$\partial x \quad \partial y \quad \partial z$

## 6.13 CYLINDERICAL POLAR COORDINATES

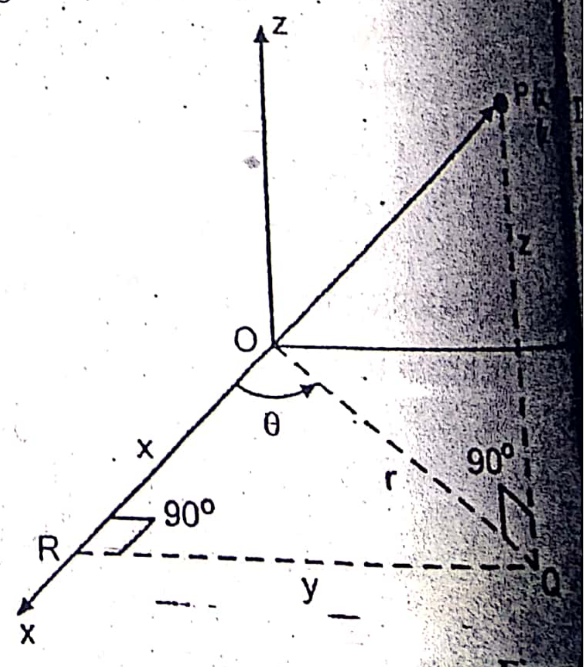
Let P ( x , y , z ) be any point whose projection on the xy–plane is Q ( x , y ) . Then the cylinderical coordinates of P are ( r , θ , z ) in which r = OQ , θ = ∠XOQ and z = QP . From the figure (6.9), the transformation equations expressing the rectangular Cartesian coordinates in terms of cylinderical polar coordinates are:

$$x = r\cos\theta \qquad (1)$$

$$y = r\sin\theta \qquad (2)$$

$$z = z \qquad (3)$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, and $-\infty < z < \infty$.




Figure

### COORDINATE SURFACES

In cylinderical coordinate system , the coordinate surfaces are:

(i) If r is held constant while θ and z vary, then the equation $r = C_1$ represents a right circular cylinder of radius $C_1$ and axis along z–axis (or z–axis if $C_1 = 0$) as shown in figure [6.10 (a)].

(ii) If θ is held constant while r and z vary, then the equation $\theta = C_2$ represents a half plane through the z–axis making an angle θ with the xz – plane as shown in figure [6.10 (b)].

(iii) If z is held constant, while r and θ vary, then the equation $z = C_3$ represents a plane perpendicular to z–axis as shown in figure [6.10 (c)].

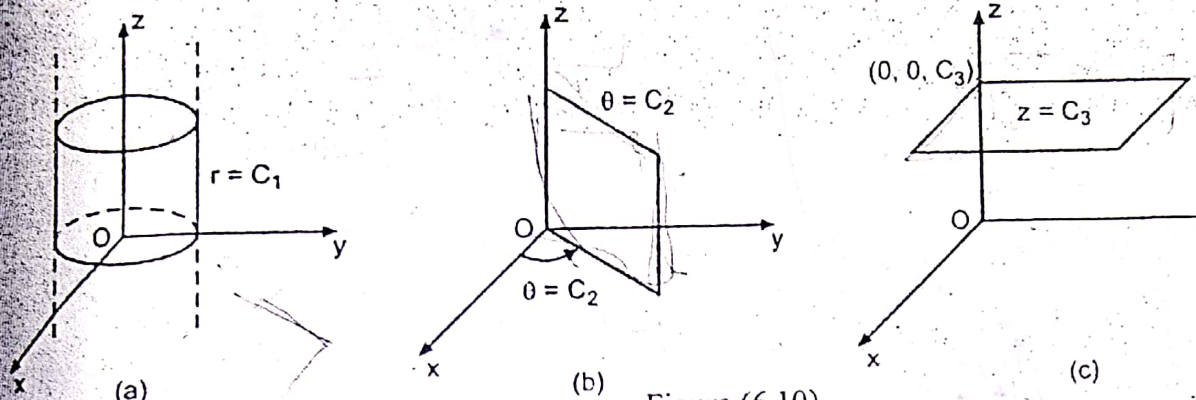


Figure (6.10)

## COORDINATE CURVES

The coordinate curves for cylinderical polar coordinate system are :

(i) If θ and z are fixed while r varies, then the intersection of $\theta = C_2$ and $z = C_3$ is a straight line called the r–coordinate curve or simply the r–curve.

(ii) If r and z are fixed while θ varies, then the intersection of $r = C_1$ and $z = C_3$ is a circle (or point) called the θ–coordinate curve or simply the θ–curve.

(iii) If r and θ are fixed while z varies, then the intersection of $r = C_1$ and $\theta = C_2$ is a straight line called the z–coordinate curve or simply the z–curve.

Thus the r–curves are straight lines radiating from and normal to the z–axis, the θ–curves are circles centered on the z–axis and parallel to the xy–plane; and the z–curves are the straight lines parallel to the z–axis as shown in figure (6.11).

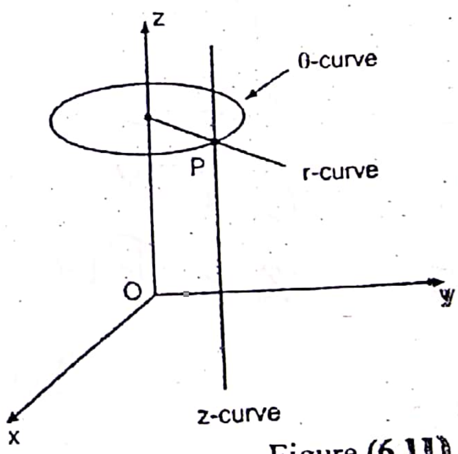


Figure (6.11)

Coordinates in terms of

## 6.14 CYLINDERICAL COORDINATES IN TERMS OF CARTESIAN COORDINATES

We know that the equations expressing the rectangular Cartesian coordinates in terms of cylinderical polar coordinates are:

$$x = r\cos\theta \qquad (1)$$

$$y = r\sin\theta \qquad (2)$$

$$z = z \qquad (3)$$

In matrix notation, equations (5) can be written as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} \qquad (6)$$

## 6.21 EXPRESSIONS FOR ARC LENGTH, AREA, AND VOLUME ELEMENTS IN CYLINDERICAL POLAR COORDINATES

In cylinderical polar coordinates, we have

$u_1 = r, \quad u_2 = \theta, \quad u_3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

### ARC LENGTH ELEMENT

In orthogonal curvilinear coordinates, the element of arc length is determined from

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$$

In cylinderical polar coordinates, this becomes

$$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (1)^2(dz)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

### ALTERNATIVE METHOD

In cylinderical polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dz = dz$$

Then
$$\begin{aligned} ds^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 + (dz)^2 \\ &= (\cos^2\theta + \sin^2\theta)(dr)^2 + (r^2\sin^2\theta + r^2\cos^2\theta)(d\theta)^2 \\ &\qquad - 2r\sin\theta\cos\theta\, dr\, d\theta + 2r\sin\theta\cos\theta\, dr\, d\theta + (dz^2)^2 \\ &= (dr)^2 + r^2(d\theta)^2 + (dz)^2 \end{aligned}$$

### AREA ELEMENT

We know that the elements of area in orthogonal curvilinear coordinates are :

$dA_1 = h_2h_3\,du_2\,du_3$; $\quad dA_2 = h_1h_3\,du_1\,du_3$ and $dA_3 = h_1h_2\,du_1\,du_2$

In cylinderical polar coordinates, these becomes

$$dA_1 = (r)(1)\,d\theta\,dz = r\,d\theta\,dz$$

$$dA_2 = (1)(1)\,dr\,dz = dr\,dz$$

and
$$dA_3 = (1)(r)\,dr\,d\theta = r\,dr\,d\theta.$$

$h_1 h_3\, d.20r$

### VOLUME ELEMENT

The volume element in orthogonal curvilinear coordinates is :

$$dV = h_1h_2h_3\,du_1\,du_2\,du_3.$$

In cylinderical polar coordinates , this becomes

$$dV = (1)(r)(1)\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

## 6.22 EXPRESSION FOR JACOBIAN IN CYLINDERICAL POLAR COORDINATE

We know that the Jacobian in orthogonal curvilinear coordinates $u_1, u_2, u_3$ is given by

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

In cylinderical coordinates $h_1 = 1$, $h_2 = r$, $h_3 = 1$ and $u_1 = r$, $u_2 = \theta$, $u_3 = z$

Therefore $J\left(\frac{x, y, z}{r, \theta, z}\right) = (1)(r)(1) = r.$

$\partial \quad \partial \quad \partial$