Chapter 1

Fluid equations

This part builds upon the knowledge you acquired from the introductory Module 911, on the fluid plasma description. Its scope is to provide qualitative insight in the building blocks of fluid theory, and relies on a qualitative picture (derivation of fluid equations from first principles) which is distinct from the analytical one introduced earlier (derivation as statistical moment evolution equations - to be iterated in detail in Part 2 of these Notes). This Chapter is therefore to be covered in a descriptive manner, by introducing terminology and setting the toolbox for the analysis of plasma as a (multi-) fluid system.

1.1 The fluid approach to describing a plasma

In principle, modern physics has the tools to accurately describe a plasma in all its detail. The motion of each (of N) charged particle(s) and their mutual electrostatic interaction via a long range Coulomb field (collisions) are governed by Maxwell's equations. However, calculating the evolution of more than a (very) small number of electrons and ions using Maxwell's equations is a daunting task even for modern day computers.

Consequently an alternative to ab-initio (from fundamental principles) calculations is required. Substantial savings in the computational effort are always possible given the right simplifications. Given the large range of parameters a plasma can have in terms of density and temperature, a variety of different theoretical approaches have been developed. These simply differ in their assumptions and simplifications and therefore have different strengths or weaknesses. The key is to understand what simplifications have been made in these models - and thus to be aware when trying to obtain results for which a given model might not be any longer or may only be partially valid.

In the following we will discover the equations governing the evolution of a plasma in the so called *fluid approximation* - often referred to as *Plasma Hydrodynamics* - from Ancient Greek $'Y\delta\omega\rho$ - water. Given that a plasma consists of a large number of charged particles (electrons, ions, and neutrals) it may come as a surprise that a plasma displays fluid-like behaviour and it would seem more natural to describe the plasma in terms of single particle motions. While models based on particle motions do exist, they are not necessarily the most efficient way of describing a plasma. The beauty of the fluid model lies in the fact that instead of tracking the behaviour of every particle, we choose to describe an ensemble of moving particles (fluid or gas) in terms of a few local parameters:

- the fluid density n
- the fluid (flow) velocity $\mathbf{u}$
- the fluid temperature T
- ... (perhaps more, depending on the complexity of the problem).

Depending on the complexity of the approach other quantities are also included such as magnetic fields, etc. Recall that all of these quantities are functions of space and time, within the fluid picture, that is $n = n(\mathbf{r}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$, $T = T(\mathbf{r}, t)$ and so forth.

As you know by now, plasmas are highly complex systems which may consist of various particle species: electrons, different types of ions, other charged ingredients (e.g. defects, dust) and neutrals (often omitted by assuming a fully ionised plasma). *Most important*, you should keep in mind that the fluid description outlined here refers to *each species* present in a multi-component plasma. Therefore, we shall speak of the electron fluid density (or velocity, or pressure), the ion fluid (same) variables, and so forth. A separate single-fluid description will be tailored in the next Chapter, combining the species into a single-fluid system.

The temporal evolution of these fluid equations is determined by the fluid equations, to be presented below. The different species are dealt with by developing one equation for each set of particles, i.e. in principle at least two, one for electrons and one for ion species. However, we frequently need to incorporate multiple species (electrons, ions, different types of ions and varying ionisation stages) as well as neutral atoms. In fact, some of the most technologically important plasmas are only very weakly ionised, with only 1 in a 1000 or less ion content.

1.2 Particle conservation: the continuity equation

One of the simplest requirements in modelling plasma is that the number of particles should remain conserved. Simply speaking, we require a set of equations that keeps the overall particle number constant. In order to formulate this in terms of an equation, let us first consider a three dimensional (3D) volume element $\Delta V = \Delta x \Delta y \Delta z$.

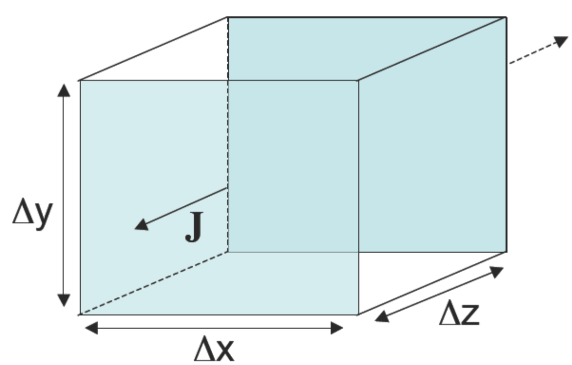

Figure 1.1: Volume element in 3D space.

If we know the local number density n we can calculate the total number of particles in the volume element ΔN, we can calculate the total number of particles in the volume ΔV as:

$$\Delta N = n \Delta V \,. \tag{1.1}$$

As long as our volume element does not change shape (*Eulerian view*[1]) and there are no physical sources or sinks of particles (so that particles may neither be 'produced' or lost), any change in the number of particles must be as a result of particle flow across the boundaries of the volume element. In other words, the change in the number of particles must only be a result of flow across the boundaries. We can easily realise that the total number of particles flowing across a boundary depends on the flow velocity **u**, the density n and the area in question.

We can describe a flux of particles with an average velocity **v** streaming across a boundary with a number density n, i.e. our flux is:

$$\mathbf{J} = n\mathbf{u}\,. \tag{1.2}$$

If we make the simplifying assumption that our plasma is only flowing along the x-axis, we can write the change in particle number as[2]

$$\frac{\partial(\Delta N)}{\partial t} = (J_{x=0} - J_{x=\Delta x})\Delta y \Delta z\,. \tag{1.3}$$

We may now express the flux through the boundary at $x = \Delta x$ in terms of both the flux $J_{x=0}$ at $x = 0$ and its derivative in terms of x, by using a Taylor expansion, i.e:

$$J_{x=\Delta x} \approx J_{x=0} + \left.\frac{\partial J}{\partial x}\right|_{x=0} \Delta x\,. \tag{1.4}$$

We can now substitute Eq. (1.4) into Eq. (1.3) to obtain an equation that relates the change in particle number to the density and velocity gradients in the plasma:

$$-\frac{\partial N}{\partial t} = (J_{x=0} - J_{x=\Delta x})\Delta y \Delta z = \frac{\partial(n u_x)}{\partial x}\Delta x \Delta y \Delta z\,, \tag{1.5}$$

[1]We shall work in Eulerian geometry throughout this part. A brief interpretation of this assumption, which determines our mathematical toolbox, is provided below, as an Appendix. A more complicated mathematical formulation exists (in the Lagrangian view), but is omitted here, for simplicity

[2]Note that we have considered a finite volume element in some fixed position in space, and are thus looking into the *time* variation of the particle density. However, all quantities of interest may also vary *in space*, hence the partial derivative employed here $\partial/\partial t$ (instead of a total derivative d/dt).

or, for the 3D (three dimensional) case,

$$-\frac{\partial N}{\partial t} = \Delta V \nabla (n\mathbf{u})\,. \tag{1.6}$$

Dividing by the volume, we can write down the continuity equation purely in terms of density and velocity as

$$\frac{\partial n}{\partial t} + \nabla (n\mathbf{u}) = 0\,. \tag{1.7}$$

The physical interpretation of (1.7) is the conservation of particle number density. We now have accomplished the first step towards a recipe to calculate the evolution of a plasma in space and time.

Note: Clearly we have made one significant simplification, by omitting particle sinks and sources. This is true in a 'textbook' picture of plasma, but may not always be permitted in the 'real world'. In other words, plasmas may consist of constituent (particles species) whose number may be variable. The most obvious of these sinks and sources is expressed by the fact that the number of electrons depends on the average ionisation Z^*. The value of Z^* relates the ion number density n_i to the electron density n_e as

$$n_e = Z^* n_i\,. \tag{1.8}$$

Since the ionisation state of a given atom is strongly dependent on the plasma temperature, an accurate plasma description requires that the average ionisation state be calculated. Any change of Z^* must then be incorporated into Eq. (1.7) as a particle sink or source.
Unless specifically stated otherwise, the plasma considered in the rest of these notes will be thought of as an ensemble of particle species of *fixed* number composition.

1.3 Momentum conservation

Of course, the continuity equation (1.7) is not enough to determine the evolution of a plasma. To derive further equations that the plasma must satisfy, we shall look into one of the most basic conservation laws in physics: momentum conservation:

$$\mathbf{p}_{tot} = \text{constant}\,, \tag{1.9}$$

which essentially expresses Newton's 2nd Law for the whole of the plasma particles.

To calculate the momentum of a plasma, we must first consider the momentum carried by the individual particles. The momentum of a given (say, $i-$th; here $i = 1, 2, ..., N$) plasma particle (ion, electron) is given by

$$\mathbf{p}_i = m_i \mathbf{u}_i \,. \tag{1.10}$$

For the overall momentum to be conserved, the sum of all the particle momenta must remain constant. For a volume element ΔV we can write:

$$\mathbf{p}_{tot} = \sum^{\Delta V} m_i \mathbf{u}_i = m_i (\Delta N) \mathbf{u} = \Delta V m_i n \mathbf{u} \tag{1.11}$$

with

$$\mathbf{p}_{tot} = \sum^{\Delta V} m_i \mathbf{u}_i = m_i (\Delta N) \mathbf{u} = \Delta V m n \mathbf{u} \tag{1.12}$$

and $\mathbf{u} = \sum^{\Delta V} \mathbf{u}_i / \Delta N$ is the mean velocity.

Note that we have made the leap from single particle quantities (the microscopic velocities $\mathbf{v_i}$) to an averaged quantity (the fluid velocity $\mathbf{u}$)) here. This step is key to the beauty and the effectiveness of the fluid description. Later you will see that when we describe a plasma in terms of the kinetics of single particle motion that we can recover the fluid description by evaluating averages (statistical ensemble averages) of the individual particle quantities. For the fluid description of a plasma it is therefore essential that we find a simple way of obtaining and measuring averages. This will be discussed at some length later in this course.

We know that for a single particle the change of momentum $\mathbf{p_i}$ is proportional to the applied force $\mathbf{F_i} = d\mathbf{p_i}/dt$ (Newton's Law). We can calculate the time dependence of the momentum from the forces acting on the particles as

$$\frac{d\mathbf{p}_{tot}}{dt} = \Delta V \frac{d}{dt}(mn\mathbf{u}) = \mathbf{F} \,. \tag{1.13}$$

Since $n\Delta V = \Delta N$ and $d(\Delta N)/dt = 0$, we can rewrite the above equation as

$$mn\Delta V \frac{d\mathbf{u}}{dt} = \mathbf{F} \,. \tag{1.14}$$

So far we have summarised the forces acting on the plasma simply in terms of the variable $\mathbf{F}$. To make quantitative predictions we need to look into this force term in a little more detail.

1.3.1 Forces acting on a fluid element

For a plasma, the dominant forces result from the electric field **E** and the magnetic field **B**. We recall that the force acting on an individual particle is simply the Lorentz force $\mathbf{F}_i = q_i(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

In fact one might argue that this should be the only term on the right hand side of the momentum equation, since electromagnetic forces dominate the interaction between charged particles and other forces (such as gravity) only play a role on enormous time and distance scales (e.g. astrophysical plasmas). While this is strictly speaking true, a detailed analysis of the field sources shows that there are very distinct contributions to the fields which operate on very different spatial scales. Coulomb collisions between charged particles take place on extremely short spatial scales while externally applied fields act on much larger length scales. Once again we face the challenge of formulating an understanding of the basic processes in a format which will allow actual calculations to be performed.

It is practical to use the average over the Lorentz force on single particles. We can therefore make our momentum conservation equation more detailed by writing:

$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}\right) = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{F}_{microscopic}\,, \qquad (1.15)$$

where the last term denotes the sum of forces acting on the plasma fluid due to microscopic effects. We shall be more specific on this in the following paragraphs.

Momentum transfer by like particles - the pressure term

Since we are averaging over a volume element to arrive at our fluid equations, collisions between like particles need not be included in our equation for momentum conservation. Collisions between like particles do play a role for non-thermal plasmas (plasmas which *do not* have a Maxwellian velocity distribution) and act to bring any velocity distribution back towards a Maxwellian distribution. They are therefore the basis on which we can use a fluid description.

However, particle movements in and out of our volume element will lead to a net transfer of momentum. This momentum transfer is exactly analogous to the concept of pressure we are familiar with from a gas. The pressure in a gas is simply the force per unit area arising from the thermal motions of the individual particles. (Note that the force per unit area has the same dimensions as energy density and that for example a magnetic field also exerts a *magnetic pressure* exactly in proportion to its energy density. We shall come back to this subtle point later.)

In single particle terms, a momentum transfer (or force) arises because of the random motion of particles entering and departing from the volume element ΔV. Similarly to our derivation of the continuity equation, we consider a volume element centred at $(x_0, \Delta y/2, \Delta z/2)$ and consider only motion along the x-axis. The number of particles passing through into the volume element is $(\Delta n_V)u_x\Delta y\Delta z$ where Δn_V denotes the number of particles per unit volume moving with velocity u_x each, i.e. with a momentum mu_x.

The momentum entering the volume element through the face located at x_0 is then the sum over all the different velocities

$$P_{x0} = \sum \Delta n_v m u_x^2 \Delta y \Delta z = \Delta y \Delta z \left(mn\overline{v}_x^2/2\right|_{x0} \tag{1.16}$$

and for the interface at $x_0 - \Delta x$

$$P_{x0-\Delta x} = \Delta y \Delta z \left(mn\overline{v}_x^2/2\right|_{x0-\Delta x} , \tag{1.17}$$

where $\overline{v}$ is the average velocity. The net gain of momentum is then

$$P_{x0} - P_{x0-\Delta x} = \Delta y \Delta z \frac{1}{2} m \left[\left(n\overline{v}_x^2\right|_{x0-\Delta x} - \left(n\overline{v}_x^2\right|_{x0}\right] = \Delta y \Delta z \frac{1}{2} m(-\Delta x)\frac{\partial}{\partial x}(n\overline{v}_x^2) . \tag{1.18}$$

The total momentum transfer is just twice this result since the contribution of particles moving in the opposite direction and carrying negative x-momentum adds an equal term, thus

$$\frac{\partial}{\partial t}(nm\overline{v}_x)\Delta x \Delta y \Delta z = -m\frac{\partial}{\partial x}(n\overline{v}_x^2)\Delta x \Delta y \Delta z . \tag{1.19}$$

What we would like to know is how we can relate this result to average quantities and since we are dealing with pressure, we would like to relate our pressure term to the local plasma temperature. We can do this by writing down the velocity of an individual particle in terms of the average velocity (the flow velocity) and a random (thermal) component.

$$\overline{v}_x = \overline{v}_x + v_{th} . \tag{1.20}$$

For a Maxwellian distribution we can relate the concept of temperature T to the average velocity via the well known relation

$$\frac{1}{2}m\overline{v}_{th}^2 = \frac{1}{2}k_B T\,, \tag{1.21}$$

which gives the average thermal velocity per degree of freedom (only motion along the $x-$ direction is considered here). Recall that k_B denotes the Boltzmann constant.

With this relation we can rewrite equation (1.19) in terms of the local temperature

$$\frac{\partial}{\partial t}(nm\overline{v}_x) = -m\frac{\partial}{\partial x}\left[n(\overline{v}_x^2 + 2\overline{u_x v_{th}} + \overline{v}_{th}^2)\right] = -m\frac{\partial}{\partial x}\left[n\left(\overline{v}_x^2 + \frac{k_B T}{m}\right)\right]. \tag{1.22}$$

After carrying out the partial differentiation we can see that we can cancel the two terms nearest the equal sign by using the continuity equation (1.7)

$$nm\frac{\partial}{\partial t}\overline{v}_x + m\overline{v}_x\frac{\partial}{\partial t}n = -m\overline{v}_x\frac{\partial}{\partial t}(n\overline{v}_x) - mn\overline{v}_x\frac{\partial}{\partial x}\overline{v}_x - \frac{\partial}{\partial x}(nk_B T)\,. \tag{1.23}$$

Using the definition of pressure as

$$p \equiv nk_B T\,, \tag{1.24}$$

we may rewrite our momentum transfer balance equation for like species in terms of pressure

$$mn\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial t}\,. \tag{1.25}$$

This is the pressure gradient force, so the change of momentum in a volume element due to pressure becomes

$$\left(\frac{dp}{dt}\right)_{pressure} = -\nabla p\,. \tag{1.26}$$

We may add, for rigour, that we have only treated the simplest case here. In principle the pressure need not be isotropic and in this case the scalar pressure must be replaced by a pressure tensor $\mathbf{P}$ and the pressure gradient force generalises to

$$\left(\frac{dp}{dt}\right)_{pressure} = -\nabla\mathbf{P}\,. \tag{1.27}$$

For isotropic pressure, the pressure tensor simply contains the scalar pressure p as the trace of the tensor with all off-diagonal elements 0. For non-isotropic pressure the off-diagonal elements remain 0, and the trace contains the pressure in x, y and z directions. In the most general case the off-diagonal elements can also become non-zero in the presence shear flow.

Momentum transfer between particle species - collisions

Coulomb collisions between different particle species *do* transfer momentum and any plasma consists of at least two distinct populations: electrons and ions. The momentum exchanged between the two species must be added to our momentum balance equation in terms of a momentum gain (loss) term. Of course, as we will see later on, the actual momentum exchanged in a specific collision depends on the collision cross section and the relative velocity of the two particles. Instead of looking at the hopeless task of including every single collision, we may again look at average relative velocities between the species and average the momentum exchanged over all possible parameters. Then the rate of momentum density exchanged between two species simply depends on their relative average velocities $(\mathbf{u}_1 - \mathbf{u}_2)$ and the frequency of their collisions ν_{12}

$$\left(\frac{d\mathbf{p}}{dt}\right)_{coll} = \nu_{12} n_1 m_1 (\mathbf{u}_1 - \mathbf{u}_2)\,, \tag{1.28}$$

whereby the collision frequency must be calculated in such a way that we get the right rate of momentum transfer, for a given physical problem.

Of course the probability of a collision must depend on both the electron and ion density as well as on the electron and ion charge. The strength of a collision depends on the particle charges and the frequency of the collisions must depend on the density of each species. For a fully ionised plasma of Z=1 and therefore expect an e^2 dependence and, since $n_e = n_i = n$ also an n^2 dependence of the momentum transfer. These two approaches should only differ by a proportionality constant η and hence

$$\left(\frac{d\mathbf{p}}{dt}\right)_{coll} = \nu_{12} n_1 m_1 (\mathbf{u}_1 - \mathbf{u}_2) = \eta e^2 n^2 (\mathbf{u}_1 - \mathbf{u}_2)\,. \tag{1.29}$$

We may therefore express the collision frequency as

$$\nu_{12} = \eta \frac{ne^2}{m} \, . \tag{1.30}$$

Taking all the above terms together we can write our momentum conservation equation as

$$\left(\frac{d\mathbf{p}}{dt}\right) = \left(\frac{d\mathbf{p}}{dt}\right)_{Lorentz} + \left(\frac{d\mathbf{p}}{dt}\right)_{coll} + \left(\frac{d\mathbf{p}}{dt}\right)_{pressure} \tag{1.31}$$

and substituting all of the above expressions

$$mn(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}) = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \nu_{12} n_1 m_1 (\mathbf{u}_1 - \mathbf{u}_2) + \nabla \mathbf{P} \, . \tag{1.32}$$

It would be useful to revise at this stage some general definition relating to collisions between any two particle species. The same definitions used for mechanical, short-range collisions apply also to Coulomb, long-range collisions, e.g. the concepts of cross section σ, mean free path λ_m, collision frequency ν_c. In particular, for a scattering process having cross section σ where particles with velocity $\mathbf{v}$ are scattered by stationary scatterers of density n_n, one has

$$\nu_c = n_n \sigma v \, . \tag{1.33}$$

1.3.2 Coulomb collisions

In order to obtain a final working expression for ν_{12}, we must calculate the parameter η accurately. This can be done by an analysis of the mechanics of a Coulomb collision between an electron and an ion. It is obvious that the exact form of η contains all the information about the intrinsic properties of the plasma, as one might expect from simple physical intuition. For instance, we may anticipate a dependence of collisions on the temperature of the plasma (affecting the particles' speed, on the average), or on the charge of the constituent particles (higher charge implies stronger interaction, hence stronger collisions).

Consider an electron with velocity $\mathbf{v}$ transiting close to an ion, which we will consider stationary during the collision. The distance between the unperturbed trajectory of the electron and the position of the ion is called the impact parameter, r_0, as indicated in Fig. 1.2. After the encounter with the ion, the electron trajectory will be modified due to the Coulomb force experienced in the proximity of the ion, and χ is the angle between the initial and final trajectories of the electron. The encounters which result in electron deflections with $\chi \sim \frac{\pi}{2}$ are called large angle collisions, and are obviously the most effective in randomizing the motion of the electrons, as after such a collision the electron will not have any memory of its original trajectory. We can derive an approximate condition on r_0 and $\mathbf{v}$ for obtaining $\chi \sim \frac{\pi}{2}$.

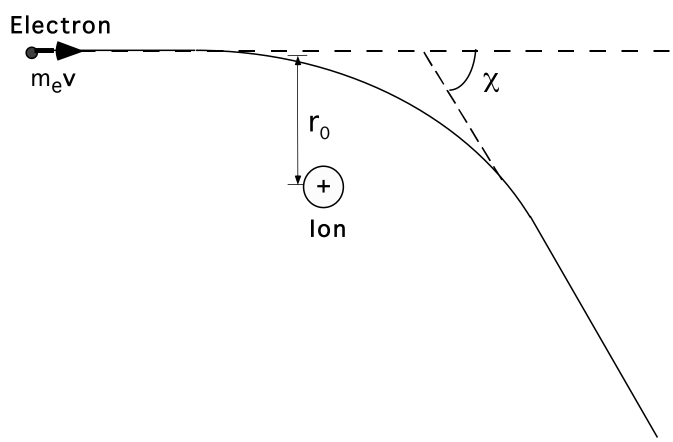

Figure 1.2: Schematic of Coulomb collision between an electron and an ion

Roughly speaking , the Coulomb force F_C on the electron is effective in deflecting it only near the ion (where $r \sim r_0$) , i.e. roughly for a time $\tau \sim \dfrac{r_0}{v}$ The change in electron momentum will be approximately equal to

$$|\Delta m_e \mathbf{v}| = F_C \tau \sim \frac{e^2}{4\pi\epsilon_0 r_0 v} \,. \tag{1.34}$$

Also, since the velocity of the electron (in module) will be conserved during the collision, and the final direction will be approximately normal to the initial one, one can also say that $\Delta(m_e\mathbf{v}) \sim m_e\mathbf{v}$. Equating the two expression and solving for r_0 we find that the condition for obtaining a large angle collision is:

$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_e v^2} \,. \tag{1.35}$$

We can now define a cross-section σ for large angle collisions:

$$\sigma = \pi {r_0}^2 = \frac{e^4}{16\pi{\epsilon_0}^2{m_e}^2v^4} \,. \tag{1.36}$$

Using the expression 1.33, and assuming Z=1 one obtains, for electron-ion collisions that

$$\nu_{ei} = n_i \sigma v = \frac{n_e e^4}{16\pi{\epsilon_0}^2{m_e}^2v^3} \,, \tag{1.37}$$

and

$$\eta = \frac{m_e}{n_e e^2}\nu_{ei} = \cancel{n_i \sigma v} = \frac{e^2}{16\pi{\epsilon_0}^2 m_e v^3} \,. \tag{1.38}$$

If we consider that in a plasma of given electron temperature T, one has $v^2 \sim k_B T/m_e$, this leads to the approximate expression

$$\eta = \frac{\pi e^2 m^{1/2}}{(4\pi\epsilon_0)^2(k_B T_e)^{3/2}} \,. \tag{1.39}$$

We call this parameter η the resistivity of the plasma (we will later see why), and this is an approximate expression based on single large-angle collisions alone.

Obviously, collisions leading to small angle deflections (i.e. $\chi < \frac{\pi}{2}$) can have an effect as well in randomizing the motion of the electrons. Particularly, small angle collisions can be more frequent, and the combined effect of many small angle collisions may also result in large deflections and large momentum exchanges. If one carries out an exact calculation, averaging over a Maxwellian distribution and integrating over all angle deflections, the results is - remarkably as our calculation was quite rough- very close to the expression of Eq. 1.39

$$\eta = \frac{\pi e^2 m^{1/2}}{(4\pi\epsilon_0)^2(k_B T_e)^{3/2}} ln\Lambda \,, \tag{1.40}$$

i.e., the expression derived above is correct, apart from a multiplicative term $\ln\Lambda$, where $\Lambda = 12\pi n_e {\lambda_D}^3$ is called the Coulomb logarithm.

Online Task 1:

- What is the maximum impact parameter that makes sense to consider when evaluating the effect of small angle collisions?

- Recall what η in Eq. (1.40) [and ν in (1.37)] represent(s) physically. What is the physical effect of its (their) value, whether small or large? Discuss the dependence of Eq. (1.40) on the plasma temperature, and the particle mass and charge. Use purely qualitative arguments (no calculation required to answer this). Is this dependence expected, from physical intuition?

1.3.3 Plasma resistivity

Note that η is also referred to as the plasma resistivity since it is the proportionality constant between the electric field $\mathbf{E}$ and the current $\mathbf{j}$ such that $\mathbf{E} = \eta\mathbf{j}$.

This can be shown by simply considering the momentum equation to analyze the case of a constant electric field applied in a plasma, driving an electron current. We will assume that the ions do not move on the timescales of interest, and that the plasma is cold ($T \sim 0$) and unmagnetized.

Under these conditions, the momentum equation for the electron fluid can be written as:

$$m_e n_e \frac{d\mathbf{u_e}}{dt} = -en_e\mathbf{E} + \eta {n_e}^2 e^2 (\mathbf{v_i} - \mathbf{v_e}) \tag{1.41}$$

Let's assume that after having applied the electric field for some time we reach a stationary state in which a constant electron current flows through the plasma. At this stage we will have that $\frac{d\mathbf{u_e}}{dt} = 0$ and:

$$\mathbf{E} = -\eta n_e e v_e = \eta \mathbf{J} \tag{1.42}$$

This is the microscopic expression of Ohm's law, proving that η is effectively an expression for the resistivity of a plasma.

Some considerations on collisions

Looking for example at equation (1.29), it might appear at first glance that for large velocity differences the momentum transfer between species becomes very efficient. However, the opposite is the case. It should also be emphasized that η implies that collisions become negligible in the limit of high temperatures since $(k_B T)^{-3/2} \propto v^{-3}$ - even if the electron and ion temperature deviate substantially. In the case of $T_e \sim T_i$ the v^{-3} scaling implies that collisions are approximately 1 million times less likely at a few hundred eV than at a few eV. Consequently we will see that many theoretical descriptions and numerical models do not include collisions without substantially affecting the accuracy of the predictions.The physical origin of this dependence is that the time during which a particle feels the coulomb force depends on the velocity of the particle and the impact parameter r_0 as $t_{coll} \approx r_0/v$ and $r_0 \approx v^{-2}$. The approximation of a *collisionless* plasma is frequently used to simplify plasma modelling - in particular in the kinetic approach.

1.4 Equation of state

In order to study the evolution of a plasma employing the fluid equation discussed previously, one also needs a functional relationship (an equation of state, EoS) between the fundamental plasma parameters p, n, T. This has been already discussed in Module 911. We will just remind here that p, n, T will in a typical equilibrium situation be assumed to satisfy the relationship $p = nk_B T$ (the state equation of a perfect gas). Furthermore, depending on the problem under investigation, either of the following conditions apply: $T = constant$ (isothermal EoS) or $P = Cn^\gamma$ (adiabatic EoS, to be used when p, T vary slowly), where $\gamma = \dfrac{f+2}{f}$, where f denotes the number of degrees of freedom of the physical problem.

If we know the right EoS for the problem we are investigating using the fluid equations, we can now specify an expression for the term $-\nabla p$ in the right end of (1.32). Particularly, we will refer to the case of the electron fluid. Let's consider a 1D case, i.e. a case in which all parameters are function of a single spatial variable x. In this case, $\nabla p_e = \frac{\partial p_e}{\partial x}$.

If the phenomenon under investigation can be described by an adiabatic EoS (e.g. it is slowly varying), then one has that $p_e = n_e k_B T_e$ and $P_e = C{n_e}^\gamma$, from which one can write:

$$\frac{\partial p_e}{\partial x} = C\gamma {n_e}^{\gamma-1} \frac{\partial n_e}{\partial x} \, .$$

However,

$$C{n_e}^{\gamma-1} = \frac{C{n_e}^\gamma}{n_e} = \frac{p_e}{n_e} = k_B T_e \, .$$

Therefore one has in this case:

$$\frac{\partial p_e}{\partial x} = \gamma K T_e \frac{\partial n_e}{\partial x} = 3 k_B T_e \frac{\partial n_e}{\partial x} \,. \tag{1.43}$$

Online Task 2: Suppose that you have to study electrostatic dynamics of an adiabatic electron plasma in 2 dimensions (2 degrees of freedom). Ions may be taken to be "frozen" (since much heavier), thus n_i = constant (no dynamics). Provide a system of fluid equations to model this problem. How is the pressure term expressed in the right-hand side of the momentum equation? You may assume that the space-dependent plasma dynamical variables are n_e and $\mathbf{u_e}$, while the pressure is given from n_e via an equation of state (which one?).
Hint(s): Consider the evolution of the electron fluid density n_e and velocity $\mathbf{u_e}$. Refer to your Module 911 notes about the Eq. of state.

Online Task 3: How many (scalar) dynamical state variables does the system in the latter question contain? How many equations? Is this a "closed" system? (i.e., one bearing so many equations are variables to solve for). If not, then how would you suggest that it may be "closed"?
Hint: Consider $\mathbf{E} = -\nabla\phi$ (magnetic field variations are neglected for electrostatic plasma excitations), and search for an equation in terms of the electric potential ϕ. Recall that the total charge density is $\rho = (n_i - n_e)e$ (why?) for ion charge state $Z = 1$.

1.5 The pressure evolution equation

Recall that the equation of state was introduced in order to "close" the system of fluid density and velocity equations, by adopting a specific exact expression for the fluid pressure variable p therein. For the sake of completeness, and yet omitting tedious algebraic details, we should add that one may also employ an evolution equation for the plasma pressure p_s (where $s = e$ denotes electrons, or i for ions).

The pressure equation, expressed in its simplest form, reads:

$$\frac{\partial p_s}{\partial t} + \mathbf{u}_s \cdot \nabla p_s \;=\; -\gamma \, p_s \, \nabla \cdot \mathbf{u}_s \,, \tag{1.44}$$

where n_s, $\mathbf{u}_s$ and p_s respectively denote the density, velocity and pressure of species s. The parameter $\gamma = c_P/c_V = 1 + 2/f$ denotes the specific heat ratio (for f degrees of freedom), e.g. $\gamma = 3$ in the one-dimensional (*1d*) case, $\gamma = 2$ in the two-dimensional (*2d*) and $\gamma = 5/3$ in the three-dimensional (*3d*) case; also, $\gamma = 1$ if an adiabatic evolution is considered.

The pressure equation can be used instead of the EoS introduced previously. Admittedly, this extended description (density, momentum, pressure evolution) adds some level of complexity to the existing one (density, momentum, plus EoS) (Maxwell's laws' contribution obviously to be added in both cases). Nevertheless, it adds some rigour to the picture provided by an EoS, which relies on (sometimes questionable) physical assumptions (e.g., adiabatic variation, or isothermal hypothesis).

A note, for rigour. As will be discussed in the second part of these Notes (kinetic theory), the pressure equation (1.44) is but a link in a long chain of equations (the momentum equations). In the picture shown here, it closes the system of fluid evolution equations, yet only because it was "forced" to do so (!), by truncating terms present in the RHS. Nevertheless, Eq. (1.44) is a good "working horse" for efficient fluid plasma study, and can be used instead of an EoS (in fact, to *avoid* the assumption of an EoS), wherever the complexity of the phenomena to be studied so require.

1.6 Appendix 1: Eulerian vs Langrangian viewpoint

When treating a plasma as a fluid there are two fundamentally different viewpoints in which we can approach the plasma. The first approach is the so called Eulerian viewpoint - more conventional in terms of the way we normally look at the world around us. Here we treat the physical quantities of interest as functions of the spatial co-ordinates (x,y,z) and time. I.e. when we try to solve for the typical quantities of interest (e.g. density and temperature) we are sitting in one place (or volume element) and seek to describe the temporal evolution in that point.

The motion of the fluid is then described by particles moving across the boundaries of our volume element. Clearly each particle leaving the volume element leads to a reduction of the local density, while particles entering the volume element lead to an increase. The change in density is then a result of net particle flow (inflow-outflow) across the boundaries and similarly, the net change in the local temperature depends on the net energy flux across the boundaries.

By contrast, in the Langrangian viewpoint, we consider the fluid from the perspective of a group of particles - in effect we are "riding" with the bulk fluid and watch the fluid properties (e.g. density, temperature) evolve as the particles move through (and actually being) the fluid. Therefore the number of particles for each fluid element is a constant (since we are 'riding' with a fixed group of particles). Changes in density are therefore described by a volume change of the volume element containing these particles, while changes in temperature are described by a change in kinetic energy of this group of particles.

A good way of visualising the two approaches is to think of a cup of coffee into which we have just put a drop of milk. From the of the Eulerian viewpoint we are watching the evolution by sticking a probe into a fixed position in a cup of coffee and watching the temperature and milk concentration change in that small volume element. From the Lagrangian perspective we might be following the evolution of the drop of milk and observe that the drop of milk starts moving more rapidly (gets heated) and increases its volume - hence becoming more diluted within the coffee.

A rigorous analytical way to formulate the Lagrangian picture is to employ, instead of the Eulerian variable $\{\mathbf{r}, t\}$, the moving frame variables

$\{\mathbf{r} - u(\mathbf{r},t)t, t\} \equiv \{\xi, \tau\}$. This is obviously a tedious mathematical formulation. The (simpler, analytically) Eulerian formalism will be adopted throughout these course Notes, unless otherwise necessary (and explicitly stated).

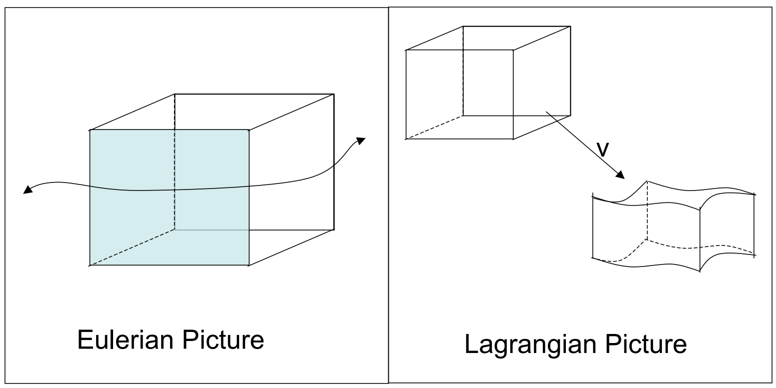

Figure 1.3: Contrast of Eulerian and Lagrangian Viewpoints. In the Eulerian Viewpoint the volume element is fixed in space and the plasma flows through it. By contrast in the Lagrangian viewpoint the volume element *itself* moves though the liquid, changing shape and size to reflect changing density. The two boxes are offset by a distance $v\Delta t$ where v is the flow velocity and $v\Delta t$ the elapsed time.

1.7 Appendix 2: The convective derivative

Consider any quantity Q that varies with both space and time in a fluid, (e.g. the temperature or density). The total derivative with regards to the time t can be rewritten as

$$\frac{dQ(x,t)}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt} = \frac{\partial Q}{\partial t} + u_x \frac{\partial Q}{\partial x}\,, \tag{1.45}$$

which is generalised in three dimensions as

$$\frac{dQ(x,t)}{dt} = \frac{\partial Q}{\partial t} + (\mathbf{u}\cdot\nabla)Q\,. \tag{1.46}$$

The right-hand side of this expression defines the so called *convective derivative*. The convective derivative accounts explicitly for quantities that vary both in time and in space. In some contexts (namely in the Lagrangian picture discuss above), one often encounters research studies wherein the fluid equations are rewritten in terms of the *operator* of the convective derivative, say D/Dt, as defined in the latter equation (1.46),

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\,. \tag{1.47}$$