From the convergence of the integral on the right in (8) it follows that

$$\lim_{n\to\infty} \int_n^\infty e^{-t} t^{z-1}\, dt = 0.$$

Hence

(9) $$\int_0^\infty e^{-t} t^{z-1}\, dt - \Gamma(z) = \lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt.$$

But, by Lemma 3 and the fact that $|t^z| = t^{\mathrm{Re}(z)}$,

$$\left| \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt \right| \leq \int_0^n \frac{t^2 e^{-t}}{n} \cdot t^{\mathrm{Re}(z)-1}\, dt$$

$$\leq \frac{1}{n} \int_0^n e^{-t} t^{\mathrm{Re}(z)+1}\, dt.$$

Now $\int_0^\infty e^{-t} t^{\mathrm{Re}(z)+1}\, dt$ converges, so $\int_0^n e^{-t} t^{\mathrm{Re}(z)+1}\, dt$ is bounded. Therefore

$$\lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt = 0,$$

and we may conclude from equation (9) that (8) is valid.

16. The Beta function. We define the Beta function $B(p, q)$ by

(1) $$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

Another useful form for this function can be obtained by putting $t = \sin^2 \varphi$, thus arriving at

(2) $$B(p,q) = 2\int_0^{\frac{1}{2}\pi} \sin^{2p-1}\varphi \cos^{2q-1}\varphi\, d\varphi, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

The Beta function is intimately related to the Gamma function. Consider the product

(3) $$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t} t^{p-1}\, dt \cdot \int_0^\infty e^{-v} v^{q-1}\, dv.$$

In (3) use $t = x^2$ and $v = y^2$ to obtain

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \exp(-x^2) x^{2p-1}\, dx \cdot \int_0^\infty \exp(-y^2) y^{2q-1}\, dy,$$

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^\infty \exp(-x^2-y^2)x^{2p-1}y^{2q-1}\,dx\,dy.$$

Next turn to polar coordinates for the iterated integration over the first quadrant in the xy-plane. Using $x = r\cos\theta$, $y = r\sin\theta$, we may write

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^{\frac{1}{2}\pi} \exp(-r^2)r^{2p+2q-2}\cos^{2p-1}\theta\sin^{2q-1}\theta\,r\,d\theta\,dr$$

$$= 2\int_0^\infty \exp(-r^2)r^{2p+2q-1}dr \cdot 2\int_0^{\frac{1}{2}\pi}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta.$$

Now put $r = \sqrt{t}$ and $\theta = \frac{1}{2}\pi - \varphi$ to obtain

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}t^{p+q-1}\,dt \cdot 2\int_0^{\frac{1}{2}\pi}\sin^{2p-1}\varphi\cos^{2q-1}\varphi\,d\varphi,$$

from which it follows that

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

THEOREM 7. *If* $\text{Re}(p) > 0$ *and* $\text{Re}(q) > 0$,

(4) $$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

By (4), $B(p,q) = B(q,p)$, a result just as easily obtained directly from (1) or (2).

Equations (2) and (4) yield a generalization of Wallis' formula of elementary calculus. In (2) put $2p - 1 = m$, $2q - 1 = n$, and use (4) to write

(5) $$\int_0^{\frac{1}{2}\pi}\sin^m\varphi\cos^n\varphi\,d\varphi = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)},$$

valid for $\text{Re}(m) > -1$, $\text{Re}(n) > -1$.

17. The value of $\Gamma(z)\Gamma(1-z)$. The important relation (4) of Section 16 suggests that the product of two Gamma functions whose arguments have the sum unity may possess some pleasant property, since if $p + q = 1$, $\Gamma(p+q) = \Gamma(1) = 1$.

If z is such that $0 < \text{Re}(z) < 1$, both z and $(1-z)$ have real part positive, and we may use (4) of Section 16 to write

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^\infty \exp(-x^2-y^2)x^{2p-1}y^{2q-1}\,dx\,dy.$$

Next turn to polar coordinates for the iterated integration over the first quadrant in the xy-plane. Using $x = r\cos\theta$, $y = r\sin\theta$, we may write

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^{\frac{1}{2}\pi} \exp(-r^2)r^{2p+2q-2}\cos^{2p-1}\theta\sin^{2q-1}\theta\, r\,d\theta\,dr$$

$$= 2\int_0^\infty \exp(-r^2)r^{2p+2q-1}dr \cdot 2\int_0^{\frac{1}{2}\pi}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta.$$

Now put $r = \sqrt{t}$ and $\theta = \frac{1}{2}\pi - \varphi$ to obtain

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}t^{p+q-1}\,dt \cdot 2\int_0^{\frac{1}{2}\pi}\sin^{2p-1}\varphi\cos^{2q-1}\varphi\,d\varphi,$$

from which it follows that

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

THEOREM 7. *If* $\mathrm{Re}(p) > 0$ *and* $\mathrm{Re}(q) > 0$,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{4}$$

By (4), $B(p,q) = B(q,p)$, a result just as easily obtained directly from (1) or (2).

Equations (2) and (4) yield a generalization of Wallis' formula of elementary calculus. In (2) put $2p-1 = m$, $2q-1 = n$, and use (4) to write

$$\int_0^{\frac{1}{2}\pi}\sin^m\varphi\cos^n\varphi\,d\varphi = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, \tag{5}$$

valid for $\mathrm{Re}(m) > -1$, $\mathrm{Re}(n) > -1$.

17. The value of $\Gamma(z)\Gamma(1-z)$. The important relation (4) of Section 16 suggests that the product of two Gamma functions whose arguments have the sum unity may possess some pleasant property, since if $p + q = 1$, $\Gamma(p+q) = \Gamma(1) = 1$.

If z is such that $0 < \mathrm{Re}(z) < 1$, both z and $(1-z)$ have real part positive, and we may use (4) of Section 16 to write

Theorem 9. *If α is neither zero nor a negative integer,*

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{3}$$

We have already had, in equation (3), page 11, the result

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{z(z+1)(z+2)\cdots(z+n-1)},$$

which can now be written in the form

(4) result use
$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n}. \tag{4}$$

Equation (4), reinterpreted in the light of Theorem 9, yields a result of value to us in the subsequent two sections.

Lemma 7. If n is integral and z is not a negative integer,

$$\lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1. \tag{5}$$

19. Legendre's duplication formula. Let us turn to Lemma 5, page 22, and use $\alpha = 2z$. We thus obtain

$$(2z)_{2n} = 2^{2n}(z)_n(z+\tfrac{1}{2})_n.$$

In view of Theorem 9 we may rewrite the above as

$$\frac{\Gamma(2z+2n)}{\Gamma(2z)} = \frac{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)}{\Gamma(z)\Gamma(z+\frac{1}{2})},$$

or

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{\Gamma(2z+2n)}{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)},$$

which, since the left member is independent of n, also implies

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \lim_{n\to\infty} \frac{\Gamma(2z+2n)}{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)}. \tag{1}$$

We next insert in the right member of (1) the appropriate factors to permit us to make use of the result in Lemma 7. From (1) we write

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})}$$
$$= \lim_{n\to\infty} \frac{\Gamma(2z+2n)}{(2n-1)!(2n)^{2z}} \cdot \frac{(n-1)!n^z}{\Gamma(z+n)} \cdot \frac{(n-1)!n^{z+\frac{1}{2}}}{\Gamma(z+\frac{1}{2}+n)} \cdot \frac{2^{2z}(2n-1)!}{2^{2n}n^{\frac{1}{2}}[(n-1)!]^2},$$

which, because of Lemma 7, becomes

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \lim_{n\to\infty} \frac{2^{2z}(2n-1)!}{2^{2n}n^{\frac{1}{2}}[(n-1)!]^2}.$$

It follows that

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})} = c,$$

in which c is independent of z. To evaluate c we use $z = \frac{1}{2}$ and find that

$$c = \frac{\Gamma(1)}{2\Gamma(\frac{1}{2})\Gamma(1)} = \frac{1}{2\sqrt{\pi}}.$$

We have thus discovered an expression for $\Gamma(2z)$ in terms of $\Gamma(z)$ and $\Gamma(z+\frac{1}{2})$. It is *Legendre's duplication formula,*

(2) $$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

20. Gauss' multiplication theorem. Following the technique used to discover and prove Legendre's duplication formula, we readily move on to a theorem of Gauss involving the product of k Gamma functions.

Lemma 6, page 22, can be written

$$(\alpha)_{nk} = k^{nk}\prod_{s=1}^{k}\left(\frac{\alpha+s-1}{k}\right)_n$$

and by Theorem 9, page 23, $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. We thus obtain

(1) $$\frac{\Gamma(\alpha+nk)}{\Gamma(\alpha)} = k^{nk}\prod_{s=1}^{k}\frac{\Gamma\left(\frac{\alpha+s-1}{k}+n\right)}{\Gamma\left(\frac{\alpha+s-1}{k}\right)}.$$

In (1) put $\alpha = kz$ and rearrange the members of the equation to arrive at

(2) $$\frac{\Gamma(kz)}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = \frac{\Gamma(kz+kn)}{k^{nk}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)}$$

$$= \lim_{n\to\infty}\frac{\Gamma(kz+kn)}{k^{nk}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)}.$$