

We shall now discuss the concept of functions of bounded variation which is closely associated to the concept of monotonic functions and has wide application in mathematics. These functions are used in Riemann-Stieltjes integrals and Fourier series.

Let a function f be defined on an interval $[a, b]$ and $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$. Consider the sum $\sum_{i=1}^n |f(x_i) - f(x_{i-1})|$. The set of these sums is infinite. It changes when we make a refinement in a partition. If this set of sums is bounded above then the function f is said to be a *bounded variation* and the supremum of the set is called the *total variation* of the function f on $[a, b]$, and is denoted by $V(f; a, b)$ or $V_f(a, b)$ and it is also affiliated as $V(f)$ or V_f .

Thus

$$V(f; a, b) = \sup \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

The supremum being taken over all the partition of $[a, b]$.

Hence the function f is said to be of *bounded variation* on $[a, b]$ if, and only, if its total variation is finite i.e. $V(f; a, b) < \infty$.

✂ Note

Since for $x \leq c \leq y$, we have

$$|f(y) - f(x)| \leq |f(y) - f(c)| + |f(c) - f(x)|$$

Therefore the sum $\sum |f(x_i) - f(x_{i-1})|$ can not be decrease (it can, in fact only increase) by the refinement of the partition.

✂ Theorem

A bounded monotonic function is a function of bounded variation.

Proof

Suppose a function f is monotonically increasing on $[a, b]$ and P is any partition of $[a, b]$ then

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = f(b) - f(a)$$

$$\therefore V(f; a, b) = \sup \sum |f(x_i) - f(x_{i-1})| = f(b) - f(a) \text{ (finite)}$$

Hence the function f is of bounded variation on $[a, b]$.

Similarly a monotonically decreasing bounded function is of bounded variation with total variation $= f(a) - f(b)$.

Thus for a bounded monotonic function f

$$V(f) = |f(b) - f(a)|$$

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**Example**

A continuous function may not be a function of bounded variation.

e.g. Consider a function  $f$ , where

$$f(x) = \begin{cases} x \sin \frac{\pi}{x} & ; \text{ when } 0 < x \leq 1 \\ 0 & ; \text{ when } x = 0 \end{cases}$$

It is clear that  $f$  is continuous on  $[0,1]$ .

Let us choose the partition  $P = \left\{ 0, \frac{2}{2n+1}, \frac{2}{2n-1}, \dots, \frac{2}{5}, \frac{2}{3}, 1 \right\}$

Then

$$\begin{aligned} \sum |f(x_i) - f(x_{i-1})| &= \left| f(1) - f\left(\frac{2}{3}\right) \right| + \left| f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right) \right| + \dots + \left| f\left(\frac{2}{2n+1}\right) - f(0) \right| \\ &= \left| \sin \pi - \frac{2}{3} \sin\left(\frac{3\pi}{2}\right) \right| + \left| \frac{2}{3} \sin\left(\frac{3\pi}{2}\right) - \frac{2}{5} \sin\left(\frac{5\pi}{2}\right) \right| + \dots \\ &\quad \dots + \left| \frac{2}{2n+1} \sin\left(\frac{(2n+1)\pi}{2}\right) - 0 \right| \\ &= \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(\frac{2}{5} + \frac{2}{7}\right) + \dots + \left(\frac{2}{2n-1} + \frac{2}{2n+1}\right) + \frac{2}{2n+1} \\ &= \left(2\left(\frac{2}{3}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{2}{7}\right) + \dots + 2\left(\frac{2}{2n+1}\right)\right) \\ &= 4\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right) \end{aligned}$$

Since the infinite series  $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$  is divergent, therefore its partial sums sequence  $\{S_n\}$ , where  $S_n = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}$ , is not bounded above.

Thus  $\sum |f(x_i) - f(x_{i-1})|$  can be made arbitrarily large by taking  $n$  sufficiently large.

$\Rightarrow V(f; 0,1) \rightarrow \infty$  and so  $f$  is not of bounded variation.  $\square$

**Remarks**

A function of bounded variation is not necessarily continuous.

e.g. the step-function  $f(x) = [x]$ , where  $[x]$  denotes the greatest integer not greater than  $x$ , is a function of bounded variation on  $[0,2]$  but is not continuous.

**Theorem**

If the derivative of the function  $f$  exists and is bounded on  $[a,b]$ , then  $f$  is of bounded variation on  $[a,b]$ .

**Proof**

$\because f'$  is bounded on  $[a,b]$

$\therefore \exists k$  such that  $|f'(x)| \leq k \quad \forall x \in [a,b]$ .

Let  $P$  be any partition of the interval  $[a,b]$  then

$$\begin{aligned} \sum |f(x_i) - f(x_{i-1})| &= \sum |x_i - x_{i-1}| f'(c) \quad , \quad c \in [a,b] \quad (\text{by M.V.T}) \\ &\leq k |b - a| \end{aligned}$$

$\Rightarrow V(f; a,b)$  is finite.  $\Rightarrow f$  is of bounded variation.  $\square$

**Note**

Boundedness of  $f'$  is a sufficient condition for  $V(f)$  to be finite and is not necessary.

**∅ Theorem**

A function of bounded variation is necessarily bounded.

**Proof**

Suppose  $f$  is of bounded variation on  $[a, b]$ .

For any  $x \in [a, b]$ , consider the partition  $\{a, x, b\}$ , consisting of just three points then

$$\begin{aligned} & |f(x) - f(a)| + |f(b) - f(x)| \leq V(f; a, b) \\ \Rightarrow & |f(x) - f(a)| \leq V(f; a, b) \end{aligned}$$

Again

$$\begin{aligned} |f(x)| &= |f(a) + f(x) - f(a)| \\ &\leq |f(a)| + |f(x) - f(a)| \\ &\leq |f(a)| + V(f; a, b) < \infty \\ \Rightarrow & f \text{ is bounded on } [a, b]. \end{aligned}$$

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