CHAPTER 2

The Gamma and Beta Functions

7. The Euler or Mascheroni constant γ. At times we need to use the constant γ, defined by

$$\gamma = \lim_{n \to \infty} (H_n - \operatorname{Log} n), \tag{1}$$

in which, as usual,

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{2}$$

We shall prove that γ exists and that $0 \leqq \gamma < 1$. Actually $\gamma = 0.5772$, approximately.

Let $A_n = H_n - \operatorname{Log} n$. Then the A_n form a decreasing sequence because

$$\begin{aligned} A_{n+1} - A_n &= H_{n+1} - H_n - \operatorname{Log}(n+1) + \operatorname{Log} n \\ &= \frac{1}{n+1} + \operatorname{Log}\frac{n}{n+1} = \frac{1}{n+1} + \operatorname{Log}\left(1 - \frac{1}{n+1}\right) \\ &= -\sum_{k=1}^{\infty} \frac{1}{(k+1)(n+1)^{k+1}} < 0. \end{aligned}$$

Furthermore, since $1/t$ decreases steadily as t increases,

$$\frac{1}{k} < \int_{k-1}^{k} \frac{dt}{t} < \frac{1}{k-1}, \qquad k \geqq 2. \tag{3}$$

We sum the inequalities (3) from $k = 2$ to $k = n$ and thus obtain

$$H_n - 1 < \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} + \cdots + \int_{n-1}^{n} \frac{dt}{t} < H_{n-1},$$

or

$$H_n - 1 < \operatorname{Log} n < H_{n-1},$$

from which it follows that

$$-1 < -H_n + \operatorname{Log} n < -\frac{1}{n},$$

or

$$1 > A_n > \frac{1}{n}.$$

Thus we see that the A_n decrease steadily, are all positive, and are less than unity. It follows that γ exists and is non-negative and less than unity.

8. The Gamma function. We follow Weierstrass in defining the function $\Gamma(z)$ by

$$(1) \qquad \frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right],$$

in which γ is the Euler constant of Section 7. The product in (1) is absolutely convergent for all finite z as was seen in Ex. 11, page 7, the special case $c = 0$ and z replaced by $(-z)$. That the product is also uniformly convergent in any closed region in the z-plane is easily shown by employing the associated series of logarithms.

We shall see in Section 15 that the function $\Gamma(z)$ defined by (1) is identical with that defined by Euler's integral; that is,

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \qquad \operatorname{Re}(z) > 0.$$

The right member of (1) is analytic for all finite z. Its only zeros are simple ones at $z = 0$ and at each negative integer. We may therefore conclude that

(a) $\Gamma(z)$ is analytic except at $z =$ nonpositive integers and $z = \infty$;

(b) $\Gamma(z)$ has a simple pole at $z =$ each nonpositive integer, $z = 0, -1, -2, -3, \cdots$;

(c) $\Gamma(z)$ has an essential singularity at $z = \infty$, a point of condensation of poles;

(d) $\Gamma(z)$ is never zero [because $1/\Gamma(z)$ has no poles].

9. A series for $\Gamma'(z)/\Gamma(z)$. By taking logarithms of each member of equation (1) of Section 8, we obtain

$$\log \Gamma(z) = -\operatorname{Log} z - \gamma z - \sum_{n=1}^{\infty}\left[\operatorname{Log}\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right].$$

Term-by-term differentiation of the members of the foregoing equation yields

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right),$$

or

(1) $$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)},$$

the series on the right being absolutely and uniformly convergent in any closed region excluding the singular points of $\Gamma(z)$, a result easily deduced by using the Weierstrass M-test and the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

10. Evaluation of $\Gamma(1)$ and $\Gamma'(1)$. In the Weierstrass definition of $\Gamma(z)$ put $z = 1$ to get

$$\begin{aligned}\frac{1}{\Gamma(1)} &= e^{\gamma}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)\exp\left(-\frac{1}{n}\right)\right]\\ &= e^{\gamma}\lim_{n\to\infty}\prod_{k=1}^{n}\left[\frac{k+1}{k}\exp\left(-\frac{1}{k}\right)\right]\\ &= e^{\gamma}\lim_{n\to\infty}(n+1)\exp(-H_n)\\ &= e^{\gamma}\lim_{n\to\infty}(n+1)\exp(-\gamma - \operatorname{Log} n - \epsilon_n),\end{aligned}$$

in which $\epsilon_n \to 0$ as $n \to \infty$. It follows that

$$\frac{1}{\Gamma(1)} = e^{\gamma}\lim_{n\to\infty}\frac{n+1}{n}e^{-\gamma} = 1,$$

so that $\Gamma(1) = 1$.

We know from the series for $\Gamma'(z)/\Gamma(z)$ obtained in Section 9 that

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 1 + \sum_{n=1}^{\infty}\frac{1}{n(n+1)},$$

so that