s) is/are true?
   (a) Closed subset of a compact space is compact .
   (b) If X is compact, F is closed subset of X and U be an open covering for F. Then, $U \cup F^C$ is an open covering for X.
   (c) A closed subset of compact space is not compact.
   (d) If X is compact, F is closed subset of X and U is an open covering for F, then U has a finite subcovering.

27. If $<f_n>$ is a sequence of mappings of a countable set D into a metric space Y such that for each $x \in D$, the closure of the set $\{f_n(x) : 0 \le n < \infty\}$ is compact, then
   (a) there is a subsequence $< f_m >$ that converges for each x in D.
   (b) there is subsequence $< f_m >$ that diverges for each x in D.
   (c) for each $x \in D$, there is a subsequence $< f_m >$ that converges.
   (d) for each $x \in D$, there is a subsequence $< f_m >$ that diverges.

28. Which of the following(s) is/are correct?
   (a) every integer $n > 1$ is either prime or product of primes.
   (b) if n is not prime, there is $1 < d < n$ such that d / n.
   (c) $n = 4,6,8,9,10,...$ are prime numbers.
   (d) $n=2,3,5,...$ are product of prime numbers.

29. Which of the following(s) is/are continuous map from $\mathbb{R}$ to $\mathbb{R}$, where $(\mathbb{R},d)$ is usual metric?
   (a) $f(x)=x^2$ (b) $f(x)=\sin x$
   (c) Both (a) and (b) (d) Neither (a) nor (b)

30. Which of the following(s) is/are true?
   (a) Any closed interval with usual metric is not compact.
   (b) Any closed interval with usual metric is compact.
   (c) The discrete space (X,d), where X is finite set, is compact.
   (d) The usual metric space $(\mathbb{R},d)$ is not compact.

31. Let $X = \{x : 0 < d(0,x) \le 1, \text{ and } x \in \mathbb{R}^2\}$, where $0=(0,0)$ and d is the usual metric on X. Then,
   (a) X is closed (b) X is bounded
   (c) X is compact (d) X is not compact

32. Which of the following(s) is/are true?
   (a) $S \subset \mathbb{R}^n$ is closed $\Rightarrow S = \bar{S}$
   (b) $S = \bar{S} \Rightarrow S \subset \mathbb{R}^n$ is closed
   (c) $S \subset \mathbb{R}^n$ is closed $\Leftrightarrow S = \bar{S}$
   (d) $S = \bar{S} \Leftrightarrow S \subset \mathbb{R}^n$ is open

ASSIGNMENT 4.2

NOTE: CHOOSE THE BEST OPTION

1. If X is a non-empty set, $\rho : X \to \mathbb{R}$, (X, ρ) is pseudo-metric, then $\forall\, x, y \in X$
   (a) $x = y \Rightarrow \rho(x, y) = 0$ (b) $x \neq y \Rightarrow \rho(x, y) = 0$
   (c) $\rho(x, y) = 0$ for some $x \neq y$ (d) none of these

2. If E is a non-empty closed set of metric space, then the closure $\overline{E}$ of E is
   (a) $\overline{E} \subseteq E$ (b) $E \subset \overline{E}$
   (c) $\overline{E} = \phi$ (d) none of these

3. If ρ is an extended metric on set X, then for some $x, y \in X$
   (a) $\rho(x, y) < \infty$ (b) $\rho(x, y) = +\infty$
   (c) $\rho(x, y) > \infty$ (d) none of these

4. If (X, ρ) is a metric space, then diameter of non-empty set $E \subset X$ is equal to
   (a) $\sup\{\rho(x, y) : x, y \in E\}$ (b) $\inf\{\rho(x, y) : x, y \in E\}$
   (c) $\{\rho(x, y) : x, y \in E\}$ (d) none of these

5. A set O of a metric space (X, ρ) is open if $x \in O$, $\exists\, \delta > 0$, such that
   (a) all y with $\rho(x, y) < \delta$ belongs to O (b) all y with $\rho(x, y) < \delta$ belongs to X
   (c) all y with $\rho(x, y) < \delta$ not belongs to O (d) none of these

6. The set E is nowhere dense if
   (a) closure of E contains no non-empty open sets
   (b) closure of E contains non-empty open sets
   (c) closure of E contains empty open set
   (d) none of these

7. A metric space X is sequentially compact if
   (a) every sequence $\langle x_n \rangle$ from X have convergent subsequence
   (b) every sequence $\langle x_n \rangle$ from X have divergent subsequence
   (c) (a) and (b) both false
   (d) (a) and (b) both true

8. A metric space X is compact, then
   (a) it is sequentially compact
   (b) it is not sequentially compact
   (c) it does not satisfies Bolzano Weierstrass
   (d) none of these

9. A metric space (X, ρ) is complete if
   (a) every sequence in X is convergent
   (b) every sequence in X is divergent
   (c) every Cauchy sequence in X is convergent
   (d) every Cauchy sequence in X is divergent

10. A compact subset of metric space is
   (a) open (b) bounded
   (c) unbounded (d) none of these

11. If f is uniformly continuous mapping of metric space X into metric space Y, then
   (a) $\langle x_n \rangle$ is Cauchy sequence in $X \Rightarrow \langle f(x_n) \rangle$ is Cauchy sequence in Y
   (b) $\langle x_n \rangle$ is a sequence in $X \Rightarrow \langle f(x_n) \rangle$ is a sequence in Y
   (c) $\langle x_n \rangle$ is Cauchy sequence in $X \not\Rightarrow \langle f(x_n) \rangle$ is Cauchy sequence in Y
   (d) none of these

12. Let $\langle x_n \rangle$ be a sequence from metric space (X, ρ)
   A : x is a limit of x_n, then x is a cluster point.
   B : x is a cluster point of x_n, then x is a limit of x_n.
   (a) only A is true (b) only B is true
   (c) A and B both true (d) A and B both false

13. If f is continuous real-valued function on compact space, then
   (a) f is unbounded (b) f is bounded
   (c) f is constant (d) none of these

14. Let E be the set of all rationals p such that $2 < p^2 < 3$. Then E is
   (a) compact in $\mathbb{Q}$ (b) closed and bounded in $\mathbb{Q}$
   (c) not compact in $\mathbb{Q}$ (d) closed and unbounded in $\mathbb{Q}$

15. Suppose $U = \{x \in \mathbb{Q} : 0 \le x \le 1\}$ and $V = \{x \in \mathbb{Q} : 0 < x < 2\}$. Let n and m be the number of connected components of U and V respectively. Then
   (a) $m = n = 1$ (b) $m = n \neq 1$
   (c) $m = 2n$, where m, n finite (d) $m > 2n$

16. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $T = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$ be the subsets of the metric space $\mathbb{R}$ with the usual metric. Then
   (a) S is complete but not T (b) T is complete but not S
   (c) both T and S are complete (d) neither T nor S is complete

17. Define $f:\mathbb{R}^2 \to \mathbb{R}$ by $f(x, y)=\begin{cases}1, & \text{if } xy=0,\\ 2, & \text{otherwise.}\end{cases}$

If $S=\{(x, y): f \text{ is continuous at the point } (x, y)\}$, then

(a) S is open (b) S is connected (c) $S=\phi$ (d) S is closed

18. Let E be a connected subset of $\mathbb{R}$ with atleast two elements. Then the number of elements in E is

(a) exactly two (b) more than two but finite
(c) countable infinite (d) uncountable

19. If (X, d) is a metric space and d(x, y) is the distance between the points x and y. Then the distance between two subsets A and B of X denoted by d(A, B) and is defined by d(A, B) =

(a) $\sup\{d(x, y) : x \in A, y \in B\}$
(b) $\inf\{d(x, y) : x \in A, y \in B\}$
(c) $\lim\sup\{d(x, y) : x \in A, y \in B\}$
(d) $\lim\inf\{d(x, y) : x \in A, y \in B\}$ where (X, d) is a metric space and d(x, y) is the distance between the points x and y.

20. Let X be a complete metric space and let $E \subseteq X$. Consider the following statements:

S_1: E is compact,
S_2: E is closed and bounded,
S_3: E is closed and totally bounded,
S_4: Every sequence in E has a subsequence converging in E.

Which one of the above statements does NOT imply all the other statements ?

(a) S_1 (b) S_2 (c) S_3 (d) S_4

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

21. The function $\rho : X \to \mathbb{R}$ is not called pseudo metric if

(a) $\rho(x, y) = 0$ iff $x = y$ and $x, y \in X$ (b) $\rho(x,y) = 0$ for some $x \neq y$; $x, y \in X$
(c) $\rho(x, y) \neq 0$ for $x = y$; $x, y \in X$ (d) $\rho(x, y) = 0$; $x, y \in X$

22. A metric space X satisfies Bolzano-Weierstrass property, then which of the following is incorrect?

(a) every infinite sequence $\langle x_n \rangle$ in X has no cluster point.
(b) every infinite sequence $\langle x_n \rangle$ in X has at least one cluster point.
(c) X is not sequentially compact.
(d) X is not compact.

23. A set F of metric space (X, ρ) is closed

(a) if it contains all its point of closure (b) its complement is open
(c) (a) is true, (b) is false (d) (a) and (b) both true

24. If X and Y are metric spaces and f is a mapping from X to Y such that for every open set $O \subset Y$, $f^{-1}(O) \subset X$ is open, then which of the following is/are not correct about the statements?

(a) f is continuous function
(b) f is constant function
(c) f is increasing function
(d) f is decreasing function

25. The empty set of a metric space (X, ρ) is

(a) open set (b) closed set
(c) both open set and closed set (d) neither open nor closed

26. If A is a non-empty subset of a metric space X is complete, then which of the following is incorrect?

(a) A is closed set
(b) A is open set
(c) Null set
(d) Complement of A is closed

27. If A is an open subset of complete metric space X, then which of the following is incorrect?

(a) A is not complete (b) A is not complete
(c) Complement of A is closed (d) A is closed.

28. The collection C of open intervals of the form $\left(\frac{1}{n}, \frac{2}{n}\right)$, n = 2, 3,... is an open covering of the open interval (0,1), then

(a) finite sub-collection covers (0, 1)
(b) finite sub-collection does not cover (0,1)
(c) the union of intervals is (0, 1)
(d) the union of intervals is (0,0)

29. Given collection C of open intervals of the form $\left(-\frac{1}{n}, \frac{1}{n}\right)$, then

(a) C is a covering of (-1,2)
(b) C is not a covering of (-1,2)
(c) union of intervals is (-1, 1)
(d) union of intervals is (1- 1)

30. Which of the following(s) is/are true?

(a) A metric space X has the Bolzano-Weierstrass property, if X is sequentially compact.
(b) A metric space X is sequentially compact, if X has Bolzano-Weierstrass property.
(c) A metric space X has the Bolzano-Weierstrass property iff X is sequentially compact.
(d) A metric space X is sequentially compact, if X does not satisfies Bolzano-Weierstrass property.

31. If metric space X is separable, then there is a countable family $\{O_i\}$ of open sets such that for any open set $O \subset X$, then

(a) $O = \cup O_i$ and $O_i \subset X$

(b) $O \subseteq \cup O_i$ and $O_i \subseteq O$

(c) $O \supseteq \cup O_i$ and $O_i \subseteq O$

(d) $O \in \cup O_i$ and $O_i \subseteq O$

32. If S is a subset of $\mathbb{R}^n$ and if S is closed and bounded, then

(a) every infinite subset of S has an accumulation point in S.

(b) every infinite subset of S has no accumulation point in S.

(c) If S is bounded, then its subset is bounded.

(d) If S is bounded, then its subset is not bounded.

33. Which of the following(s) is/are correct?

(a) Cantor set is a perfect set.

(b) Cantor set is nowhere dense set.

(c) Measure of cantor set is zero.

(d) none of the above.

34. Which of the following(s) is/are true?

(a) The euclidean space $\mathbb{R}^n$ is separable.

(b) The metric space l_∞ of all bounded sequences with submetric is not separable.

(c) Every dense subset is uncountable in l_∞.

(d) none of the above.

35. Let $(\mathbb{Q},d)$ be a metric space, where $\mathbb{Q}$ is the set of rational numbers and metric d is defined by $d(x, y) = |x - y|, \forall x, y \in Q$ then,

(a) the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$ is a Cauchy sequence.

(b) the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$ converges to a point $\mathbb{Q}$.

(c) The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a Cauchy sequence.

(d) The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ does not converge to a point in $\mathbb{Q}$.

36. Which of the following(s) is/are correct?

(a) The discrete space (X,d) is a complete metric space.

(b) The space $(\mathbb{R},d)$ is a complete metric space, where d is usual metric.

(c) The space $\mathbb{R}^n$ of all n-tuples with metric d, defined by $d(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$ is a complete metric space.

(d) none of the above

37. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following is always true?

(a) $f^{-1}(U)$ is open for all open sets $U \subseteq \mathbb{R}$

(b) $f^{-1}(C)$ is closed for all closed sets $C \subseteq \mathbb{R}$

(c) $f^{-1}(K)$ is compact for all compact sets $K \subseteq \mathbb{R}$

(d) $f^{-1}(G)$ is connected for all connected sets $G \subseteq \mathbb{R}$

38. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
$Y = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$ and
$Z = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$. Then

(a) X is not homeomorphic to Y.

(b) Y is not homeomorphic to Z.

(c) X is not homeomorphic to Z.

(d) no two of X, Y or Z are homeomorphic.

ANSWERS TO EXERCISES

(PRACTICE SET)

Exercise 1 : (c)	**Exercise 2 :** (b)	**Exercise 3 :** (a)
Exercise 4 : (a)	**Exercise 5 :** (a,c)	**Exercise 6 :** (b,d)
Exercise 7 : (b,d)	**Exercise 8 :** (a,d)	**Exercise 9 :** (c)
Exercise 10 :(a,b,c)		

ANSWERS TO ASSIGNMENTS

ASSIGNMENT – 4.1

1. (a)	2. (c)	3. (a)	4. (a)	5. (b)	6. (a)	7. (a)
8. (a)	9. (c)	10. (b)	11. (a)	12. (a)		
13. (b,c,d)	14. (a,c,d)	15. (a,d)	16. (a,d)	17. (a,c)	18. (a,d)	19. (a,b)
20. (a,d)	21. (a,d)	22. (a,c,d)	23. (a,b,c,d)	24. (b,d)	25. (a,c)	26. (a,b,c,d)
27. (a,c)	28. (a,b)	29. (a,b,c)	30. (b,c,d)	31. (b,d)	32. (a,b,c)	

ASSIGNMENT – 4.2

1. (c)	2. (a)	3. (b)	4. (a)	5. (a)	6. (a)	7. (a)
8. (a)	9. (c)	10. (b)	11. (a)	12. (a)	13. (b)	14. (a)
15. (b)	16. (c)	17. (a)	18. (d)	19. (b)	20. (b)	
21. (a,c,d)	22. (a,c,d)	23. (a,b,d)	24. (b,c,d)	25. (a,b,c)	26. (b,c,d)	27. (a,d)
28. (b,c)	29. (b,c)	30. (a,b,c)	31. (a,b,c)	32. (a,c)	33. (a,b,c)	34. (a,b,c)
35. (a,b,c,d)	36. (a,b,c)	37. (a,b,c,d)	38. (b,c)			

CHAPTER – 5
RIEMANN INTEGRATION, IMPROPER INTEGRAL AND MEASURE THEORY

INTRODUCTION

We already have studied the integrals of the form $\int_a^b f(x)dx$, where a and b are finite and f(x) is bounded.

Integration is used to find the area under a curve $y = f(x)$, above the x-axis and two boundary ordinates $x = a$ and $x = b$. Now the question arises do all the functions have graphical representation. Answer is No, there are many functions such as $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ that cannot be represented graphically. Riemann have discussed about integration of such functions. In this section we will discuss about such functions, whether they are integrable or not? Next question arises, about the convergence of integrals, when either one or both limits of integrals are infinite or when integrand is unbounded. Such integrals are called improper integrals. In this section we will also discuss about convergence of improper integrals. Also, we will discuss some basic concepts of measure theory in this section. We already have studied how to calculate length of an interval, we will discuss how to calculate the length of other sets that are not intervals in this section.

5.1. RIEMANN INTEGRATION

5.1.1. Some definitions:

(1) **Partition:** $P = \{x_0, x_1, x_2, ..., x_n : a = x_0 < x_1 < x_2 < ... < x_n = b\}$ is called a partition of closed interval $[a,b]$.

(2) **Norm of a partition:** Let $P = \{x_0, x_1, x_2, ..., x_n : a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$.
For each $i = 1, 2, 3, ..., n$, let $\Delta x_i = x_i - x_{i-1}$ is the length of the subinterval $[x_{i-1}, x_i]$.
Norm or mesh of a partition P is denoted by $||P||$ and defined as $||P|| = \max_i \Delta x_i$.

(3) **Refinement of a partition:** If P and P* are partitions of [a, b], then P* is called a refinement of P if $P \subset P^*$.

(4) **Common refinement of P_1 and P_2:** If P_1 and P_2 are partitions of [a,b], then $P = P_1 \cup P_2$ is called a common refinement of P_1 and P_2.

(5) Let f be a bounded real valued function defined on closed interval [a, b] and let $P = \{x_0, x_1, x_2, ..., x_n : a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of [a, b]. For each $i = 1, 2, 3, ..., n$, let $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$, $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$

(6) **Upper Riemann sum:** $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ is called upper Riemann sum of f with respect to the partition P of $[a, b]$.

(7) **Lower Riemann sum:** $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ is called lower Riemann sum of f with respect to the partition P of $[a, b]$.

(8) **Oscillatory sum:** $W(P,f) = U(P,f) - L(P,f)$ is called the oscillatory sum with respect to the partition P of $[a, b]$.

(9) **Upper Riemann integral:** $\inf_P U(P, f)$ is denoted by $\int_a^{\bar{b}} f(x)\,dx$ and is called upper Riemann integral of f on $[a, b]$.

(10) **Lower Riemann integral:** $\sup_P L(P, f)$ is denoted by $\int_{\underline{a}}^{b} f(x)\,dx$ and is called lower Riemann integral of f on $[a, b]$.

(11) **Riemann integrable:** A bounded real valued function f defined on $[a, b]$ is said to be Riemann integrable, if $\int_{\underline{a}}^{b} f(x)\,dx = \int_a^{\bar{b}} f(x)\,dx$. The common value is denoted by $\int_a^b f(x)\,dx$ and is called **Riemann integral** of f on $[a, b]$.

If f is **Riemann integrable** on $[a, b]$, we write $f \in R\,[a, b]$.

Results:

(1) If f is a bounded function on $[a, b]$ and P is a partition of $[a, b]$, then

$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$, where $m = \inf_{a \le x \le b} f(x)$, $M = \sup_{a \le x \le b} f(x)$, $\int_{\underline{a}}^{b} f(x)\,dx$, $\int_a^{\bar{b}} f(x)\,dx$ exist in

$\mathbb{R}$ and $m(b-a) \le \int_{\underline{a}}^{b} f(x)\,dx \le \int_a^{\bar{b}} f(x)\,dx \le M(b-a)$

(2) Every constant function on $[a, b]$ is Riemann integrable on $[a, b]$.

(3) If f is a bounded real function on $[a, b]$ such that $|f(x)| \le k \ \forall\, x \in [a, b]$ and a partition P^* of $[a, b]$ is a refinement of a partition P of $[a, b]$ containing r more points than P, show that
$L(P,f) \le L(P^*,f) \le L(P,f) + 2rk\|P\|$ and $U(P, f) \ge U(P^*, f) \ge U(P, f) - 2r\,k\|P\|$.

(4) If f is a bounded function on $[a, b]$ and a partition P^* of $[a, b]$ is refinement of a partition P of $[a, b]$, then $L(P,f) \le L(P^*,f) \le U(P^*,f) \le U(P,f)$ and hence $U(P^*,f) - L(P^*,f) \le U(P,f) - L(P,f)$

(5) If f is a bounded real valued function defined on $[a, b]$ and P_1, P_2 are partitions of $[a, b]$, then $L(P_1,f) \le U(P_2, f)$, hence deduce that $\int_{\underline{a}}^{b} f(x)\,dx \le \int_a^{\bar{b}} f(x)\,dx$.

(6) **Darboux's Theorem:**
If f is a bounded real valued function defined on $[a, b]$, then $\forall \in > 0$, $\exists\ \delta > 0$ such that
$U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \in$ and $L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \in$, for every partition P of $[a, b]$ with $\|P\| < \delta$.

Hence deduce that $\int_a^{\bar{b}} f(x) = \lim_{\|P\| \to 0} U(P,f)$ and $\int_{\underline{a}}^{b} f(x)\,dx = \lim_{\|P\| \to 0} L(P, f)$.

(7) A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if $\forall \in > 0, \exists \delta > 0$ such that $U(P, f) - L(P, f) < \in$ for every partition P of $[a, b]$ with $\|P\| < \delta$.

(8) A bounded function on $[a, b]$ is Riemann integrable if and only if

$$\lim_{\|P\| \to 0} L(P, f) = \lim_{\|P\| \to 0} U(P, f) = \int_a^b f(x)dx.$$

(9) A bounded real valued function f defined on $[a, b]$ is Riemann integrable on $[a, b]$, i.e., $f \in R[a, b]$ if and only if $\forall \in > 0$, $\exists$ a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \in$.

(10) If f is a bounded function defined on $[a, b]$ and $P = \{x_0, x_1, x_2, ..., x_n\}$ is a partition of $[a, b]$ such that $U(P, f) - L(P, f) < \in$, then
(a) $U(P^*, f) - L(P^*, f) < \in$ for every refinement P^* of P.
(b) $\sum_{i=1}^{n} |f(s_i) - f(t_i)|\,\Delta x_i < \in$, where $s_i, t_i \in [x_{i-1}, x_i]$.

(11) A bounded function on $[a, b]$ is Riemann integral if and only if $\lim_{\|P\| \to 0} S(P, f)$ exists and

$$\lim_{\|P\| \to 0} S(P, f) = \int_a^b f(x)\,dx.$$

(12) If f is an integrable function on $[a, b]$ and g is a function on $[a, b]$ such that $g(x) = f(x)$ for every x except finite number of points, then g is integrable on $[a, b]$ and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

(13) A continuous function on $[a, b]$ is Riemann integrable on $[a, b]$.

(14) A monotonic function on $[a, b]$ is Riemann integrable on $[a, b]$.

(15) *If a bounded function f defined on [a, b] has only finitely many points of discontinuity on [a, b], then f is Riemann integrable on [a, b].*

(16) *If the set of points of discontinuities of a bounded function f on [a,b] is countable, then f is Riemann integrable on [a,b].*

(17) *If $f \in R[a, b]$ and $c \in \mathbb{R}$, then $cf \in R[a, b]$ and* $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

(18) *If $f, g \in R[a, b]$, then $f + g \in R[a, b]$ and* $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

(19) *If $f \in R[a, b]$ and $a < c < b$, then, $f \in R[a, c]$ and $f \in R[c, b]$ and* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

(20) *If $f: [a, b] \to [c, d]$ is a bounded integrable function and $g: [c, d] \to \mathbb{R}$ is a continuous function, then $h = g \circ f$ is an integrable function on [a, b]*

(21) *If f is an integrable function on [a, b] and $f \geq 0$, i.e., $f(x) \geq 0\ \forall\ x \in [a,b]$, then* $\int_a^b f(x)dx \geq 0$.

(22) *If f, g are integrable functions on [a, b] and $f \leq g$, i.e., $f(x) \leq g(x)\ \forall\ x \in [a, b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(23) *If f is Riemann integrable function on [a, b], then |f| is Riemann integrable on [a, b] and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx.$$

Converse is not true, i.e., if |f| is R-integrable, then f need not be.

For example. *Given an example of a bounded function f defined on a closed interval such that |f| is R-integrable but f is not.*

Solution: *Consider the function f defined on [0, 1] by*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

Now $-1 \leq f(x) \leq 1\ \forall\ x \in [0, 1]$

$\therefore$ *f is bounded on [0, 1]*

Let $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\}$ *be any partition of [0,1]*

For any sub-interval $[x_{i-1}, x_i]$, *we have,* $m_i = -1$, $M_i = 1$ *for* $i = 1, 2, 3, \ldots, n$

[$\because$ *every sub-interval contains rational as well as irrational points]*

$$L(P, f) = \sum_{i=1}^{n} m_i \delta_i = (-1)(\delta_1) + (-1)(\delta_2) + \ldots + (-1)(\delta_n)$$

$$= -(\delta_1 + \delta_2 + \ldots + \delta_n) = -(1-0) = -1$$

$$U(P, f) = \sum_{i=1}^{n} M_i \delta_i = 1.\delta_1 + 1.\delta_2 + \ldots + 1.\delta_n$$

$$= (\delta_1 + \delta_2 + \ldots + \delta_n) = (1-0) = 1$$

Now $\underline{\int_0^1} f(x)\,dx = \underset{\|P\|\to 0}{Lt} L(P, f) = -1$ and $\overline{\int_0^1} f(x)\,dx = \underset{\|P\|\to 0}{Lt} U(P, f) = 1$

$$\therefore \underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$$

$\Rightarrow$ *f is not R-integrable.*

Now $|f|(x) = |f(x)| = 1\ \forall\ x \in [0, 1]$.

$\Rightarrow$ *|f| is a constant function and hence continuous on [0, 1].*

$\Rightarrow$ *|f| is R-integrable on [0, 1].*

(24) *If f is Riemann integrable on [a, b], then f^2 is Riemann integrable on [a, b].*

(25) *If f, g are Riemann integrable on [a, b], then fg is Riemann integrable on [a, b].*

(26) *If the set of points of discontinuity of a bounded function f on [a,b] has finite number of limit points then f is Riemann integrable on [a,b].*

Example 5.1.1.1. *Give an example of a bounded function, which is not Riemann integrable on [a, b].*

Solution: *Let f be a function on [a, b] defined by* $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$

Let $P = \{x_0, x_1, x_2, \ldots, x_n : a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ *be a partition of [a, b].*

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) = 1$$

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x) = -1$$

$$\therefore U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i$$

[$\because M_i = 1$ *for all i*]

$$= \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n$$

$$= x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \ldots + x_n - x_{n-1}$$

$$= x_n - x_0 = b - a.$$

and $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$

$= -\sum_{i=1}^{n} \Delta x_i$ [$\because$ $m_i = -1 \ \forall i$]

$= -(b-a)$

$\overline{\int_a^b} f(x)\,dx = \inf_P U(P, f) = (b-a)$

[$\because$ $U(P, f) = b-a$, for every partition P of $[a, b]$].

and $\underline{\int_a^b} f(x)\,dx = \sup_P L(P, f) = -(b-a)$.

[$\because$ $L(P,f) = -(b-a)$, for every partition P of $[a,b]$]

$\therefore \underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$.

Hence, f is not Riemann integrable on $[a, b]$

Example 5.1.1.2. *Let f be a function defined on $[0, 1]$ by* $f(x) = \begin{cases} 1, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$

Show that f is Riemann integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$.

Solution: *Let $\in > 0$ be given.*

Let $P = \{x_0, x_1, x_2, ..., x_n : a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$ such that $x_j - 1 < \frac{1}{2} < x_j$ and

$x_j - x_{j-1} < \in$.

$m_i = \inf_{x_{i-1} \le x \le x_i} f(x) = \begin{cases} 1 & if \ i \neq j \\ 0 & if \ i = j \end{cases}$ and

$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) = 1$.

$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$

$= \sum_{i \neq j} (M_i - m_i) \Delta x_i + (M_j - m_j) \Delta x_j$

$= \sum_{i \neq j} (1-1) \Delta x_i + (1-0) \Delta x_j = \Delta x_j = x_j - x_{j-1} < \in$.

Thus $\forall \in > 0$, $\exists$ partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \in$.

Hence, f is Riemann integrable on $[0, 1]$.

Now, $\int_0^1 f(x)dx = \overline{\int_0^1} f(x)dx = \int_0^1 f(x)dx$ ($\because$ f is Riemann integrable on $[0, 1]$)

As $M_i = 1$, for every partition P of $[0, 1]$.

$\Rightarrow U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1-0 = 1$, for every partition P of $[a, b]$.

$\therefore \int_0^1 f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \inf_P U(P, f) = 1$

Example 5.1.1.3. Let the function f be defined on $[0, 1]$ by the conditions

$f(x) = 2rx$, when $\frac{1}{r+1} < x < \frac{1}{r}$, $r = 1,2,3...$

$f(0) = f(1/r) = 0$ for $r = 1,2,3,...$

Show that $f \in R\ [0, 1]$ and $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

Solution: *The given function f is not defined at the set of points* $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{r}, ...\right\}$. ...(1)

We may, however, define f at these points in any manner, provided f remains bounded. Now the set (1) of points of discontinuity of f has only one limit point at $x = 0$ and consequently $f \in R\ [0, 1]$.

Consider the function F defined by $F(h) = \int_h^1 f(x)\,dx$. *We know that F is a continuous function so that*

$F(0) = \int_0^1 f(x)\,dx = \lim_{h \to 0} F(h)$.

But $h = 1/n$ so that $h \to 0$ as $n \to \infty$.

We have

$\int_{1/n}^1 f(x)\,dx = \int_{1/2}^1 f + \int_{1/3}^{1/2} f + ... + \int_{1/(r+1)}^{1/r} f + ... + \int_{1/n}^{1/(n-1)} f$.

Now $\int_{1/(r+1)}^{1/r} f(x)\,dx = \int_{1/(r+1)}^{1/r} 2r\,x\,dx = \frac{2r+1}{r(r+1)^2}$.

Therefore, $\int_{1/n}^1 f(x)\,dx = \sum_{r=1}^{n-1} \frac{2r+1}{r(r+1)^2} = \sum_{r=1}^{n-1} \left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right)$

$= \sum_{r=1}^{n-1} \left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1} \frac{1}{(1+r)^2} = 1 - \frac{1}{n} + \sum_{r=1}^{n-1} \frac{1}{(r+1)^2}$.

Now the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$ is known to be convergent and has the sum $\pi^2/6$.

Hence $\int_0^1 f(x)\,dx = \lim_{n \to \infty} \int_{1/n}^1 f(x)\,dx = 1 - 0 + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right) = 1 + \frac{\pi^2}{6} - 1 = \frac{\pi^2}{6}$.

5.1.2. The Integral as a Limit of the Sum:

Definition. *Let f be a function defined on $[a, b]$ and let $P = \{x_0, x_1, x_2, ..., x_n : a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$, then*

(i) $S(P, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$, where $t_i \in [x_{i-1}, x_i]$, is called a **Riemann sum** of f on $[a, b]$ relative to P.

(ii) $\lim_{\|P\|\to 0} S(P, f) = I$ if $\forall \in > 0, \exists \delta > 0$ such that $|S(P, f) - I| < \in$, for every Riemann sum of f on $[a, b]$ relative to partition P of $[a, b]$ with $\|P\| < \delta$.

Note:

1) If f is Riemann integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} hf(a+ih) = \lim_{n\to\infty}\sum_{i=1}^{n} hf(a+ih)$,
where $nh = b - a$.

2) If f is integrable on $[a, b]$, $a \neq 0$, then $\int_a^b f(x)\,dx = a(r-1)\lim_{n\to\infty}\sum_{i=1}^{n-1} f(ar^i)r^{i-1}$, where $r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$.

3) If $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$, then that $\alpha f + \beta g \in R[a,b]$ and

4) If $f, g \in R[a, b]$, then $f - g \in R[a, b]$ and $\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)dx$.

For example. Find the sum of the series $\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+(n-1)^2} + \dots$, using integral.

Solution: Here general term $= \frac{n}{n^2+r^2}, r = 0,1,2,\dots$

$$= \frac{1}{n}\left[\frac{1}{1+\left(\frac{r}{n}\right)^2}\right] \quad \dots(1)$$

$= \frac{dx}{1+x^2}$ [Replace $\frac{1}{n}$ by dx and $\frac{r}{n}$ by x]

For limits of integration,

lower limit $= \lim_{n\to\infty}\frac{n}{n^2} = 0$ [Put $r = 0$ in general term]

upper limit $= \lim_{n\to\infty}\frac{n}{n^2+(n-1)^2} = 1$ [Put $r = n-1$ in general term]

$\therefore$ Required sum $= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}$

5.2. FUNCTION OF BOUNDED VARIATION

5.2.1. Total variation: Let $P = \{x_0, x_1, x_2, \dots, x_n : a = x_0 < x_1 < \dots < x_n = b\}$, be a partition of $[a, b]$ and sub-intervals can be taken as $[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$

If $f(x)$ is a function defined on $[a, b]$, then corresponding to each partition P of $[a, b]$, $\exists$ a sum i.e.

$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ Supremum of such sums i.e. $\sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ (where supremum is taken over P) is known as variation or total variation in $[a, b]$.

Notation: $v(f, a, b) = \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

Properties:

(1) $v(a) = 0$.
(2) $v(b)$ = total variation of f over $[a, b]$.
(3) If $x_1 < x_2 \Rightarrow v(x_1) < v(x_2)$, i.e., $v(x)$ is non-decreasing function.
(4) $v(x)$ is continuous iff $f(x)$ is continuous.

5.2.2. Function of bounded variation: Let f be a function and if variation of the function is bounded in a given interval, then function is known as function of bounded variation in that interval.

Note: A function of bounded variation need not be continuous in that interval.

Results on bounded variation:

1) If $f(x)$ is bounded monotonic function in $[a,b]$, then $f(x)$ is a function of bounded variation.

2) If a function is differentiable in $[a, b]$ and its derivative is bounded in $[a, b]$, then $f(x)$ is a function of bounded variation and also the variation is given by $V(f,a,b) \leq M(b-a)$, where M is upper limit of derivative of a function.

3) If $f(x)$ is a function of bounded variation in an interval, then it is bounded in that interval, i.e., function of bounded variation $\Rightarrow$ function is bounded.

4) If $f(x)$ is function of bounded variation in $[a, b]$ and if $c \in (a, b)$ be any point, then $f(x)$ is a function of bounded variation in $[a, c]$ and also in $[c, b]$. Moreover $v(f, a, b) = v(f, a, c) + v(f, c, b)$.

5) If $f(x)$ and $g(x)$ are functions of bounded variation in $[a, b]$ and $\exists$ a positive real number M such that $|g(x)| \geq M \;\forall x \in [a, b]$, then $\frac{f(x)}{g(x)}$ is also function of bounded variation in $[a,b]$.
Total variation = variation function = $v(x) = v(f[a,b])$.

6) If $f(x)$ is of bounded variation on $[a, b]$, then it can be expressed as difference of two monotonic non-decreasing functions and conversely [This is necessary and sufficient condition]

For example: $f(x) = \{x\}$; $x \in [0, 5]$

$f(x) = x - [x]$, is non-decreasing function

$\Rightarrow f(x)$ is difference of two non-decreasing functions $\Rightarrow f(x)$ is of bounded variation.

7) If f is of bounded variation, then it is R-integrable, but converse not true.

8) If f and g are functions of bounded variation, then
(i) $|f|$ and $|g|$ are of bounded variation.
(ii) $f \pm g$ are of bounded variation.
(iii) $f.g$ is of bounded variation.
(iv) f/g need not to be of bounded variation.

9) If $f(x)$ is a function of bounded variation in $[a, b]$ and if for each $x \in [a, b]$ $|f(x)| \geq M$, where M is a positive real number, then $\frac{1}{f(x)}$ is also a function of bounded variation in $[a, b]$.

Example 5.2.2.1. Consider $f(x) = \sin x$ in $\left[0, \frac{\pi}{2}\right]$. Is $f(x)$ is a function of bounded variation? If so find the total variation?

Solution: As $\sin x$ is monotonically increasing in $\left[0, \frac{\pi}{2}\right] \Rightarrow f(x)$ is a function of bounded variation in $\left[0, \frac{\pi}{2}\right]$

and then $v\left(\sin x, 0, \frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \sin 0 = 1.$

Thus the given function is a function of bounded variation and its total variation is 1.

Example 5.2.2.2. Consider $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Is function $f(x)$ of bounded variation in $[-5, 5]$?

Solution:

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Now } |f'(x)| = |2x\sin\frac{1}{x} - \cos\frac{1}{x}|$$

$$\leq 2|x||\sin\frac{1}{x}| + |\cos\frac{1}{x}|$$

$$\leq 2|x| + 1$$

$|f'(x)| \leq 11$ in $[-5, 5]$

$\Rightarrow f(x)$ is a function of bounded variation with variation,

i.e., $v(f, a, b) \leq 11\,[5 - (-5)] = 110.$

Example 5.2.2.3. Consider $f(x) = \sin x$ in $[0, 2\pi]$. Is this function a function of bounded variation in $[0, 2\pi]$?

Solution: $f(x)$ is a function of bounded variation as it has bounded derivative

Now $f'(x) = \cos x$.

Change in sign of $f'(x) = \cos x$ are given by

$0 \quad + \quad \frac{\pi}{2} \quad - \quad \frac{3\pi}{2} \quad + \quad 2\pi$

As $f'(x)$ is monotonic increasing in $\left[0, \frac{\pi}{2}\right]$, decreasing in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, increasing in $\left[\frac{3\pi}{2}, 2\pi\right]$

$\Rightarrow v(\sin x, 0, 2\pi) = v\left(\sin x, 0, \frac{\pi}{2}\right) + v\left(\sin x, \frac{\pi}{2}, \frac{3\pi}{2}\right) + v\left(\sin x, \frac{3\pi}{2}, 2\pi\right) = |1-0| + |-1-1| + |1-0| = 4$

$\Rightarrow f(x)$ is a function of bounded variation with variation = 4.

Example 5.2.2.4. $f(x) = \begin{cases} x^{\alpha}\left(\sin\frac{1}{x}\right)^{\beta}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x \in [0, 1]$

$f(x)$ is of bounded variation iff $\frac{\alpha}{\beta} > 1$ i.e. $\alpha > \beta$. This result is also true for cosine function.

Example 5.2.2.5. Dirichlet's and Riemann functions are not of bounded variation.

<u>PRACTICE SET – I</u>

Exercise 1. The value of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + kn}}$ is **(IIT-JAM 2011)**

(a) $2(\sqrt{2} - 1)$ (b) $2\sqrt{2} - 1$

(c) $2 - \sqrt{2}$ (d) $\frac{1}{2}(\sqrt{2} - 1)$

Exercise 2. If $f : [1,2] \to \mathbb{R}$ is a non negative Reimann Integrable function such that $\int_1^2 \frac{f(x)}{\sqrt{x}}dx = k\int_1^2 f(x)dx \neq 0$ **(GATE-2009)**

then k belongs to the interval

(a) $\left[0, \frac{1}{3}\right]$ (b) $\left(\frac{1}{3}, \frac{2}{3}\right]$

(c) $\left(\frac{2}{3}, 1\right]$ (d) $\left(1, \frac{4}{3}\right]$

Exercise 3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x\cos(\pi/(2x)) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then

(a) f is continuous on [0, 1].
(b) f is of bounded variation on [0, 1].
(c) f is differentiable on the open interval (0, 1) and its derivative f' is bounded on (0, 1).
(d) f is Reimann integrable on [0, 1].

Exercise 4. Let f be a function defined on [a, b]. Then which of the following is correct?
(a) If f is continuous on [a, b] then 'f' is R-integrable on [a,b].
(b) If f is R-integrable on [a, b], then 'f' is continuous on [a, b].
(c) 'f' is R-integrable on [a, b] iff 'f' is continuous on [a, b].
(d) 'f' is R-integrable on [a, b] iff 'f' is continuous on [a, b] almost everywhere.

Exercise 5. A function f is defined on [0, 1] by $f(x) = \frac{1}{n}$ for $\frac{1}{n} \geq x > \frac{1}{(n+1)}$, $n = 1, 2, 3, \ldots$. Then show that

$f \in R[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$.

Exercise 6. Any countable set of points on the real axis has
(a) no measure (b) measure one (c) measure zero (d) none of these

Exercise 7. Which of the following is/are true?
(a) Closed set is Lebesgue measurable.
(b) Closed set is not Lebesgue measurable.
(c) Open set is Lebesgue measurable.
(d) Open set is not Lebesgue measurable.

5.3. IMPROPER INTEGRAL

5.3.1 Definitions:

(1) Improper integrals: A function f on [a, b] is Riemann Integrable if
(i) Domain of the function f is the closed interval [a, b], $a, b \in \mathbb{R}$, $a < b$.

(ii) f is bounded on [a, b].

If either a or b (or both) become infinite or f is not bounded on [a, b], then $\int_a^b f(x)\,dx$ is called an improper integral.

(2) Proper integral: If f is a bounded function on [a, b] such that f is Riemann on [a, b], then $\int_a^b f(x)\,dx$ is called a proper integral.

(3) Point of infinite discontinuity: Let f be a real function with domain E and $c \in \mathbb{R}$, c is called a point of infinite discontinuity of the function f if $\forall\, s > 0$, f is not bounded on $(c-s, c+s) \cap E$.

For example, c is a point of infinite discontinuity of the function $f(x) = \frac{1}{x-c}$.

Note: It is assumed once for all that if a function f has no point of infinite discontinuity in [a, b], $a, b \in \mathbb{R}$, $a < b$, then f is bounded and integrable on [a, b] and $\int_a^b f(x)\,dx$ is a proper integral.

5.3.2 Types of Improper Integral:

(I) Improper integral of first kind: A definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of first kind if either a or b or both are infinite so that interval is unbounded but the function is bounded.

Examples. $\int_0^\infty \sin x^2\,dx$, $\int_0^\infty \frac{\sin x}{x^2}$, $\int_{-\infty}^\infty x^3\,dx$

(i) When 'b' is infinite:

Let 'f' be bounded and integrable for $x \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx.$$

Convergence: If $\lim_{t\to\infty}\int_a^t f(x)dx$ exists and is finite, then the improper integral $\int_a^\infty f(x)dx$ is said to converge, otherwise it is called divergent.

(ii) When 'a' is infinite:

Let 'f' be bounded and integrable for $x \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$.

Convergence: If $\lim_{t\to-\infty}\int_t^b f(x)dx$ exists and is finite, then $\int_{-\infty}^b f(x)dx$ is said to converge, otherwise it is called divergent.

(iii) When both 'a' and 'b' are infinite:

Let 'f' be bounded function and integrable on $\mathbb{R}$, then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$, where 'c' is only real number

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)dx + \lim_{t_2 \to \infty} \int_{c}^{t_2} f(x)dx.$$

Convergence: *If* $\lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)dx$ *and* $\lim_{t_2 \to \infty} \int_{c}^{t_2} f(x)dx$ *both exists and are finite, then* $\int_{-\infty}^{\infty} f(x)dx$ *is said to converge, otherwise it is called divergent.*

(II) ***Improper integral of second kind:*** *The definite integral* $\int_a^b f(x)dx$ *is said to be improper integral of second kind if 'a' and 'b' are finite but f has one or more point of infinite discontinuity, i.e., f is not bounded on [a,b].*

Examples. $\int_0^1 \frac{dx}{x^2}, \int_1^2 \frac{dx}{2-x}, \int_1^4 \frac{dx}{(x-1)(x-4)}, \int_0^2 \frac{dx}{(x-1)^2}.$

(i) *Let 'a' be the only point of infinite discontinuity of a function f in [a, b]. The improper integral* $\int_a^b f(x)\,dx$ *is defined as* $\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx.$

Convergence: *If* $\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx$ *exists and is finite, then the improper integral* $\int_a^b f(x)\,dx$ *is said to converge (exist) at a, otherwise it is called divergent.*

(ii) *Let 'b' be the only point of infinite discontinuity of a function in [a, b]. The improper integral* $\int_a^b f(x)\,dx$ *is defined as* $\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx.$

Convergence: *If* $\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx$ *exist and is finite, then the improper integral* $\int_a^b f(x)\,dx$ *is said to converge (exist) at b, otherwise it is called divergent.*

(iii) *If 'a' and 'b' are the only points of infinite discontinuity of a function f in [a, b]. The improper integral* $\int_a^b f(x)\,dx$ *is defined as* $\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{c} f(x)\,dx + \lim_{\epsilon' \to 0^+} \int_a^{b-\epsilon'} f(x)\,dx$, *where* $a < c < b$.

Convergence: *The improper integral* $\int_a^b f(x)\,dx$ *is said to converge if both* $\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{c} f(x)\,dx$ *and* $\lim_{\epsilon' \to 0^+} \int_c^{b-\epsilon'} f(x)dx$ *exist and are finite, otherwise it is called divergent.*

We may write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b.$

Note: *In the improper integral of second kind the existence, convergence or divergence of improper integral* $\int_a^b f(x)\,dx$, *(where a, b are the only points of infinite discontinuity), any* $c \in (a, b)$ *may be taken.*

For $c, d \in (a, b)$, $a < c < d < b$, $\int_c^d f(x)\,dx$ *is a proper integral and hence is convergent.*

$\int_a^d f(x)dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx$ *and* $\int_c^b f(x)dx = \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$

$\therefore$ $\int_a^d f(x)\,dx$ *converges at a iff* $\int_a^c f(x)\,dx$ *converges at 'a' and* $\int_c^b f(x)\,dx$ *converges at 'b' iff* $\int_d^b f(x)\,dx$ *converges at b. However a convenient point c,* $a < c < b$, *may be taken, for example in the improper integral* $\int_{-1}^{1} f(x)\,dx$, *we may take c = 0 and in the improper integral* $\int_0^2 f(x)\,dx$, *we may take c = 1*

Note: *If there are finite number of points* $c_1, c_2, \ldots, c_m$ *of infinite discontinuity of a function f in [a, b] and* $a \le c_1 < c_2 < \ldots < c_m \le b$, *then the improper integral* $\int_a^b f(x)\,dx$ *is defined by*

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_m}^{b} f(x)\,dx$$

Convergence: $\int_a^b f(x)\,dx$ *is said to be convergent if all the improper integrals* $\int_a^{c_1} f(x)\,dx, \int_{c_1}^{c_2} f(x)\,dx, \ldots \int_{c_m}^{b} f(x)\,dx$ *are convergent, otherwise it is divergent.*

(III) ***Improper integral of third kind:*** *The definite integral* $\int_a^b f(x)dx$ *is said to be an improper integral of third kind if the integrand of 'f' is unbounded, i.e., 'a' or 'b' or both are infinite and f is also unbounded on [a, b]*

For Example. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx, \int_1^\infty \frac{dx}{(x-1)^2}, \int_{-\infty}^{0} \frac{1}{(x+2)}dx$ *are improper integrals of third kind.*

Results on Improper integral:

I) *Let 'a' be the only point of infinite discontinuity of function f in [a, b] and f is positive in [a, b], i.e.,* $f(x) > 0\ \forall\ x$. *The improper integral* $\int_a^b f(x)\,dx$ *converges at a if and only if* $\exists$ *a positive real number M such that* $\int_{a+\epsilon}^{b} f(x)\,dx < M,\ 0 < \epsilon < b - a.$

2) Let 'b' be the only point of infinite discontinuity of a function f in [a, b] and f is positive in [a, b], i.e., $f(x) > 0\ \forall\ x$. The improper integral $\int_a^b f(x)\,dx$ converges at b if and only if, $\exists$ a positive real number M such that $\int_a^{b-\epsilon} f(x)\,dx < M,\ 0 < \epsilon < b - a$.

3) **Comparison test:** Let $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ be two improper integrals, where

- a f and g has only point of discontinuity at x = a
- f and g be two non negative functions
- $f(x) \le g(x)\ \forall\ x \in [a, b]$, then
  - (i) $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ converges.
  - (ii) $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ diverges.

4) **limit comparison Test :** Let $I_1 = \int_a^b f(x)dx$ and $I_2 = \int_a^b g(x)dx$ are two improper integrals, f and g defined on [a, b] and x = a is the only point of infinite discontinuity for both f and g, where f and g are positive functions on [a,b].

If $\lim_{x\to a} \frac{f(x)}{g(x)} = l$ [l is a non-zero finite], then both integrals converges or diverges together.

5) If f and g are two positive functions in [a, b] and 'a' is the only point of infinite discontinuity of f in [a, b], then

(i) If $\lim_{x\to a^+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ converges at a, then $\int_a^b f(x)\,dx$ converges at a.

(ii) If $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ diverges at a, then $\int_a^b f(x)\,dx$ diverges at a.

6) **Cauchy's Integral Test:** If for $x \ge 1$, f(x) is a non-negative monotonically decreasing integrable function of x such that $f(n) = u_n$ for all positive integral value of n, then the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

7) Every absolutely convergent integral is convergent ,i.e., $\int_a^b f(x)\,dx$ converges if $\int_a^b |f(x)|\,dx$ converges.

Examples.

(a) The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges if and only if $n < 1$.

(b) The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ converges if and only if $n < 1$.

Example 5.3.2.1. Examine the convergence of $\int_1^2 \frac{dx}{(2-x)^3}$.

Solution: Note that '2' is the only point of infinite discontinuity.

$$\therefore \int_1^2 \frac{dx}{(2-x)^3} = \lim_{h\to 0^+} \int_1^{2-h} \frac{dx}{(2-x)^3}$$

$$= \lim_{h\to 0^+} \left[\frac{(2-x)^{-3+1}}{(-1)(-3+1)}\right]_1^{2-h}$$

$$= \lim_{h\to 0^+} \frac{1}{2}[(2-(2-h))^{-2} - (2-1)^{-2}]$$

$$= \frac{1}{2} \lim_{h\to 0^+} \left[\frac{1}{h^2} - 1\right] = \infty$$

Hence $\int_1^2 \frac{dx}{(2-x)^3}$ is divergent ,i.e., $\int_1^2 \frac{dx}{(2-x)^3}$ does not converge.

Example 5.3.2.2. Examine the convergence of $\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution: Note that '0' is the only point of infinite discontinuity.

$$\therefore \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{h\to 0^+} \int_h^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{h\to 0^+} \left[\frac{x^{\frac{-1}{2}+1}}{-\frac{1}{2}+1}\right]_h^1 = \lim_{h\to 0^+} 2\ ((1)^{1/2} - h^{1/2})$$

$$= 2 - 2.0 = 2.$$

Hence $\int_0^1 \frac{dx}{\sqrt{x}}$ converges.

Example 5.3.2.3. Examine the convergence of the integral $\int_1^2 \frac{x}{\sqrt{x-1}}dx$.

Solution: Note that '1' is the only point of infinite discontinuity.

$$\therefore \int_1^2 \frac{x}{\sqrt{x-1}}dx = \lim_{h\to 0^+}\int_{1+h}^2 \frac{(x-1)+1}{\sqrt{x-1}}dx$$

$$= \lim_{h\to 0^+}\int_{1+h}^2\left(\frac{x-1}{\sqrt{x-1}}+\frac{1}{\sqrt{x-1}}\right)dx$$

$$= \lim_{h\to 0^+}\left(\int_{1+h}^2 (x-1)^{\frac{1}{2}}dx + \int_{1+h}^2 (x-1)^{-\frac{1}{2}}dx\right)$$

$$= \lim_{h\to 0^+}\left[\frac{(x-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{1+h}^2 + \lim_{h\to 0^+}\left[\frac{(x-1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{1+h}^2$$

$$= \lim_{h\to 0^+}\frac{2}{3}((2-1)^{3/2}-(1+h-1)^{3/2}) + \lim_{h\to 0^+}2((2-1)^{1/2}-(1+h-1)^{1/2})$$

$$= \frac{2}{3}\lim_{h\to 0^+}(1-h^{3/2}) + 2\lim_{h\to 0^+}(1-h^{1/2})$$

$$= \frac{2}{3}(1-0) + 2(1-0)$$

$$= \frac{2}{3}+2 = \frac{8}{3}$$

$\therefore \int_1^2 \frac{x}{\sqrt{x-1}}dx$ converges.

Example 5.3.2.4. *Examine the convergence of* $\int_0^1 \log x\, dx$.

Solution: *Note that '0' is the only point of infinite discontinuity.*

$$\therefore \int_0^1 \log x\, dx = \lim_{h\to 0^+}\int_h^1 \log x\, dx$$

$$= \lim_{h\to 0^+}\left([x\log x]_h^1 - \int_h^1 x\frac{1}{x}dx\right) \quad \textit{(Integrating by parts)}$$

$$= \lim_{h\to 0^+}[1\log 1 - h\log h - (x)_h^1] = \lim_{h\to 0^+}[0 - h\log h - (1-h)]$$

$$= -\lim_{h\to 0^+} h\log h - (1-0) = -\lim_{h\to 0^+}\frac{\log h}{\frac{1}{h}} - 1$$

$$= -\lim_{h\to 0^+}\frac{\frac{1}{h}}{-\frac{1}{h^2}} - 1 \qquad \textit{(using L' Hospital Rule)}$$

$$= -\lim_{h\to 0^+}(-h) - 1 = 0 - 1 = -1.$$

Hence, $\int_0^1 \log x\, dx$ *converges.*

Example 5.3.2.5. *Discuss the convergence of* $\int_0^{1/e}\frac{dx}{x(\log x)^2}$.

Solution: *Note that '0' is the only point of infinite discontinuity*

$$\therefore \int_0^{1/e}\frac{dx}{x(\log x)^2} = \lim_{h\to 0^+}\int_h^{1/e}(\log x)^{-2}\frac{1}{x}dx$$

$$= \lim_{h\to 0^+}\left[\frac{(\log x)^{-1}}{-1}\right]_h^{1/e} \quad \left[\because \frac{d(\log x)}{dx} = \frac{1}{x}\right]$$

$$= -\left[\lim_{h\to 0^+}\left(\left(\log\frac{1}{e}\right)^{-1} - (\log h)^{-1}\right)\right] = -\lim_{h\to 0^+}\left(-1-\frac{1}{\log h}\right)$$

$$= \lim_{h\to 0^+}\left(1+\frac{1}{\log h}\right) = 1$$

Hence, $\int_0^{1/e}\frac{dx}{x(\log x)^2}$ *converges.*

Example 5.3.2.6: *Discuss the convergence of* $\int_0^e \frac{dx}{x(\log x)^3}$.

Solution: *'0' is the only point of infinite discontinuity*

$$\therefore \int_0^e \frac{dx}{x(\log x)^3} = \lim_{h\to 0^+}\int_h^e \frac{dx}{x(\log x)^3}$$

$$= \lim_{h\to 0^+}\int_h^e (\log x)^{-3}\frac{1}{x}dx$$

$$= \lim_{h\to 0^+}\left[\frac{(\log x)^{-2}}{-2}\right]_h^e$$

$$= \lim_{h\to 0^+} -\frac{1}{2}((\log e)^{-2} - (\log h)^{-2}) = \frac{-1}{2}.$$

Hence, given integral converges.

Example 5.3.2.7. *Examine the convergence of* $\int_0^1 \frac{dx}{\sqrt{x}(1+x)^2}$.

Solution: *Here, '0' is the only point of infinite discontinuity*

Let $f(x) = \frac{1}{\sqrt{x}(1+x)^2}$ *and* $g(x) = \frac{1}{\sqrt{x}}$.

Note that $f(x) > 0$, $g(x) > 0$ *for* $0 < x \le 1$

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{\sqrt{x}(1+x)^2}}{\frac{1}{\sqrt{x}}} = \frac{1}{(1+x)^2}$$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{1}{(1+x)^2} = 1$, *which is finite and non-zero.*

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}dx$ *converges* $[\because \frac{1}{2} < 1]$

$\therefore \int_0^1 f(x) = \int_0^1 \frac{1}{\sqrt{x}(1+x)^2}$ *converges.*

Example 5.3.2.8. *Examine the convergence of* $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}\,dx.$

Solution: *If* $p \le 1$, *then* $\int_0^{\pi/2} \frac{\sin x}{x^p}dx$ *is a proper integral and hence converges.*

If $p > 1$, *then* $\int_0^{\pi/2} \frac{\sin x}{x^p}dx$ *is an improper integral and 0 is the only point of infinite discontinuity.*

Let $f(x) = \frac{\sin x}{x^p}$ *and* $g(x) = \frac{1}{x^{p-1}}$

$f(x) > 0,\ g(x) > 0$ *for* $0 < x \le \frac{\pi}{2}$

$$\frac{f(x)}{g(x)} = \frac{\frac{\sin x}{x^p}}{\frac{1}{x^{p-1}}} = \frac{\sin x}{x}$$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$, *which is finite and non-zero.*

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^{p-1}}dx$ *converges if and only if* $p - 1 < 1$, *i.e.* $p < 2$.

$\Rightarrow \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{\sin x}{x^p}dx$ *converges if and only if* $p < 2$.

Example 5.3.2.9. *Show that* $\int_1^2 \frac{\sqrt{x}}{\log x}$ *is divergent.*

Solution: *Here '1' is the only point of infinite discontinuity.*

let $f(x) = \frac{\sqrt{x}}{\log x}$, $g(x) = \frac{1}{x-1}$.

Note that $f(x) > 0$, $g(x) > 0$ *for* $1 < x \le 2$

$$\frac{f(x)}{g(x)} = \frac{\frac{\sqrt{x}}{\log x}}{\frac{1}{x-1}} = \frac{(x-1)\sqrt{x}}{\log x}$$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)\sqrt{x}}{\log x}$$

$$= \lim_{x\to 1^+} \frac{(x-1)\frac{1}{2}x^{-1/2} + x^{1/2}}{\frac{1}{x}} \left(\frac{0}{0}\text{form}\right)$$

$= 1$, *which is finite and non-zero.*

But $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{x-1}$ *is divergent*

$\therefore \int_1^2 f(x)\,dx = \int_1^2 \frac{\sqrt{x}}{\log x}dx$ *is divergent.*

Example.5.3.2.10. $\int_0^1 \frac{\sin^m x}{x^n};\ m, n > 0.$

Solution: $\int_0^1 \frac{\sin^m x}{x^n} = \int_0^1 \frac{\sin^m x}{x^m} \cdot \frac{1}{x^{n-m}}.$

Here, $f(x) = \int_0^1 \frac{\sin^m x}{x^m} \cdot \frac{1}{x^{n-m}}$ *and* $g(x) = \frac{1}{x^{n-m}}$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin^m x}{x^m} = 1 \ne 0$ *and finite.*

So, both behave alike

$\int_0^1 \frac{1}{x^{n-m}}$ *is convergent if* $n - m < 1$

$\int_0^1 \frac{\sin^m x}{x^n}dx$ *is convergent if* $n < 1 + m$ *and divergent if* $n \ge 1 + m$.

5.4 MEASURE THEORY

5.4.1 Lengths of open and closed sets:

(1) ***Length of open subset:*** *If G is any open subset of [a, b], then there will exist a countable family of disjoint open intervals say* $\{I_n\}$ *such that* $G = \cup_n I_n$.

The length $|G|$ of the open set G is defined as the sum of the lengths of the intervals of this family i.e., $|G| = \sum_n |I_n|$.

Obviously if $G_1 \supseteq G_2$, then $|G_1| \geq |G_2|$.

Also if $G_1 \cap G_2 = \phi$, then $|G_1 \cup G_2| = |G_1| + |G_2|$.

(2) **Length of closed subset:** If F is any closed subset of $[a, b]$, then the length $|F|$ of the closed set F is defined as $|F| = |G| - |G - F|$, where G is any open subset of $[a, b]$ such that $F \subseteq G$.

Note: Since F is closed $\Rightarrow (G - F)$ will be an open set and hence $|G - F|$ is defined.
In particular, $|F| = (b - a) - |F^c|$.

(3) **Outer Lebesgue Measure of a Set :** The outer Lebesgue measure of a set $A \subset \mathbb{R}$ is defined as

$$m^*(A) = \begin{cases} 0 & , \text{ if } A = \phi \\ \inf\{y : y = \sum |I_i|\} & , \text{ otherwise} \end{cases}$$

where $\{I_i\}$ is a countable family of open intervals such that $\cup I_i \supseteq A$, where $A \neq \phi$ and is denoted by $m^*(A)$. It is also known as Lebesgue exterior measure or more briefly the outer measure and is also denoted by $m_e(A)$ or $\overline{m}(A)$.

(4) **Lebesgue Inner Measure :** The inner measure of a set A, denoted by $m_*(A)$ or $m_i(A)$ is defined by $m_*(A) = b - a - m^*(A^c)$, where Ac is the complement of A relative to interval $[a, b]$ such that $A \subseteq [a, b]$.

(5) **Lebesgue measurable:**

➢ **Definition 1 :** The set A is said to be Lebesgue measurable if for each set $E \subseteq \mathbb{R}$, we have $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$.

➢ **Definition 2 :** The set A is said to be Lebesgue measurable, if $m^*(A) = m_*(A)$ and the common value is called its measure and is denoted by $m(A)$. Thus, if A is Lebesgue measurable, then $m^*(A) = m_*(A) = m(A)$.

Results:

(1) If $\{A_1, A_2, A_3, \ldots\}$ is a countable family of subsets of $\mathbb{R}$, then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

(2) A is countable $\Rightarrow m^*(A) = 0$.
The converse of the above theorem is not always true ,i.e., A set with outer measure zero may or may not be countable. (See Cantor's ternary set which is uncountable but has the outer measure zero).

(3) A set with outer measure different from zero is uncountable. For Example : As the set of all rational numbers is countable, its outer Lebesgue measure is zero.

(4) If A and B are any two disjoint subsets of $\mathbb{R}$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

(5) The outer measure of an interval is equal to its length.

(6) A linear set A of outer measure zero is Lebesgue measurable.

(7) Any subset of A (whose outer measure is zero) is also measurable.
For $B \subseteq A \Rightarrow m^*(B) \leq m^*(A) \Rightarrow m^*(B) \leq 0$. Also $m^*(B) \geq 0 \Rightarrow m^*(B) = 0$. Hence by above theorem, B is measurable.

(8) The necessary and sufficient condition for a set E to be measurable with measure zero is that $m^*(E) = 0$

(9) Every countable set is Lebesgue measurable, and its measure is zero.

(10) Every bounded open set, and every bounded closed set is measurable.

(11) Union of two measurable sets is also measurable.

(12) $m^*(A) \geq m_*(A)$, i.e., $m_e(A) \geq m_i(A)$ for any set $A \subset \mathbb{R}$.

5.4.2. Measurable Functions:

(1) **Measurable Function:** An extended real-valued function f defined over a measurable set E is said to be measurable in the sense of Lebesgue, if set $E(f > a) = \{x \in E : f(x) > a\}$ is measurable for all extended real numbers a.

This definition states that f is a measurable function if for every real number a, the inverse image of (a, ∞) under f, i.e., $f^{-1}(a, \infty)$ is a measurable set. Measure of set $E(f > a)$ may be finite or infinite.

(2) **Characteristic Function:** The characteristic function ϕ_A (or Ψ_A or χ_A) of the set A is defined as

$$\phi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \text{ but } x \in E - A \end{cases}$$

A being a subset of the measurable set E. This function is also called the **indicator function of A**.

(3) **Simple function:** A real – valued function $\psi : E \to \mathbb{R}$ is said to be a simple function if $\exists$

(i) a finite collection $\{A_1, A_2, \ldots, A_n\}$ of disjoint measurable subsets of E such that $\bigcup_{i=1}^{n} A_i = E$.

(ii) n non –zero real numbers $k_1, k_2, \ldots, k_n$ such that $\psi(x) = k_i, \forall x \in A_i, (i = 1, 2, \ldots, n)$.

If ψ is a simple function and assumes the values $k_1, k_2, ..., k_n$, then we can write

$\psi \equiv \sum_{i=1}^{n} k_i \phi_{A_i}$, where ϕ_{A_i} is the characteristic function of the set $A_i = \{x : \psi(x) = k_i\}$.

This representation of ψ is called **canonical representation** and such a representation of ψ is not unique. For example, the characteristic function of a set A is a simple function as it takes only two values (0 or 1) on two disjoint sets A and Ac.

Note that the sum, differences and product of simple functions are also simple functions.

(4) **Step Function:** A real valued function s(x) defined on an interval [a, b] is said to be a step function if there exists a partition $a = x_0 < x_1 < ... < x_n = b$ of [a, b] such that the function assumes one and only one value in each interval, i.e., $s(x) = c_i$, $\forall x \in (x_{i-1}, x_i]$, $i = 1, 2, ..., n$.

Thus step functions also assumes finite number of values like simple functions but here the sets $\{x : s(x) = c_i\}$ are intervals for each i.

Obviously every step function is also a simple function, but the converse is not true as the function

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is a simple function but not step as the sets of rational and irrationals are not intervals.

Results:

1) Let f be a measurable function defined over a measurable set E_r, $\forall r \in \mathbb{N}$, then f is measurable on E, where $E = \bigcup_{r=1}^{\infty} E_r$.

2) Characteristic function of a set A is measurable, iff A is measurable.

3) Let f and g be measurable functions defined over a measurable set E. Then $f + g$, $f - g$, fg, are measurable functions over E.

5.4.3. The Lebesgue Integral of a Function:

The definition of Lebesgue integral is parallel to the definition of Riemann integral except that for Riemann integral we use subdivisions of closed intervals [a, b] into sub intervals, while for Lebesgue integral we use subdivisions of [a, b] into much more general kinds of measurable sets.

(1) **Partition:** By a measurable partition of [a, b], we mean any finite collection $\{S_r, r = 1, 2, 3, ..., n\}$ of measurable subsets of [a, b] such that $\bigcup_{r=1}^{n} S_r = [a,b]$ and $m(S_i \cap S_j) = 0$, $(i \neq j)$ $\forall i, j = 1,2,...,n$.

The partition of an interval is denoted by P. Then the sets $S_1, S_2, S_3, ..., S_n$ are called **components** of the partition P. Symbolically, $P = \{S_1, S_2, ..., S_n\}$. Thus, a measurable partition P of an interval [a, b] is nothing but a finite collection of subsets of [a, b] whose union is the whole [a, b] and whose pairwise intersections have the measure zero.

Now let us observe the difference in the above subdivision of [a, b] and that subdivision of [a, b], which we did for defining the Riemann integral.

If $y_0, y_1, y_2, ..., y_n$ are the real numbers such that $a = y_0 < y_1 < y_2 < ... < y_n = b$, then $\sigma = \{y_0, y_1, y_2, ..., y_n\}$ gives Riemann subdivision of [a, b] with component intervals $I_1 = [y_0, y_1]$, $I_2 = [y_1, y_2]$, ..., $I_n = [y_{n-1}, y_n]$.

Note that the collection $P = \{I_1, I_2, ..., I_n\}$ is forming a measurable partition of [a, b].

Thus in case of Riemann subdivision the components are necessarily intervals while in case of Lebesgue partition, the components are any sets.

Many measurable partitions do not have interval components. For example, the set $\{E_1, E_2\}$ is a measurable partition of [a, b], where E_1 and E_2 are respectively the sets of all the rational and irrational numbers of the interval [a, b].

(2) **Refinement of a partition:** Let P be a measurable partition of the closed interval [a, b] and P* be another measurable partition of [a, b] such that every component of P* is contained in some components of P, then P* is called a **refinement of P.**

Obviously P* can be constructed by breaking up the components of P.

Symbolically it is written as $P \subset P^*$.

If $P = \{S_1, S_2, ..., S_n\}$ and $Q = \{T_1, T_2, ..., T_m\}$ be any two measurable partitions of [a, b], then the partition PQ whose components are the sets $S_i \cap T_j$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$ is a common refinement of both P and Q. Thus Partition $PQ = \{S_i \cap T_j : i = 1, 2, ...n; j = 1, 2, ..., m\}$.

Note: Let E be a subset of [a, b] and f be a bounded function on [a, b]. Then we define

$M[f ; E] = \sup \{f(x) : x \in E\}$, $m [f ; E] = \inf \{f(x) : x \in E\}$.

(3) **Lower and Upper Lebesgue Sums:** Let $P = \{S_1, S_2, ..., S_n\}$ be a measurable partition of the closed interval [a, b] and f be a bounded variation defined on [a, b] then we define

$U [f;P]$ = Upper Lebesgue sum = $\sum_{r=1}^{n} M[f ; S_r] m(S_r)$ and

$L [f; P]$ = Lower Lebesgue sum = $\sum_{r=1}^{n} m[f ; S_r] . m(S_r)$, where $m(S_r)$ is the Lebesgue measure of S_r

It is obvious that $L [f ; P] \leq U [f ; P]$.

Also $L [-f, P] = -U [f, P]$ and $U [-f, P] = -L [f, P]$.

Further note that if $S_1, S_2, ..., S_n$ are the interval components of a Riemann subdivision σ of the interval [a, b], then the above defined upper – Lebesgue sum U [f: P] becomes same as the upper Riemann sum $U [f, \sigma] = \sum M [f; S_r]$, (length of interval S_r)], since in case of interval "length of the interval = Lebesgue measure of the interval."

It shows that every $U [f ; \sigma]$ gives a Lebesgue upper sum $U [f ; P]$ and hence the set of number $U[f ; \sigma]$ for all the Riemann subdivisions of [a, b], is a subset of the set of numbers $U [f ; P]$ for all measurable partitions P of the [a, b]. A similar conclusion also holds for $L [f ; \sigma]$ and $L [f ; P]$.

(4) **Upper and Lower Lebesgue Integrals:** Let f be a bounded function defined on [a, b]. The supremum of $L [f : Q]$ is called the lower Lebesgue integral on [a, b] and is denoted by $L\underline{\int_a^b} f(x) dx$, supremum is taken over all measurable partitions Q of [a, b].

Similarly, the infimum of U [f : P] is called the upper Lebesgue integral and is denoted by

$L\overline{\int_a^b} f(x)\,dx.$

Thus, $L\underline{\int_a^b} f(x)\,dx = \sup\{L[f;Q] : Q$ *is a measurable partition of [a, b] }*

and $L\overline{\int_a^b} f(x)\,dx = \inf\{U[f;P]$: *P is a measurable partition of [a, b]}.*

For simplicity, sometimes we denote the lower and upper integrals of $L\underline{\int_a^b} f$ and $L\overline{\int_a^b} f$.

Since $L[-f;Q] = -U[f;Q]$, *and* $U[-f,P] = -L[f,P]$,

$$L\underline{\int_a^b}(-f) = -\left(L\overline{\int_a^b} f\right),\quad L\overline{\int_a^b}(-f) = -\left(L\underline{\int_a^b} f\right)$$

Remarks:

(i) $\underline{\int_a^b} f \geq L[f;Q]\ \forall$ *measurable partitions Q and* $L\overline{\int_a^b} f \leq U[f;P]\ \forall$ *measurable partitions P.*

(ii) *For every* $\in > 0$*, however small, there always exists at least one partition P' such that*

$L\overline{\int_a^b} f + \varepsilon > U[f;P']$.

(iii) *Similarly, for every* $\in > 0$*, however small, there always exists at least one partition Q' such that*

$L\underline{\int_a^b} f - \varepsilon < L[f;Q']$.

(5) **Lebesgue Integral:** *Let f be a bounded function defined on the interval [a, b]; we say that f is Lebesgue integrable on [a, b], iff* $L\underline{\int_a^b} f = L\overline{\int_a^b} f$ *and their common value is called the L-integral of f on [a, b] and is denoted by* $\int_a^b f$.

We shall denote by L [a, b], the class of all bounded functions f which are Lebesgue integrable on [a, b]. Thus $f \in L[a, b] \Leftrightarrow f$ *is L – integrable on [a, b]. The numbers a and b are called the lower and upper limits of integration respectively.*

Results:

1) *Let f be a bounded function defined on [a, b]. Then every upper sum is greater than or equal to every lower sum for f.*

2) *If f is a bounded function on [a, b] and if P is a measurable partition of [a, b], then* $\sup_Q L[f:Q] \leq \inf_P U[f;P]$, *where supremum and infimum are taken over all measurable partitions Q and P of [a, b].*

3) *If f is a bounded function defined on [a, b] and f is R – integrable on [a, b], then f is also L- integrable on [a, b] and* $L\int_a^b f = R\int_a^b f$.

4) *Let f be a bounded measurable real valued function such that* $a \leq f(x) \leq b$ *on a measurable set* $E\,[p, q] \subset \mathbb{R}$. *Then* $a.m(E) \leq \int_E f(x)\,dx \leq b.m(E)$.

5) *Let f be a bounded function defined on [a, b]. Then the function f is L-integrable, iff for each* $\in > 0$*, however small, there exists a measurable partition P of [a, b] such that* $U[f;P] - L[f;P] < \in$

6) *Every bounded measurable function f defined on [a, b] is Lebesgue-integrable over [a, b].*

7) *Let f be a bounded function defined on [a, b] and let f be L-integrable over [a, b]. If* $a < c < b$*, then f is L-integrable over [a, c], f is L-integrable over [c, b] and* $\int_a^b f = \int_a^c f + \int_c^b f$.

8) *If set of discontinuities of a function f is countably infinite in E, then 'f' is Lebesgue integrable in E.*

5.4.4. The Lebesgue Integral of Unbounded Functions:

Suppose f(x) is an unbounded, measurable and non-negative real-valued function defined on [a, b]. Let $n \in \mathbb{N}$ *be arbitrary. We define a function* $[f(x)]_n$ *on [a, b], such that*

$$[f(x)]_n = \begin{cases} f(x), & \text{when } f(x) \leq n \\ n, & \text{when } f(x) > n \end{cases}$$

Thus $[f(x)]_n = \min\{f(x), n\}$.

Note: $[f(x)]_n$ *as defined above is bounded and measurable over [a, b].*

5.4.5 Convergence of Sequences of Measurable Functions:

(1) Convergence almost everywhere:

Let $\langle f_n \rangle$ *be a sequence of measurable functions defined over a measurable set E. Then* $\langle f_n \rangle$ *is said to converge almost everywhere in E if there exists a subset* E_0 *of E such that*

(i) $f_n(x) \to f(x),\ \forall x \in E - E_0,$

(ii) $m(E_0) = 0.$

(2) **Pointwise Convergence:** *Let* $\langle f_n \rangle$ *be a sequence of measurable functions on a measurable set E. Then* $\langle f_n \rangle$ *is said to converge "pointwise" in E, if* $\exists$ *a measurable function f on E such that*

$f_n(x) \to f(x)$, $\forall\ x \in E$ or $\lim_{n\to\infty} f_n(x) = f(x)$

i.e., for arbitrarily chosen positive quantity $\in$, however small we must get a number $n_0(\in, x) \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \in\ \forall\ n \geq n_0$,

Thus the number $n_0(\in, n)$ will depend upon '$\in$' and 'x' both i.e., for different $x \in E$, we may get different numbers n_0, but we must get n_0 for every $x \in E$.

(3) **Convergence in Measure :** Let $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E. Then the sequence $\langle f_n \rangle$ is said to converge in measure to the function f, written as $f_n \xrightarrow{m} f$, if

(i) f is measurable function on E s.t. $f(x) < \infty$ a.e. on the set E.

(ii) $\lim m\,[E(|f_n - f| \geq \in)] = 0$, for $\in > 0$, however small, i.e., for each $\in > 0$, $\exists$ a number $n_0(\delta)$ such that $\forall\ n \geq n_0(\delta)$, we have $m\,[E\,(|f_n - f| \geq \in] < \delta$.

The above condition (ii) is also said as follows:

For each $\in > 0$ and $\delta > 0$, there must exist a positive number n_0 such that $\forall\ n \geq n_0$

$m[E(|f_n - f| \geq \in)] < \delta$.

(4) **Uniform, almost everywhere convergence :** Let $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E. Then the sequence $\langle f_n \rangle$ is said to converge uniformly a.e. to f, if $\exists$ a set $E_0 \subset E$ such that

(i) $m(E_0) = 0$,

(ii) $\langle f_n \rangle$ converges uniformly to f on the set $E - E_0$,

i.e. for arbitrarily chosen small quantity $\in > 0$, we can find a number $n_0(\in) \in \mathbb{N}$, depending upon $\in$ alone and independent of x, such that $\forall\ n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \in$, $\forall\ x \in E - E_0$.

(5) **Lebesgue Bounded Convergence Theorem:** Let $\langle f_n \rangle$ be a sequence of bounded measurable functions defined on a set E of finite measure, if $\exists$ a positive real number M such that $|f_n(x)| \leq M$, $\forall\ n \in \mathbb{N}$ and $\forall\ x \in E$ and $\langle f_n \rangle$ converges in measure to a measurable function f on the set E, then

$\lim_{n\to\infty} \int_E f_n(x) = \int_E f(x)\,dx$.

(6) **Lebesgue Dominated Convergence tTheorem:** Let $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E, such that $|f_n(x)| < \psi(x)$, $\forall\ x \in E$ and $\forall\ n \in \mathbb{N}$ where $\psi(x)$ is an integrable function over E. Let $\langle f_n \rangle$ converge in measure to a measurable function f over E. Then

$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)$.

(7) **Lebesgue Monotone Convergence Theorem:** Let $\langle f_n \rangle$ be a non – decreasing sequence of integrable functions defined over a measurable set E. Let $\lim_{n\to\infty} f_n(x) = f(x)$ be integrable over E, then

$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$.

PRACTICE SET – II

Exercise 1. Let S be a non-measurable subset of $\mathbb{R}$ and T be a measurable subset of $\mathbb{R}$ such that $S \subset T$. Denote the outer measure of a set U by $m^*(U)$. Then, (GATE-2006)

(a) $m^*(T/S) = 0$ and $m^*(S) = 0$
(b) $m^*(T/S) > 0$ and $m^*(S) > 0$
(c) $m^*(T/S) > 0$ and $m^*(S) = 0$
(d) $m^*(T/S) = 0$ and $m^*(S) > 0$

Exercise 2. Let A and B be disjoint subsets of $\mathbb{R}$ and let m^* denote the Lebesgue outer measure on $\mathbb{R}$. Consider the statements:

P: $m^*(A \cup B) = m^*(A) + m^*(B)$

Q: Both A and B are Lebesgue measurable

R: One of A and B is Lebesgue measurable. Which one of the following is correct?

(a) If P is true, then Q is true
(b) If P is NOT true, then R is true
(c) If R is true, then P is NOT true
(d) If R is true, then P is true

Exercise 3. For $0 \leq x \leq 1$, let $f_n(x) = \begin{cases} \frac{n}{1+n}, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$ and $f(x) = \lim_{n\to\infty} f_n(x)$. Then, on the interval [0,1]

(a) f is measurable and Riemann integrable
(b) f is measurable and Lebesgue integrable
(c) f is not measurable
(d) f is not Lebesgue integrable

Exercise 4. Which of the following integrals are convergent?

(a) $\int_0^\infty \frac{dx}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{2}}\right)}$
(b) $\int_0^\infty \frac{dx}{(1+x)^3}dx$
(c) $\int_0^\infty \frac{x^2\,dx}{(1+x)^3}$
(d) $\int_2^\infty \frac{dx}{\sqrt{x^2 - 1}}$

Exercise 5. Choose the correct statement (s):

(a) There is a continuous surjective function from [0, 1) to $\mathbb{R}$.
(b) $\mathbb{R}$ and [0, 1) are homeomorphic to each other.
(c) There is a bijective function from [0, 1) to $\mathbb{R}$.
(d) Bounded subspace of $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}$.

Exercise 6. Consider the following improper integrals:

$$I_1 = \int_1^\infty \frac{dx}{(1+x^2)^{1/2}} \text{ and } I_2 = \int_1^\infty \frac{dx}{(1+x^2)^{3/2}}$$

Then

(a) I_1 converges but not I_2.
(b) I_2 converges but not I_1.
(c) both I_1 and I_2 converge.
(d) neither I_1 nor I_2 converges.

KEY POINTS

- A bounded function f defined on [a, b] is Reimann integrable on [a, b] if and only if $\forall \in > 0, \exists \delta > 0$ such that $U(P, f) - L(P, f) < \in$ for every partition P of [a, b] with $||P|| < \delta$.
- If f is an integrable function on [a, b] and g is a function on [a, b] such that g(x) = f(x) for every x except finite number of points, then g is integrable on [a, b] and $\int_a^b g(x)dx = \int_a^b f(x)dx$.
- A continuous function on [a, b] is Riemann integrable on [a, b].
- A monotonic function on [a, b] is Riemann integrable on [a, b].
- If the set of points of discontinuities of a bounded function f on [a,b] is countable then f is Riemann integrable on [a, b].
- If the set of points of discontinuity of a bounded function f on [a, b] has finite number of limit points then f is Riemann integrable on [a, b].
- If f(x) is bounded monotonic function in [a, b], then f(x) is a function of bounded variation.
- If a function is differentiable in [a, b], the f(x) is a function of bounded variation.
- Every absolutely convergent integral is convergent, i.e., $\int_a^b f(x)dx$ converges if $\int_a^b |f(x)|dx$ converges.
- The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges if and only if $n < 1$.

- The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ converges if and only if $n < 1$.
- A is countable $\Rightarrow m^*(A) = 0$.
- A set with outer measure different from zero is uncountable.
- If A and B are only two disjoint subsets of $\mathbb{R}$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.
- Every countable set is Lebesgue measurable and its measure is zero.

SOLVED QUESTIONS FROM PREVIOUS YEAR PAPERS

Example 1. The limit $\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k^3 - k}$ is equal to ________ **(JAM-2015)**

Solution: 0.25

As $\frac{1}{k^3-k} = \frac{1}{k(k^2-1)} = \frac{1}{k}\left[\frac{1}{2(k-1)} - \frac{1}{2(k+1)}\right]$

$\Rightarrow \lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{k^3-k} = \lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{2k}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)$

Let $a_n = \frac{1}{2n}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$

$\Rightarrow a_2 = \frac{1}{2.2}\left(1-\frac{1}{3}\right)$

$a_3 = \frac{1}{2.3}\left(\frac{1}{2}-\frac{1}{4}\right)$

$a_4 = \frac{1}{2.4}\left(\frac{1}{3}-\frac{1}{5}\right), \ldots, a_n = \frac{1}{2n}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$

$S_n = \sum_{k=2}^{n} a_k = \frac{1}{2.2}\left(1-\frac{1}{3}\right) + \frac{1}{2.3}\left(\frac{1}{2}-\frac{1}{4}\right) + \frac{1}{2.4}\left(\frac{1}{3}-\frac{1}{5}\right) + \ldots + \frac{1}{2n}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{3.4} \ldots + \frac{1}{n(n-1)} - \frac{1}{n(n+1)}\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n(n+1)}\right]$

$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=2}^{n} a_k = \lim_{n\to\infty}\frac{1}{2}\left[\frac{1}{2} - \frac{1}{n(n+1)}\right] = \frac{1}{4}$

$\Rightarrow Ans = \frac{1}{4}$

Example 2. *Let E be a non-measurable subset of [0, 1]. If f : [0, 1] →ℝ is defined by*

$$f(x) = \begin{cases} \frac{-1}{2}, & x \in E \\ 0, & \text{otherwise} \end{cases}. \text{ Then,}$$ *(GATE-2006)*

(a) *f is measurable but not |f|* (b) *|f| is measurable but not f*
(c) *both f and |f| are measurable* (d) *neither f nor |f| is measurable*

Solution: (d) *As* $f^{-1}\left(-\frac{1}{2}\right)$ *and* $|f|^{-1}\left(\frac{1}{2}\right)$ *are not measurable.*

Example 3. *Let* $\{E_n : n = 1, 2, ...\}$ *be a decreasing sequence of Lebesgue measurable sets on ℝ and let F be a Lebesgue measurable set on ℝ such that* $E_1 \cap F = \phi$. *Suppose that F has Lebesgue measure 2 and the Lebesgue measure of* E_n *equals* $\frac{2n+2}{3n+1}, n = 1, 2, \ldots$. *Then the Lebesgue measure of the set* $\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F$ *equals* *(GATE-2007)*

(a) $\frac{5}{3}$ (b) 2 (c) $\frac{7}{3}$ (d) $\frac{8}{3}$

Solution: (d)

$\{E_n : n = 1, 2, ...\}$ *is a decreasing sequence* $E_1 \cap F = \phi$ *so* $E_n \cap F = \phi \ \forall\, n \geq 1$.

$\{E_n \cup F\}_{n \geq 1}$ *is a decreasing sequence of sets. Now* $\mu(F) = 2$, μ *is Lebesgue-measure*

$$\mu\left[\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F\right] = \mu\left[\bigcap_{n=1}^{\infty} (E_n \cup F)\right]$$

$= \lim_{n\to\infty} \mu(E_n \cup F)$ *(Since* $\{E_n \cup F\}$ *is a decreasing sequence)*

$= \lim_{n\to\infty} [\mu(E_n) + \mu(F)]$ *(Since* $E_n \cap F = \phi$*)*

$$= \lim_{n\to\infty}\left[\frac{2n+2}{3n+1} + 2\right] = \lim_{n\to\infty}\left[\frac{2 + \frac{2}{n}}{3 + \frac{1}{n}} + 2\right] = \frac{2}{3} + 2 = \frac{8}{3}.$$

Example 4. *For a positive real number p, let* $(f_n : n = 1, 2, \ldots)$ *be sequence of functions defined on [0,1] by*

$$f_n(x) = \begin{cases} n^{p+1}x, & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{1}{x^p}, & \text{if } \frac{1}{n} < x \leq 1 \end{cases}.$$ *Let* $f(x) = \lim_{n\to\infty} f_n(x), x \in [0,1]$. *Then, on [0, 1].* *(GATE-2007)*

(a) *f is Riemann integrable.*

(b) *the improper integral* $\int_0^1 f(x)dx$ *converges for* $p \geq 1$.

(c) *the improper integral* $\int_0^1 f(x)dx$ *converges for* $p < 1$.

(d) f_n *converges uniformly.*

Solution: (c)

For option (d)

Here $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \begin{cases} 0, & x = 0 \\ \frac{1}{x^p}, & 0 < x \leq 1 \end{cases}$$

f(x) is discontinuous at x = 0

∴ $\{f_n(x)\}$ *doesn't converges uniformly.*

Thus option (d) is incorrect.

For option (c)

Note the fact about convergence of $\int_a^b \frac{dx}{(x-a)^p}$ *for real* $p > 0$.

Case I: $p = 1$

$$\lim_{\varepsilon\to0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)} = \lim_{\varepsilon\to0^+} [\log(b-a) - \log\varepsilon] = 0$$

Case II: $p \in (0,1)$

$$\lim_{\varepsilon\to0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^p} = \frac{(b-a)^{1-p}}{1-p}.$$

Hence integral is convergent.

Case III: $1 < p < \infty$,

$$\lim_{\varepsilon\to0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^p} = \lim_{\varepsilon\to0^+}\left[\frac{(b-a)^{1-p}}{1-p} - \frac{\varepsilon^{1-p}}{1-p}\right] \text{ does not exist.}$$

In our question a = 0, b = 1

∴ *By case II, option (c) is correct.*

By III option (b) is incorrect.

(Note here that in definition of $f(x)$, $0 \leq x \leq \frac{1}{n}$ *does not contribute when* $n \to \infty$. *So in the consideration of f sufficient to see* $\frac{1}{n} < x < 1$).

Example 5. *A function f: ℝ→ ℝ need not be Lebesgue measurable if* *(GATE-2009)*

(a) *f is monotone.*

(b) $\{x \in \mathbb{R} : f(x) \geq a\}$ *is measurable for each* $\alpha \in \mathbb{Q}$.

(c) $\{x \in \mathbb{R} : f(x) \geq a\}$ *is measurable for each* $\alpha \in \mathbb{R}$.

(d) *for each open set G in ℝ,* $f^{-1}(G)$ *is measurable.*

Solution: (c) *Every monotone function f is measurable because for $f^{-1}(\alpha,\infty)$ is an interval for any $\alpha \in \mathbb{R}$. So (a) is not the correct answer. Similarly (b) and (d) are not correct by definition of measurable function. Let's give counter example to show that (c) is the right answer. Define $f:\mathbb{R}\to\mathbb{R}$ by following formula where E is a non – measurable set in $[0,1]$*

$$f(x)=\begin{cases}4 & \text{if } x\in(-\infty,0)\cup(1,\infty)\\ x & \text{if } x\in E\\ x+2 & \text{if } x\in[0,1]/E\end{cases}$$

Then f is not Lebesgue measurable and $\{x\in\mathbb{R} \mid f(x)=\alpha\}$ is measurable for each α.

Example 6. *Let $f,g:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}x & \text{if } x=\frac{1}{n} \text{ for } n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}$ and $g(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0, & \text{otherwise}\end{cases}$*

Then **(GATE-2013)**

(a) both f and g are Riemann integrable.
(b) f is Riemann integrable and g is Lebesgue integrable.
(c) g is Riemann and f is Lebesgue integrable.
(d) neither f nor g is Riemann integrable.

Solution: (b)

Let $f,g:[0,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}x & \text{if } x=\frac{1}{n} \text{ for } n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad g(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0, & \text{otherwise}\end{cases}$$

Then f is Riemann integrable as set of points of discontinuity has unique limit point and g(x) is Lebesgue integrable as g(x) is discontinuous at countably many points.

Example 7. *Define $f:[0,1]\to[0,1]$ by $f(x)=\frac{2^k-1}{2^k}$ for $x\in\left[\frac{2^{k-1}-1}{2^{k-1}},\frac{2^k-1}{2^k}\right], k\geq 1$. Then f is a Riemann – integrable function such that* **(CSIR UGC NET DEC-2011)**

(a) $\int_0^1 f(x)dx=\frac{2}{3}$ (b) $\frac{1}{2}<\int_0^1 f(x)dx<\frac{2}{3}$

(c) $\int_0^1 f(x)dx=1$ (d) $\frac{2}{3}<\int_0^1 f(x)dx<1$

Solution: (a)

$f:[0,1]\to[0,1]$

$f(x)=\frac{2^k-1}{2^k}$

$$\int_0^1 f(x)dx=\sum_{k=1}^{\infty}\int_{\frac{2^{k-1}-1}{2^{k-1}}}^{\frac{2^k-1}{2^k}} f(x)dx$$

$$=\sum_{k=1}^{\infty}\frac{2^k-1}{2^k}\left[\frac{2^k-1}{2^k}-\frac{2^{k-1}-1}{2^{k-1}}\right]$$

$$=\sum_{k=1}^{\infty}\frac{2^k-1}{2^k}\times\frac{1}{2^{k-1}}\left[\frac{2^k-1-2^k+2}{2}\right]$$

$$=\sum_{k=1}^{\infty}\frac{2^k-1}{2^{2k-1+1}}\times 1$$

$$=\sum_{k=1}^{\infty}\frac{2^k-1}{2^{2k}}=\sum_{k=1}^{\infty}\frac{1}{2^k}-\sum_{k=1}^{\infty}\frac{1}{2^{2k}}$$

$$=\frac{\frac{1}{2}}{1-\frac{1}{2}}-\frac{\frac{1}{4}}{1-\frac{1}{4}}=1-\frac{1}{3}=\frac{2}{3}$$

Example 8. *Which of the conditions below imply that a function $f:[0,1]\to\mathbb{R}$ is necessarily of bounded variation?* **(CSIR UGC NET DEC-2012)**

(a) f is a monotone function on [0,1].
(b) f is a continuous and monotone function on [0,1].
(c) f' has a derivative at each $x\in(0,1)$.
(d) f' has a bounded derivative on the interval (0,1).

Solution: (a,b,d)

We know that if 'f' is a monotonic function on [a,b]. Then 'f' is of bounded variation on [a,b]

∴ options (a) and (b) are correct.

Also, if derivative of f(x) is bounded on (a,b), then 'f' is of bounded variation.

∴ option (d) is also correct.

Example 9. *Let $L=\int_0^1\frac{dx}{1+x^8}$. Then* **(CSIR UGC NET DEC-2013)**

(a) $L<1$ (b) $L>1$

(c) $L<\frac{\pi}{4}$ (d) $L>\frac{\pi}{4}$

Solution: (a,d)

Given, $L=\int_0^1\frac{dx}{1+x^8}$

As $0<x<1,\ x^8<x^2\Rightarrow 1+x^8<1+x^2$

$\Rightarrow \frac{1}{1+x^8}>\frac{1}{1+x^2}$

$$L=\int_0^1\frac{dx}{1+x^8}>\int_0^1\frac{dx}{1+x^2}=\tan^{-1}x\Big|_0^1=\tan^{-1}1-\tan^{-1}0\Rightarrow L>\frac{\pi}{4}$$

$\therefore$ option (d) is correct.

Also, $\dfrac{1}{1+x^8}$ is maximum when $x=0$

$$\Rightarrow \frac{1}{1+x^8} < \frac{1}{1+0} \Rightarrow \int_0^1 \frac{dx}{1+x^8} < \int_0^1 dx \Rightarrow L < 1$$

$\therefore$ option (a) is correct.

Example 10. Let α, p be real numbers and $\alpha > 1$. **(CSIR UGC NET JUNE-2014)**

(a) If p>1, then $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$.

(b) If $p > \frac{1}{\alpha}$, then $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$.

(c) If $p < \frac{1}{\alpha}$, then $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$.

(d) For any $p \in \mathbb{R}$, we have $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx = \infty$.

Solution: (d) We will check by options

For option (a)

Given $\alpha > 1$ and $p > 1$

Take $\alpha = 2, p = 2$

$$\therefore I = \int_{-\infty}^{\infty} \frac{1}{x^4} dx = \int_{-\infty}^{0} \frac{1}{x^4} dx + \int_{0}^{\infty} \frac{1}{x^4} dx$$
$$= I_1 + I_2$$

Clearly, I_1 and I_2 both are divergent

$$\therefore \int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx = \infty$$

Thus, option (a) is incorrect

With above example option (b) is also incorrect.

For option (c)

Take, $\alpha = 2, p = -2$

$$\therefore I = \int_{-\infty}^{\infty} \frac{1}{|x|^{-4}} dx = \int_{-\infty}^{\infty} x^4 dx$$

Clearly $I = \infty$

$\therefore$ option (c) is also incorrect

As all other options are incorrect

$\therefore$ option (d) is correct.

Example 11. Let a be positive real number. Which of the following integrals are convergent?

(a) $\int_0^a \frac{1}{x^4} dx$ (b) $\int_0^a \frac{1}{\sqrt{x}} dx$ (c) $\int_4^\infty \frac{1}{x \log_e x} dx$ (d) $\int_5^\infty \frac{1}{x(\log_e x)^2} dx$

Solution: (b,d)

We know that integral $\int_0^a \frac{1}{x^p} dx$ converges for $p < 1$ and diverges for $p \geq 1$

In option (a), $p = 4 > 1$ $\therefore \int_0^a \frac{1}{x^4} dx$ diverges

Thus, option (a) is incorrect

In option (b), $p = \frac{1}{2} < 1$ $\therefore \int_0^a \frac{1}{\sqrt{x}} dx$ converges

Thus, option (b) is convergent.

Further, by Cauchy's integral test, the series $\sum_{n=1}^{\infty} \frac{1}{n(\log e^n)^p}$ and integral $\int_a^\infty \frac{1}{x(\log_e x)^p} (a>0) dx$ behave alike and we know the series $\sum_{n=1}^{\infty} \frac{1}{n(\log_e n)^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Thus, $\int_a^\infty \frac{1}{x(\log_e x)^p} dx$ converges for $p > 1$ and diverges for $p \leq 1$

In option (c), $p = 1$ and in option (d), $p = 2 > 1$

Thus $\int_4^\infty \frac{1}{x \log_e x} dx$ diverges and $\int_5^\infty \frac{1}{x(\log_e x)^2} dx$ converges

Example 12. Consider the improper Riemann integral $\int_0^x y^{-1/2} dy$. This integral is:

(CSIR UGC NET JUNE-2016)

(a) continuous in $[0, \infty)$ (b) continuous only in $(0, \infty)$
(c) discontinuous in $(0, \infty)$ (d) discontinuous only in $(1/2, \infty)$

Solution: (a) The given integral is $\int_0^x y^{\frac{-1}{2}} dy$

Let $f(x) = \int_0^x y^{\frac{-1}{2}} dy$

$\Rightarrow f(x) = [2\sqrt{y}]_0^x \Rightarrow f(x) = 2\sqrt{x}$, which is continuous in $[0, \infty)$

$\therefore$ Option (a) is correct.

ASSIGNMENT 5.1

NOTE: CHOOSE THE BEST OPTION

1. If f_1 is Riemann integrable on [a, c], then $\int_a^b f_1 dx + \int_b^c f_1\, dx$, is

(a) $\int_a^c f_1\, dx + c$ (b) $\int_a^b f_1\, dx + c$

(c) $\int_a^c f_1 dx$ (d) none of these

2. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then

(a) $\int_a^b |f(x)|\, dx \le \left|\int_a^b f(x)dx\right|$ (b) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\, dx$

(c) $\int_a^b |f(x)|\, dx = \left|\int_a^b f(x)dx\right|$ (d) none of these

3. $f : [a, b] \to \mathbb{R}$, P and Q are partitions of [a, b] such that $P \subset Q$, then
(a) $L(P, f) \le L(Q, f)$ (b) $L(P,f) \ge L(Q, f)$
(c) $L(P, f) \ge L(Q, f)$ (d) $L(P,f) \ge L(Q,f)$

4. A function f is Riemann integrable on [a, b] iff

(a) Only $\int_a^{\bar{b}} f(x)dx$ exist (b) Only $\int_{\bar{a}}^{b} f(x)dx$ exist

(c) $\int_a^{\bar{b}} f(x)dx \ne \int_{\bar{a}}^{b} f(x)dx$ (d) $\int_a^{\bar{b}} f(x)dx = \int_{\bar{a}}^{b} f(x)dx$

5. If $\int_{\bar{a}}^{b} f(x)dx$ and $\int_a^{\bar{b}} f(x)dx$ are lower and upper Riemann integral on [a, b]. Then

(a) $\int_{\bar{a}}^{b} f(x)dx \ge \int_a^{\bar{b}} f(x)dx$ (b) $\int_{\bar{a}}^{b} f(x)dx = \int_a^{\bar{b}} f(x)dx$

(c) $\int_{\bar{a}}^{b} f(x)dx \le \int_a^{\bar{b}} f(x)dx$ (d) none of these

6. If $f : [a, b] \to \mathbb{R}$ is monotonic, then
(a) f is of bounded variation. (b) f is unbounded.
(c) the set of discontinuities of f are uncountable. (d) none of these.

7. If f is of bounded variation on [a, b], then total variation of f on [a, b] is
(a) finite number (b) infinite
(c) zero (d) none of these

8. If the total variation of f on [a, b] is zero i.e. $V_f(a, b) = 0$, then
(a) a = b (b) f is constant on [a, b]
(c) f is monotonic on [a, b] (d) none of these

9. If f is Riemann integrable with respect to α on [a, b], then
(a) f is increasing and α is bounded function. (b) f is bounded and α is increasing function.
(c) f and α are both bounded. (d) f and α are both increasing.

10. If f is of bounded variation on [a, b] and $c \in (a,b)$, then
(a) f is of bounded variation on [a, c] and on [c, b].
(b) f is not of bounded variation on [a, c] and on [c, b].
(c) f is constant on [a, c] and [c, b].
(d) none of these.

11. If f is constant function on [a, b], then total variation of f on [a, b]
(a) $V_f(a, b) = 0$ (b) $V_f(a, b) \ge 0$
(c) $V_f(a, b) \le 0$ (d) none of these

12. The total variation on [a, b] is
(a) $V_f(a, b) < 0$ (b) $V_f(a, b) \ge 0$
(c) $V_f(a, b) = 0$ (d) none of these

13. The total variation on [a, b] is
(a) non-negative finite number. (b) non-positive finite number.
(c) extended real number. (d) none of these.

14. The value of integral $\int_1^4 [\log x]dx$, where [ ] denotes greatest integer is

(a) log 4 (b) ½ (c) 4-e (d) 4+e

15. The limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}$ exists and is the Riemann integral of the function

(a) $f(x) = x^2 \quad 0 \le x \le 1$ (b) $f(x) = k$ for all $x, 0 \le x \le 1$
(c) $f(x) = 3x \quad 0 \le x \le 1/3$ (d) $f(x) = x \quad 0 \le x \le 1$

16. A function of bounded variation is
(a) necessarily bounded. (b) necessarily unbounded.
(c) may be bounded or unbounded. (d) none of these.

17. Let $f : [0, 1] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f$ is

(a) Riemann integrable.
(b) continuous.
(c) Lebesgue integrable, but not Riemann integrable.
(d) none of these.

18. Given set $S \subset \mathbb{R}$ with the property that for every $x \in S$ the intersection of S with any interval containing x is countable, then S is

(a) uncountable and not of measure zero.
(b) uncountable but of measure zero.
(c) countable.
(d) none of these.

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

19. If f_1 and f_2 are Riemann integral functions, then $\int_a^b f_1\,dx + \int_a^b f_2\,dx$ is not equal to

(a) $\int_b^a (f_1 + f_2)\,dx$
(b) zero
(c) $\int_a^b (f_1 + f_2)\,dx$
(d) one

20. The length of the greatest of all the sub-intervals $[x_{r-1}, x_r]$ of the division D is called

(a) norm
(b) subnorm
(c) step function
(d) signnum function.

21. $L(P, f)$ is the lower Riemann sum over all partitions on [a, b], then choose the incorrect option

(a) $\int_a^b f(x)\,dx = l.u.b\,\{L(P, f)\}$
(b) $\int_a^b f(x)\,dx = g.l.b\,\{L(P, f)\}$
(c) $\int_a^b f(x)\,dx = L(P, f)$
(d) none of these

22. If f is a Riemann integrable function on [a, b] and λ is any real number then which of the following does not hold?

(a) $\lambda \int_a^b f(x)\,dx = \int_a^b \lambda f(x)\,dx$
(b) $\lambda \int_a^b f(x)\,dx \neq \int_a^b \lambda f(x)\,dx$
(c) $\lambda \int_a^b f(x)\,dx \geq \int_a^b \lambda f(x)\,dx$
(d) all of these

23. A real valued bounded function $f(x)$ is not Riemann integrable on [a, b]. Then may be

(a) $\int_{\bar{a}}^{b} f(x)dx$ and $\int_{a}^{\bar{b}} f(x)dx$ does exist.
(b) $\int_{\bar{a}}^{b} f(x)dx = \int_{a}^{\bar{b}} f(x)dx$.
(c) $\int_{a}^{\bar{b}} f(x)dx \neq \int_{\bar{a}}^{b} f(x)dx$.
(d) $\int_{a}^{\bar{b}} f(x)dx < \int_{\bar{a}}^{b} f(x)dx$.

24. If f is real valued bounded function on [a, b] and m, M are greatest lower bound and least upper bound respectively, then correct option may be

(a) $m(b-a)=M(b-a)$
(b) $m(b-a) \geq M(b-a)$
(c) $m(b-a) \leq M(b-a)$
(d) none of these

25. If f is Riemann integrable on closed interval [a,b], then the function F defined by $F(x) = \int_a^x f(t)dt$ is

(a) continuous function of bounded variation on [a,b].
(b) discontinuous function of bounded variation on [a, b].
(c) function of bounded variation.
(d) continuous function.

26. If f is of bounded variation on [a, b], then

(a) total variation is the sum of positive and negative variations of f over [a, b].
(b) total variation is equal to positive variations of f over [a, b].
(c) total variation is equal to negative variations of f over [a, b].
(d) total variation is not equal to the sum of positive and negative variations of f over [a, b].

27. A function f is of bounded variation on [a b]

(a) if it is the difference of two monotonic real functions on [a, b].
(b) if it is not the difference of two real-valued functions is on [a, b].
(c) Both (a) and (b) are true.
(d) Neither (a) nor (b) is true.

28. Which of the following(s) is/are true?

(a) The set [0,1] is not countable.
(b) The set [0,1] is countable.
(c) $m^*[0,1]=1$.
(d) $m^*[0,1]=0$.

29. Which of the following(s) is/are true?

(a) The interval $[a,\infty)$ is measurable.
(b) Every interval is measurable.
(c) Every open set in $\mathbb{R}$ is measurable.
(d) Every closed set in $\mathbb{R}$ is measurable.

30. Which of the following(s) is/are correct?

(a) Singleton set is measurable.
(b) Measure of singleton set is zero.
(c) Countable set is measurable.
(d) Measure of countable set is zero.

ASSIGNMENT 5.2

NOTE: CHOOSE THE BEST OPTION

1. If $f : [a, b] \to \mathbb{R}$ is continuous and monotonic function, then
   (a) f is Riemann integrable on $[a, b]$.
   (b) f is not Riemann integrable on $[a, b]$.
   (c) f is Riemann integrable on $\mathbb{R}$.
   (d) none of these.

2. $f : [0, 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$. Then
   (a) the upper and lower integrals of f does not exist.
   (b) f is not Riemann integrable .
   (c) f is Riemann integrable.
   (d) none of these.

3. $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$
   (a) the statement is true.
   (b) the statement is false.
   (c) neither true nor false.
   (d) partially true.

4. If f_1 and f_2 are two real valued bounded functions defined on $[a, b]$. Then for every partition P on $[a, b]$
   (a) $U(P, f_1 + f_2) = U(P, f_1) + U(P, f_2)$
   (b) $U(P, f_1 + f_2) \le U(P, f_1) + U(P, f_2)$.
   (c) $U(P, f_1 + f_2) \ge U(P, f_1) + U(P, f_2)$
   (d) none of these.

5. If f and g are of bounded variation, then which of the following is false?
   (a) $f + g$ is of bounded variation.
   (b) $f - g$ is of bounded variation.
   (c) fg is of bounded variation.
   (d) f/g is of bounded variation if $g \ne 0$.

6. If f is of bounded variation on $[a, b]$ and $c \in (a, b)$, then
   (a) $V_f(a, b) \le V_f(a, c) + V_f(c, b)$.
   (b) $V_f(a, b) \ge V_f(a, c)+V_f(a, b)$.
   (c) $V_f(a, b)=V_f(a, c)+V_f(c, b)$.
   (d) none of these.

7. f is of bounded variation on $[a, b]$ iff
   (a) f is the difference of two monotone real valued function on $[a, b]$.
   (b) f is the product of two monotone real valued function on $[a, b]$.
   (c) f is the quotient of two monotone real valued functions on $[a, b]$.
   (d) none of these.

8. If f is continuous on $[a, b]$. Then f is of bounded variation on $[a, b]$ iff
   (a) f is the difference of two monotone continuous function on $[a, b]$.
   (b) f is the product of two monotone continuous functions on $[a, b]$.
   (c) f is the quotient of two monotone continuous functions on $[a, b]$.
   (d) none of these

9. If f is an absolutely continuous function on $[a, b]$, then
   (a) f is of bounded variation
   (b) f is not of bounded variation.
   (c) f is not continuous.
   (d) none of these.

10. If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$, then
   (a) f is constant almost everywhere.
   (b) f is non-constant.
   (c) f is not continuous.
   (d) none of these.

11. If $\int_a^b f(x)dx$ exists, then the function f is bounded and integrable over.
   (a) $[a, b]$
   (b) (a,b)
   (c) $[a, b)$
   (d) $(a,b]$

12. If a function f is Riemann integrable, then Lebesgue integration is equal to
   (a) Definite integral
   (b) Indefinite integral
   (c) Riemann integral
   (d) none of these

13. A function of bounded variation is
   (a) not necessarily continuous.
   (b) necessarily continuous.
   (c) both (a) and (b).
   (d) none of these.

14. $f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & \text{if } x>0 \\ 0, & \text{if } x \le 0 \end{cases}$, then in $[0, 1]$
   (a) $f(x)$ is bounded variation and $f'(x)$ exists.
   (b) $f(x)$ is of bounded variation $f'\left(\frac{1}{2}\right)$ does not exist.
   (c) $f(x)$ is not of bounded variation and $f(x)$ is continuous.
   (d) $f(x)$ is discontinuous at $x = \frac{2}{\pi}$.

15. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational in } [0,1] \\ -1, & \text{if } x \text{ is irrational in } [0,1] \end{cases}$, then in $[0,1]$
   (a) $f(x)$ is continuous everywhere.
   (b) $f(x)$ is Riemann integrable.
   (c) $f(x)$ is not Riemann integrable.
   (d) $f(x)$ is continuous only at the irrationals.

16. The integral $\int_{-1}^{1} \frac{dx}{x^3}$
   (a) does not exist
   (b) exist
   (c) divergent
   (d) oscillating finitely

17. A continuous function
(a) is always a function of bounded variation.
(b) is never a function of bounded variation.
(c) may or may not be a function of bounded variation.
(d) none of these.

18. If $\alpha > 0$ and $\beta > 0$, the function $f(x)=\begin{cases} x^{\alpha}\left(\sin\frac{1}{x}\right)^{\beta}, & \text{when } x\neq 0 \\ 0, & \text{when } x=0 \end{cases}$ then f(x) is of bounded variation in [0,1] if
(a) $\alpha > \beta$ (b) $\alpha < \beta$ (c) $1+\alpha < \beta$ (d) none

19. Let $f:[0,2]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} 0 & \text{if } x \text{ irrational} \\ 1/2 & \text{if } x \text{ is rational} \end{cases}$.
Let A and B be the upper Riemann integral and the lower Riemann integral respectively of f on [0,2], then
(a) A=B=0
(b) A=1 and B=0
(c) A=1/2 and B=0
(d) A=B=1/2

20. The total variation of the function sin x on the interval $[0,2\pi]$ is
(a) 1 (b) 2 (c) 2π (d) 4

21. Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ if $x \neq 0$, $f(0) = 0$. Then
(a) f is of bounded variation in [0, 1] but $f'(0)$ does not exist.
(b) f is of bounded variation in [0, 1] and $f'(0)$ exist.
(c) f is neither of bounded variation in [0, 1] nor $f'(0)$ exist.
(d) f is not of bounded variation but $f'(0)$ exists.

22. Let $f_n(x)=\begin{cases} 1/x, & n<x<n+1 \\ 0, & \text{otherwise} \end{cases}$. If $f(x)=\lim_{n\to\infty} f_n(x)$, then $\int_{(0,\infty)} f(x)dx = \lim_{n\to\infty}\int_{(0,\infty)} f_n(x)dx$ follows by
(a) bounded convergence theorem.
(b) monotone convergence theorem.
(c) dominated convergence theorem.
(d) none of these.

23. Let S be a non empty Lebesgue measurable subset of $\mathbb{R}$ such that every subset of S is measurable. Then the measure of S is equal to the measure of any
(a) subset of S.
(b) countable subset of S.
(c) bounded subset of S.
(d) closed subset of S.

24. Suppose S_1, S_2 and S_3 are measurable subsets of [0,1] each of measure 3/4 such that measure of $S_1\cup S_2\cup S_3$ is 1. Then, the measure of $S_1\cap S_2\cap S_3$ lies in
(a) $\left[0,\frac{1}{16}\right]$ (b) $\left[\frac{1}{16},\frac{1}{8}\right]$ (c) $\left[\frac{1}{8},\frac{1}{4}\right]$ (d) $\left[\frac{1}{4},1\right]$

25. Suppose E is a non-measurable subset of [0,1]. Let $P = E^{o}\cup\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$ and $Q=\overline{E}\cup\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$, where E^{o} is the interior of E and $\overline{E}$ is the closure of E. Then
(a) P is measurable but not Q.
(b) Q is measurable but not P.
(c) both P and Q are measurable.
(d) neither P nor Q is measurable.

NOTE: MORE THAN ONE OPTION MAY BE CORRECT

26. $f(x)=\begin{cases} x\sin\frac{\pi}{x}, & \text{when } 0<x\leq 1 \\ 0, & \text{when } x=0 \end{cases}$, then in [0,1]
(a) f(x) is continuous on [0,1] and f is of bounded variation on [0, 1].
(b) f(x) is continuous on [0, 1].
(c) f is not of bounded variation on [0, 1].
(d) f(x) is not continuous on [0, 1].

27. f(x) = [x], then in [0, 2]
(a) f is not continuous.
(b) f is of bounded variation.
(c) f is not of bounded variation and continuous.
(d) f is not of bounded variation and not continuous.

28. $\int_0^1 \frac{dx}{x^3}$ is not
(a) divergent
(b) convergent
(c) oscillating infinitely
(d) oscillating finitely

29. $\int_0^1 \frac{dx}{\sqrt{x-x^2}}$ is
(a) convergent and converges to $\frac{\pi}{2}$.
(b) convergent and converges to π.
(c) divergent and diverges to ∞.
(d) convergent.

30. $\int_0^{\infty} \frac{\sin^2 x}{x^2}dx$ is not
(a) convergent
(b) divergent to $+\infty$
(c) divergent to $-\infty$
(d) oscillating

31. $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}dx$ is not

(a) convergent (b) divergent
(c) oscillating finitely (d) oscillating infinitely

32. If the derivative f' exists and is bounded on $[a, b]$ then
(a) the function f is of bounded variation on $[a, b]$.
(b) the function f is not of bounded variation on $[a, b]$.
(c) the function f is not necessarily continuous.
(d) the function f is necessarily continuous.

33. Let A and B be two sets of positive real numbers bounded above. Let $a = \text{Sup}A$ and $b = \text{Sup}B$ and $C = \{xy : x \in A \text{ and } y \in B\}$, then
(a) $x > 0, y > 0 \Rightarrow xy \le ab$ (b) ab is an upper bound for C
(c) $ab = \text{Sup } C$ (d) ab is an lower bound for C

34. Let f be bounded and integrable in $[-\pi, \pi]$ and monotonic in $[-\alpha, 0)$ and $(0, \alpha]$, (not necessarily in the same sense), where α, is some positive number less than π. Then,

(a) $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(+0)+f(-0)}{2}$

(b) $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \neq \frac{f(+0)+f(-0)}{2}$

(c) $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)dx + \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sum_{n=1}^{\infty}\cos n\pi x\, dx$

(d) $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{1}{\pi}[f(-0)+f(+0)]\int_0^{\infty}\frac{\sin x}{x}dx$ as $m \to \infty$

35. Let $f : [0, 1] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f$ is
(a) Riemann integrable.
(b) continuous.
(c) Lebesgue integrable, but not Riemann integrable.
(d) not necessarily measurable.

36. Let $f:[a,b] \to (0, \infty)$ be continuous. Let $G_f = \{(x, y) : y = f(x)\}$ be a graph of f. Then

(a) G_f is measurable with measure zero in $\mathbb{R}^2$.
(b) G_f is measurable only if f is differentiable in (a, b).
(c) G_f is measurable and the measure of G_f lies between $(b-a)f(a)$ and $(b-a)f(b)$.
(d) G_f need not be measurable.

37. Let A be the set of points in the interval $(0,1)$ representing the numbers whose expansion as infinite decimals do not contain the digit 7. Then the measure of A is
(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) ∞

38. A uniformly continuous function is
(a) measurable (b) not measurable
(c) measurable and simple (d) integrable and simple

39. Which of the following(s) is/are true?
(a) The set of rational numbers is Lebesgue measurable.
(b) The set of rational numbers have Lebesgue outer measure equal to zero.
(c) The set of rational numbers is not Lebesgue measurable.
(d) The set of rational numbers have Lebesgue outer measure equal to one.

40. Which of the following(s) is/are correct?
(a) If f is measurable, then $|f|$ is measurable.
(b) If f is a measurable function on $[a,b]$ and if $k \in \mathbb{R}$ then $f+k$ and kf are measurable.
(c) Constant functions are measurable.
(d) None of the above

41. If $\{f_n\}$ is a sequence of measurable functions on $[a,b]$ such that the sequence $\{f_n(x)\}$ is bounded for every $x \in [a,b]$. Then, the functions
(a) $g(x) = \text{lub}\{f_1(x), f_2(x), f_3(x), \ldots\}$ is measurable.
(b) $g(x) = \text{glb}\{f_1(x), f_2(x), f_3(x), \ldots\}$ is measurable.
(c) $g(x) = \text{lub}\{f_1(x), f_2(x), f_3(x), \ldots\}$ is not measurable.
(d) $g(x) = \text{glb}\{f_1(x), f_2(x), f_3(x), \ldots\}$ is not measurable.

42. Let $< f_n >$ be a sequence of measurable functions (with the same domain), then
(a) $\inf\{f_1, f_1, \ldots, f_n\}$ is measurable. (b) $\inf\{f_1, f_1, \ldots, f_n\}$ is not measurable.
(c) $\inf_n f_n$ is measurable. (d) $\inf_n f_n$ is not measurable.

43. If $< f_n >$ is a sequence of measurable functions (with the same domain), then
(a) $\sup\{f_1, f_2, \ldots, f_n\}$ is measurable. (b) $\sup\{f_1, f_2, \ldots, f_n\}$ is not measurable.
(c) $\sup_n f_n$ is measurable. (d) $\sup_n f_n$ is not measurable.

44. c is fixed point in $[0,1]$. A function $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 1$ if $x < c = 2$, otherwise it is given that the Riemann integral $\int_0^1 f(x)\,dx$ is 3/2. Then the value of c is not equal
(a) 1/2 (b) 1/3 (c) 1/4 (d) 1

ANSWERS TO EXERCISES

(PRACTICE SET – I)

Exercise 1: (a) *Exercise 2:* (c) *Exercise 3:* (a,d) *Exercise 4:* (a,d)

Exercise 5: $\frac{\pi^2}{6}-1$ *Exercise 6:* (c) *Exercise 7:* (a,c)

(PRACTICE SET – II)

Exercise 1: (b) *Exercise 2:* (d) *Exercise 3:* (b) *Exercise 4:* (b,d)

Exercise 5: (a,c) *Exercise 6:* (c)

ANSWERS TO ASSIGNMENTS

ASSIGNMENT 5.1

1. (c)	2. (b)	3. (a)	4. (d)	5. (c)	6. (a)	7. (a)
8. (b)	9. (b)	10. (a)	11. (a)	12. (b)	13. (a)	14. (c)
15. (d)	16. (a)	17. (a)	18. (c)			
19. (a,b,d)	20. (a)	21. (b,c)	22. (b,c,d)	23. (a,c,d)	24. (a,c)	25. (a,c,d)
26. (a)	27. (a)	28. (a,c)	29. (a,b,c)	30. (a,b,c,d)		

ASSIGNMENT 5.2

1. (a)	2. (b)	3. (a)	4. (b)	5. (d)	6. (c)	7. (a)
8. (a)	9. (a)	10. (a)	11. (a)	12. (c)	13. (a)	14. (a)
15. (c)	16. (a)	17. (c)	18. (a)	19. (b)	20. (d)	21. (b)
22. (c)	23. (a)	24. (d)	25. (a)	26. (b,c)		
27. (a,b)	28. (b,c,d)	29. (b,d)	30. (b,c,d)	31. (a,c,d)	32. (a,d)	33. (a,b,c)
34. (a,c)	35. (a)	36. (a)	37. (b)	38. (a)	39. (a,b)	40. (a,b,c)
41. (a,b)	42. (a,c)	43. (a,c)	44. (b,c,d)			

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MY NOTES