$$\lim_{n\to\infty} \frac{(n+1)^z}{n^z} = 1,$$

we can equally well write the result (3) in the form

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}. \tag{4}$$

12. The difference equation $\Gamma(z+1) = z\Gamma(z)$. From Euler's product for $\Gamma(z)$ we obtain

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{z}{z+1} \frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}\right]}{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]}$$

$$= \frac{z}{z+1}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{z}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}\right]$$

$$= \frac{z}{z+1}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{k+1}{k}\cdot\frac{k+z}{k+z+1}\right)$$

$$= \frac{z}{z+1}\lim_{n\to\infty}\frac{n+1}{1}\cdot\frac{1+z}{n+z+1} = z.$$

Therefore

$$\Gamma(z+1) = z\Gamma(z) \tag{1}$$

for all finite z except for the poles of $\Gamma(z)$.

If $z = m$, a positive integer, iterated use of the equation (1) yields $\Gamma(m+1) = m!$. Since $\Gamma(1) = 1$, this is another of the many reasons we define $0! = 1$.

13. The order symbols o and O. Let R be a region in the complex z-plane. If and only if

$$\lim_{z\to c \text{ in } R} \frac{f(z)}{g(z)} = 0,$$

we write

$$f(z) = o[g(z)], \qquad \text{as } z \to c \text{ in } R.$$

If and only if $\left|\frac{f(z)}{g(z)}\right|$ is bounded as $z \to c$ in R, we write

$$f(z) = O[g(z)], \qquad \text{as } z \to c \text{ in } R.$$

$$\prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\cdots(n+a_s)}{(n+b_1)(n+b_2)\cdots(n+b_s)} = \frac{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_s)}.$$

If one or more of the a_k is a negative integer, the product on the left is zero, which agrees with the existence of one or more poles of the denominator factors on the right.

Example: Evaluate

$$\prod_{n=1}^{\infty} \frac{(c-a+n-1)(c-b+n-1)}{(c+n-1)(c-a-b+n-1)}.$$

Since $(c-a-1)+(c-b-1)=(c-1)+(c-a-b-1)$, we may employ Theorem 5 if no one of the quantities c, $c-a$, $c-b$, $c-a-b$ is either zero or a negative integer. With those restrictions we obtain

$$(4)\qquad \prod_{n=1}^{\infty} \frac{(c-a+n-1)(c-b+n-1)}{(c+n-1)(c-a-b+n-1)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

15. Euler's integral for $\Gamma(z)$. Elementary treatments of the Gamma function are usually based on an integral definition. Theorem 6 connects the function $\Gamma(z)$ defined by the Weierstrass product with that defined by Euler's integral.

Theorem 6. *If* $\mathrm{Re}(z) > 0$,

$$(1)\qquad \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt.$$

We shall establish four lemmas intended to break the proof of Theorem 6 into simple parts.

Lemma 1. *If* $0 \leqq \alpha < 1$, $1+\alpha \leqq \exp(\alpha) \leqq (1-\alpha)^{-1}$.

Proof: Compare the three series

$$1+\alpha=1+\alpha, \qquad \exp(\alpha)=1+\alpha+\sum_{n=2}^{\infty}\frac{\alpha^n}{n!}, \qquad (1-\alpha)^{-1}=1+\alpha+\sum_{n=2}^{\infty}\alpha^n.$$

Lemma 2. If $0 \leqq \alpha < 1$, $(1-\alpha)^n \geqq 1-n\alpha$, *for n a positive integer.*

Proof: For $n=1$, $1-\alpha = 1-1\cdot\alpha$, as desired. Next assume that

$$(1-\alpha)^k \geqq 1-k\alpha,$$

and multiply each member by $(1-\alpha)$ to obtain

$$(1-\alpha)^{k+1} \geqq (1-\alpha)(1-k\alpha) = 1-(k+1)\alpha+k\alpha^2,$$

so that

$$(1 - \alpha)^{k+1} \geqq 1 - (k + 1)\alpha.$$

Lemma 2 now follows by induction.

Lemma 3. If $0 \leqq t < n$, *n a positive integer,*

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq \frac{t^2 e^{-t}}{n}.$$

Proof: Use $\alpha = t/n$ in Lemma 1 to get

$$1 + \frac{t}{n} \leqq \exp\left(\frac{t}{n}\right) \leqq \left(1 - \frac{t}{n}\right)^{-1}$$

from which

(2) $$\left(1 + \frac{t}{n}\right)^n \leqq e^t \leqq \left(1 - \frac{t}{n}\right)^{-n}$$

or

$$\left(1 + \frac{t}{n}\right)^{-n} \geqq e^{-t} \geqq \left(1 - \frac{t}{n}\right)^n,$$

so that

(3) $$e^{-t} - \left(1 - \frac{t}{n}\right)^n \geqq 0.$$

But also

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left[1 - e^t\left(1 - \frac{t}{n}\right)^n\right]$$

and, by (2), $e^t \geqq \left(1 + \frac{t}{n}\right)^n$. Hence

(4) $$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq e^{-t}\left[1 - \left(1 - \frac{t^2}{n^2}\right)^n\right].$$

Now Lemma 2 with $\alpha = t^2/n^2$ yields

$$\left(1 - \frac{t^2}{n^2}\right)^n \geqq 1 - \frac{t^2}{n}$$

which may be used in (4) to obtain

(5) $$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq e^{-t}\left[1 - 1 + \frac{t^2}{n}\right] = \frac{t^2 e^{-t}}{n}.$$

The inequalities (3) and (5) constitute the result stated in Lemma 3.

Lemma 4. If n is integral and $\text{Re}(z) > 0$,

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{6}$$

Proof: In the integral on the right in (6) put $t = n\beta$ and thus obtain

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = n^z \int_0^1 (1-\beta)^n \beta^{z-1}\, d\beta. \tag{7}$$

An integration by parts gives us the reduction formula

$$\int_0^1 (1-\beta)^n \beta^{z-1}\, d\beta = \frac{n}{z}\int_0^1 (1-\beta)^{n-1}\beta^z\, d\beta,$$

iteration of which yields

$$\int_0^1 (1-\beta)^n\beta^{z-1}\, d\beta = \frac{n(n-1)(n-2)\cdots 1}{z(z+1)(z+2)\cdots(z+n-1)}\int_0^1 \beta^{z+n-1}\, d\beta$$

$$= \frac{n!}{z(z+1)(z+2)\cdots(z+n)}.$$

Now (7) becomes

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}$$

so that

$$\lim_{n\to\infty}\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)} = \Gamma(z)$$

by equation (4), page 12.

We are now in a position to prove Theorem 6, which states that

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\, dt, \qquad \text{Re}(z) > 0. \tag{8}$$

The integral on the right in (8) converges for $\text{Re}(z) > 0$. With the aid of Lemma 4, write

$$\int_0^\infty e^{-t}t^{z-1}\, dt - \Gamma(z) = \lim_{n\to\infty}\left[\int_0^\infty e^{-t}t^{z-1}\, dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt\right]$$

$$= \lim_{n\to\infty}\left[\int_0^n \left\{e^{-t} - \left(1 - \frac{t}{n}\right)^n\right\} t^{z-1}\, dt + \int_n^\infty e^{-t}t^{z-1}\, dt\right].$$

From the convergence of the integral on the right in (8) it follows that

$$\lim_{n\to\infty} \int_n^\infty e^{-t} t^{z-1}\, dt = 0.$$

Hence

$$(9)\qquad \int_0^\infty e^{-t} t^{z-1}\, dt - \Gamma(z) = \lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt.$$

But, by Lemma 3 and the fact that $|t^z| = t^{\mathrm{Re}(z)}$,

$$\left| \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt \right| \leq \int_0^n \frac{t^2 e^{-t}}{n} \cdot t^{\mathrm{Re}(z)-1}\, dt$$

$$\leq \frac{1}{n} \int_0^n e^{-t} t^{\mathrm{Re}(z)+1}\, dt.$$

Now $\int_0^\infty e^{-t} t^{\mathrm{Re}(z)+1}\, dt$ converges, so $\int_0^n e^{-t} t^{\mathrm{Re}(z)+1}\, dt$ is bounded. Therefore

$$\lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1}\, dt = 0,$$

and we may conclude from equation (9) that (8) is valid.

16. The Beta function. We define the Beta function $B(p, q)$ by

$$(1)\qquad B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

Another useful form for this function can be obtained by putting $t = \sin^2 \varphi$, thus arriving at

$$(2)\qquad B(p,q) = 2\int_0^{\frac{1}{2}\pi} \sin^{2p-1}\varphi \cos^{2q-1}\varphi\, d\varphi, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

The Beta function is intimately related to the Gamma function. Consider the product

$$(3)\qquad \Gamma(p)\Gamma(q) = \int_0^\infty e^{-t} t^{p-1}\, dt \cdot \int_0^\infty e^{-v} v^{q-1}\, dv.$$

In (3) use $t = x^2$ and $v = y^2$ to obtain

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \exp(-x^2) x^{2p-1}\, dx \cdot \int_0^\infty \exp(-y^2) y^{2q-1}\, dy,$$