$$|z+x|^2 \geqq [x+\operatorname{Re}(z)]^2 + |z|^2 \sin^2\delta, \qquad \text{for } \operatorname{Re}(z) < 0.$$

It follows that

$$\int_0^\infty \frac{[P^2(x) - \frac{1}{12}]\,dx}{(z+x)^2} = O\left(\frac{1}{|z|}\right),$$

as $|z| \to \infty$ in $|\arg z| \leqq \pi - \delta$, $\delta > 0$.

We have shown that as $|z| \to \infty$ in $|\arg z| \leqq \pi - \delta$, $\delta > 0$,

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \operatorname{Log} z - z + \tfrac{1}{2} \operatorname{Log}(2\pi) + o(1). \tag{4}$$

Indeed we showed a little more than that, but (4) is itself more precise than is needed later in this book.

From (4) we obtain at once the actual result to be employed in Chapter 5.

THEOREM 13. *As $|z| \to \infty$ in the region where $|\arg z| \leqq \pi - \delta$ and $|\arg(z+a)| \leqq \pi - \delta$, $\delta > 0$,*

$$\log \Gamma(z+a) = (z + a - \tfrac{1}{2}) \operatorname{Log} z - z + O(1). \tag{5}$$

EXERCISES

1. Start with $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \dfrac{1}{z} - \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$,

prove that

$$\frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2 \operatorname{Log} 2,$$

and thus derive Legendre's duplication formula, page 24.

2. Show that $\Gamma'(\tfrac{1}{2}) = -(\gamma + 2 \operatorname{Log} 2)\sqrt{\pi}$.

3. Use Euler's integral form $\Gamma(z) = \displaystyle\int_0^\infty e^{-t} t^{z-1}\,dt$ to show that $\Gamma(z+1) = z\Gamma(z)$.

4. Show that $\Gamma(z) = \displaystyle\lim_{n\to\infty} n^z B(z, n)$.

5. Derive the following properties of the Beta function:
   (a) $pB(p, q+1) = qB(p+1, q)$;
   (b) $B(p, q) = B(p+1, q) + B(p, q+1)$;
   (c) $(p+q)B(p, q+1) = qB(p, q)$;
   (d) $B(p, q)B(p+q, r) = B(q, r)B(q+r, p)$.

6. Show that for positive integral n, $B(p, n+1) = n!/(p)_{n+1}$.

7. Evaluate $\displaystyle\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx$.

Ans. $2^{p+q-1}B(p, q)$.

Solution 11. *Let*

$$
\begin{aligned}
1+a_n(z) &= \left(1-\frac{z}{c+n}\right)\exp\left(\frac{z}{n}\right)\\
&= \left(1+\frac{z}{n}+\frac{1}{2!}\frac{z^2}{n^2}+\frac{1}{3!}\frac{z^3}{n^3}+\dots\right)-\frac{1}{c+n}\left(z+\frac{z^2}{n}+\frac{1}{2!}\frac{z^3}{n^2}+\frac{1}{3!}\frac{z^4}{n^3}+\dots\right)\\
&= 1+\left(\frac{1}{n}-\frac{1}{c+n}\right)z+\left(\frac{1}{2!n^2}-\frac{1}{n(c+n)}\right)z^2+\left(\frac{1}{3!n^3}-\frac{1}{2!n^2(c+n)}\right)z^3+\dots\\
&= 1+\frac{c}{n(c+n)}z+\frac{c-n}{2n^2(c+n)}z^2+\sum_{k=3}^{\infty}\frac{c-(k-1)n}{k!n^k(c+n)}z^k\\
&= 1+\frac{c}{n(c+n)}z-\frac{1}{2n(c+n)}z^2+\mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Thus, for c not a negative integer and for any finite z, there is a constant M such that $|a_n(z)| \le \dfrac{M}{n^2}$ and so by Theorems 3 and 4 the product converges absolutely and uniformly.

2. Chapter 2 Solutions

Problem 1. *Start with* (†) $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \dfrac{1}{z} - \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$, *prove that*

$$\frac{2\Gamma'(2z)}{\Gamma(2z)}-\frac{\Gamma'(z)}{\Gamma(z)}-\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}=2\log 2,$$

and thus derive Legendre's duplication formula, page 24.

Solution 1. *Applying* (†) *three times and simplifying yields*

$$
\begin{aligned}
&\frac{2\Gamma'(2z)}{\Gamma(2z)}-\frac{\Gamma'(z)}{\Gamma(z)}-\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\\
&= -2\gamma-\frac{2}{2z}+\gamma+\frac{1}{z}+\gamma+\frac{1}{z+\frac{1}{2}}-\sum_{n=1}^{\infty}\left(\frac{2}{2z+n}-\frac{2}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{z+\frac{1}{2}+n}-\frac{1}{n}\right)\\
&= \frac{2}{2z+1}-\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\frac{2}{2z+k}-\frac{2}{k}\right)+\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{z+k}-\frac{1}{k}\right)+\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{2z+1+2k}-\frac{1}{k}\right)\\
&= \frac{2}{2z+1}+\lim_{n\to\infty}\left[\sum_{k=1}^{2n}\frac{-2}{2z+k}+2H_{2n}+\sum_{k=1}^{n}\frac{2}{2z+2k}-H_n+\sum_{k=1}^{n}\frac{2}{2z+2k+1}-H_n\right]\\
&= \frac{2}{2z+1}+\lim_{n\to\infty}\left[\sum_{k=1}^{2n}\frac{-2}{2z+k}+\sum_{k=2}^{2n+1}\frac{2}{2z+k}+2H_{2n}-2H_n\right]\\
&= \frac{2}{2z+1}+\frac{-2}{2z+1}+\lim_{n\to\infty}\frac{2}{2z+2n+1}+\lim_{n\to\infty}(2H_{2n}-2H_n)\\
&= 0+0+2\lim_{n\to\infty}[(H_{2n}-\log 2n)-(H_n-\log n)+\log 2n-\log n]\\
&= 2[\gamma-\gamma+\log 2]\\
&= 2\log 2. \quad \square
\end{aligned}
$$

Problem 2. *Show that* $\Gamma'(\frac{1}{2}) = -(\gamma+2\log 2)\sqrt{\pi}$.

Solution 2. *By Problem 1, we know that*

$$\frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\log 2.$$

Now let $z = \dfrac{1}{2}$ *to get*

$$2\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} - \frac{\Gamma'(1)}{\Gamma(1)} = 2\log 2,$$

and so, algebra yields

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{\Gamma'(1)}{\Gamma(1)} - 2\log 2.$$

But $\Gamma(1) = 1, \Gamma'(1) = -\gamma, \Gamma(\frac{1}{2}) = \sqrt{\pi}$, *hence*

$$\frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} = -\frac{\gamma}{1} - 2\log 2,$$

and by rearrangement,

$$\Gamma'(\frac{1}{2}) = -(\gamma + 2\log 2)\sqrt{\pi}. \quad \square$$

Problem 3. *Use Euler's integral form* $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}dt$ *to show that*

$$\Gamma(z+1) = z\Gamma(z).$$

Solution 3. *From* $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}dt$, *for* $\mathscr{R}(z) > 0$, *integration by parts yields*

$$\begin{array}{ll|l} u = t^z & dv = e^{-t}dt & \Gamma(z+1) = \displaystyle\int_0^\infty e^{-t}t^z dt \\ du = zt^{z-1} & v = -e^{-t} & \quad = [-t^z e^{-t}]_0^\infty + z\displaystyle\int_0^\infty e^{-t}t^{z-1}dt \\ & & \quad = 0 + z\Gamma(z), \end{array}$$

where $\lim\limits_{t\to\infty} -t^z e^{-t}$ *converges for* $\mathscr{R}(z) > 0$ $\square$.

Problem 4. *Show that* $\Gamma(z) = \lim\limits_{n\to\infty} n^z B(z,n)$.

Solution 4. *From page 28 (1), we know*

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!n^z}{(z)_n},$$

but

$$B(z,n) = \frac{\Gamma(z)\Gamma(n)}{\Gamma(z+n)} = \frac{\Gamma(z)(n-1)!}{(z)_n\Gamma(z)} = \frac{(n-1)!}{(z)_n}.$$

Hence

$$\Gamma(z) = \lim_{n\to\infty} n^z B(z,n). \quad \square$$

Problem 5. *Derive the following properties of the beta function:*
(a) $pB(p,q+1) = qB(p+1,q)$;
(b) $B(p,q) = B(p+1,q) + B(p,q+1)$;
(c) $(p+q)B(p,q+1) = qB(p,q)$;
(d) $B(p,q)B(p+q,r) = B(q,r)B(q+r,p)$.

Solution 5. *(a) We know* $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, *so*

$$pB(p,q+1) = \frac{p\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \frac{\Gamma(p+1)q\Gamma(q)}{\Gamma(p+1+q)} = qB(p+1,q).$$

(note: $p \to q$ and $q \to p$ – is this the symmetric property?)

(b)

$$\begin{aligned}
B(p,q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\
&= \frac{\Gamma(p)\Gamma(q)}{\dfrac{\Gamma(p+q+1)}{p+q}} \\
&= \frac{(p+q)\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \\
&= \frac{p\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} + \frac{q\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \\
&= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1) + \dfrac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}} \\
&= B(p+1,q) + B(p,q+1).
\end{aligned}$$

(c)

$$\begin{aligned}
(p+q)B(p,q+1) &= \frac{(p+q)\Gamma(p))\Gamma(q+1)}{\Gamma(p+q+1} \\
&= \frac{(p+q)\Gamma(p)\Gamma(q+1)}{(p+q)\Gamma(p+q)} \\
&= \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q)} \\
&= \frac{\Gamma(p)q\Gamma(q)}{\Gamma(p+q)} \\
&= qB(p,q).
\end{aligned}$$

(d)

$$\begin{aligned}
B(p,q)B(p+q,n) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\frac{\Gamma(p+q)\Gamma(n)}{\Gamma(p+q+n)} \\
&= \frac{\Gamma(p)\Gamma(q)\Gamma(n)}{\Gamma(p+q+n)} \\
&= \frac{\Gamma(q)\Gamma(n)}{\Gamma(q+n)}\frac{\Gamma(p)\Gamma(q+n)}{\Gamma(p+q+n)} \\
&= B(q,n)B(q+n,p). \quad \square
\end{aligned}$$

Solution 5. *(a) We know* $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, *so*

$$pB(p,q+1) = \frac{p\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \frac{\Gamma(p+1)q\Gamma(q)}{\Gamma(p+1+q)} = qB(p+1,q).$$

(note: $p \to q$ and $q \to p$ – is this the symmetric property?)

(b)

$$\begin{aligned} B(p,q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\ &= \frac{\Gamma(p)\Gamma(q)}{\dfrac{\Gamma(p+q+1)}{p+q}} \\ &= \frac{(p+q)\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \\ &= \frac{p\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} + \frac{q\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \\ &= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1) + \dfrac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}} \\ &= B(p+1,q) + B(p,q+1). \end{aligned}$$

(c)

$$\begin{aligned} (p+q)B(p,q+1) &= \frac{(p+q)\Gamma(p))\Gamma(q+1)}{\Gamma(p+q+1} \\ &= \frac{(p+q)\Gamma(p)\Gamma(q+1)}{(p+q)\Gamma(p+q)} \\ &= \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q)} \\ &= \frac{\Gamma(p)q\Gamma(q)}{\Gamma(p+q)} \\ &= qB(p,q). \end{aligned}$$

(d)

$$\begin{aligned} B(p,q)B(p+q,n) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\frac{\Gamma(p+q)\Gamma(n)}{\Gamma(p+q+n)} \\ &= \frac{\Gamma(p)\Gamma(q)\Gamma(n)}{\Gamma(p+q+n)} \\ &= \frac{\Gamma(q)\Gamma(n)}{\Gamma(q+n)}\frac{\Gamma(p)\Gamma(q+n)}{\Gamma(p+q+n)} \\ &= B(q,n)B(q+n,p). \quad \square \end{aligned}$$

Problem 6. *Show that for positive integral n, $B(p, n+1) = \dfrac{n!}{(p)_{n+1}}$.*

Solution 6. *For integer n and using Theorem 9 (pg. 23),*

$$\begin{aligned} B(p, n+1) &= \frac{\Gamma(p)\Gamma(n+1)}{\Gamma(p+n+1)} \\ &= \frac{\Gamma(p)\Gamma(n+1)}{(p+1)_n\Gamma(p+1)} \\ &= \frac{\Gamma(p)n!}{(p+1)_n p\Gamma(p)} \\ &= \frac{n!}{p(p+1)_n} \\ &= \frac{n!}{(p)_{n+1}}. \quad \square \end{aligned}$$

Problem 7. *Evaluate $\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}dx$.*

Solution 7. *Let $A = \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}dx$. Now let $y = \dfrac{1+x}{2}$, $x = 2y - 1$, $1 - x = 2 - 2y = 2(1-y)$. Hence*

$$\begin{aligned} A &= \int_0^1 2^{p-1}y^{p-1}2^{q-1}(1-y)^{q-1}2dy \\ &= 2^{p+q-1}\int_0^1 y^{p-1}(1-y)^{q-1}dy \\ &= 2^{p+q-1}B(p,q). \quad \square \end{aligned}$$

Problem 8. *Show that for $0 \le k \le n$,*

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}.$$

Note particularly the special case $\alpha = 1$.

Solution 8. *Consider $(\alpha)_{n-k}$ for $0 \le k \le n$. Then*

$$\begin{aligned} (\alpha)_{n-k} &= \alpha(\alpha+1)\dots(\alpha+n-k-1) \\ &= \frac{\alpha(\alpha+1)\dots(\alpha+n-k-1)[(\alpha+n-k)(\alpha+n-k+1)\dots(\alpha+n-1)]}{(\alpha+n-1)(\alpha+n-2)\dots(\alpha+n-k)} \\ &= \frac{(\alpha)_n}{(\alpha+n-k)_k} \\ &= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)_k} \\ &= \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}. \end{aligned}$$

Note for $\alpha = 1$, that $(n-k)! = \dfrac{(-1)^k n!}{(-n)_k}$. $\square$