# PROBLEMS ON THE CURL:

## PROBLEM 29:

$\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$ $\quad\quad \nabla \times \nabla \times \vec{A}$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \left[2z + 2x\right]\hat{i} + \left[0 - 0\right]\hat{j} + \left[-2z - x^2\right]\hat{k}$$

$$\nabla \times \vec{A} = (2z + 2x)\hat{i} + 0\hat{j} + (-2z - x^2)\hat{k}$$

$$\nabla \times \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2z+2x) & 0 & (-2z-x^2) \end{vmatrix}$$

$$= \left[0 - 0\right]\hat{i} + \left[2 + 2x\right]\hat{j} + \left[0 - 0\right]\hat{k}$$

$$\nabla \times \nabla \times \vec{A} = (2x+2)\hat{j}$$

## PROBLEM 30:

(i) $\nabla \times \vec{r} = \vec{0}$ $\quad\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \left[\partial z/\partial y - \partial y/\partial z\right]\hat{i} + \left[\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right]\hat{j} + \left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]\hat{k}$$

$$= (0 - 0)\hat{i} + (0 - 0)\hat{j} + (0 - 0)\hat{k}$$

$$\nabla \times \vec{r} = \vec{0}$$

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(ii) $\nabla\times[f(r)\vec{r}]=\vec{0}$

Using $\nabla\times(\phi\vec{A})=\phi(\nabla\times\vec{A})-\nabla\phi\times\vec{A}$

$$\nabla\times[f(r)\vec{r}]=f(r)[\nabla\times\vec{r}]-[\nabla f(r)]\times\vec{r}$$

$$=f(r)[\nabla\times\vec{r}]-\left[\frac{f'(r)\vec{r}}{r}\right]\times\vec{r} \qquad \because \nabla f(r)=\frac{f'(r)\vec{r}}{r}$$

$$=f(r)[\nabla\times\vec{r}]-\frac{f'(r)}{r}(\vec{r}\times\vec{r})$$

$$=f(r)(\nabla\times\vec{r})-\vec{0} \qquad \because \vec{r}\times\vec{r}=\vec{0}$$

$$=f(r)(\vec{0}) \qquad \because \nabla\times\vec{r}=\vec{0}$$

$$\nabla\times[f(r)\vec{r}]=\vec{0}$$

(iii) $\nabla\times(r^n\vec{r})=\vec{0}$

$$\nabla\times[f(r)\vec{r}]=\vec{0}$$

put $f(r)=r^n \Rightarrow \nabla\times[r^n\vec{r}]=\vec{0}$

(iv) $\nabla\times\left(\frac{\vec{r}}{r^2}\right)=\vec{0}$

As $\nabla\times(r^n\vec{r})=\vec{0}$

put $n=-2$ $\quad \nabla\times(r^{-2}\vec{r})=\vec{0}$

$$\nabla\times(\vec{r}/r^2)=\vec{0}$$

**PROBLEM 31 :**

Show that $\vec{V}=\vec{r}/r^2$ is irrotational.

Find $\phi$ such that $\vec{V}=-\nabla\phi$ and such that $\phi(a)=0$ $a>0$

For $\vec{V}$ is irrotational $\Rightarrow \nabla\times\vec{V}=\vec{0}$

$$\nabla\times\vec{V}=\nabla\times(\vec{r}/r^2)$$

$$=\nabla\times(r^{-2}\vec{r})=\vec{0} \qquad \because \nabla\times(r^n\vec{r})=\vec{0}$$

Hence $\vec{V}$ is irrotational.

$$\vec{V} = -\nabla\phi$$

$$\Rightarrow \vec{r}/r^2 = -\nabla\phi$$

$$\Rightarrow \nabla\phi = -\vec{r}/r^2$$

$$\Rightarrow \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = -\vec{r}/r^2$$

$$\Rightarrow \frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial x}\hat{i} + \frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial y}\hat{j} + \frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial z}\hat{k} = -\vec{r}/r^2$$

$$\Rightarrow \frac{\partial\phi}{\partial r}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right) = -\vec{r}/r^2.$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = y/r, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \frac{\partial\phi}{\partial r}\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right] = -\vec{r}/r^2$$

$$\Rightarrow \frac{\partial\phi}{\partial r}\left[\frac{\cancel{\vec{r}}}{\cancel{r}}\right] = \frac{-\cancel{\vec{r}}}{r^{\cancel{2}}}$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \frac{-1}{r}$$

Integrating w.r.t $r$.

$$\phi = -\log r + c.$$

$$\Rightarrow \phi = -\log r + c$$ Hence required.

Now

when $\phi(a) = 0$

$$\Rightarrow -\log a + c = 0. \qquad a > 0$$

$$\Rightarrow c = \log a.$$

Hence $\phi = -\log r + \log a.$

$$\Rightarrow \phi = \log(a/r)$$

Hence required.

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# PROBLEM 32:

(i) $a = ?$ $b = ?$ $c = ?$

$$\vec{V} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational.

As $\vec{V}$ is irrotational $\Rightarrow \nabla \times \vec{V} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = \vec{0}$$

$\Rightarrow [c+1]\hat{i} + [a-4]\hat{j} + [b-2]\hat{k} = \vec{0}$

$\Rightarrow (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow c+1 = 0$ , $a - 4 = 0$ , $b - 2 = 0$

$\Rightarrow \boxed{c = -1}$ , $\boxed{a = 4}$ , $\boxed{b = 2}$

(ii) show that $\vec{V}$ can be expressed as the gradient of a scalar function.

let us suppose that

$$\vec{V} = \nabla \phi$$

$$\vec{V} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$\Rightarrow (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - 1y + 2z)\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$\Rightarrow \frac{\partial \phi}{\partial x} = x + 2y + 4z$ —— (1)

$\frac{\partial \phi}{\partial y} = 2x - 3y - z$ —— (2)

$\frac{\partial \phi}{\partial z} = 4x - 1y + 2z$ —— (3)

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Now from ①

$\partial\phi/\partial x = x + 2y + 4z$

Integrating w.r.t x. Considering y, z as constants.

$\phi = x^2/2 + 2xy + 4xz + g(y,z)$

Partial Diff. w.r.t y.

$\partial\phi/\partial y = 0 + 2x + 0 + \partial g/\partial y$

$\cancel{2x} - 3y - z = \cancel{2x} + \partial g/\partial y$

$-3y - z = \partial g/\partial y$

Integrating w.r.t y Considering z constant.

$h(z) - 3y^2/2 - yz = g(y,z)$

$\Rightarrow g(y,z) = h(z) - 3y^2/2 - yz$

Now

$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + h(z)$

Partial Diff. w.r.t z.

$\frac{\partial\phi}{\partial z} = 0 + 0 + 4x - 0 - y + h'(z)$

$\cancel{4x} - \cancel{y} + 2z = \cancel{4x} - \cancel{y} + h'(z)$

$h'(z) = 2z$

Integrating w.r.t z.

$h(z) = z^2 + c.$

$\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2 + c$ (constant)

**PROBLEM 33:**

Since $\phi$ satisfies Laplace's Equation therefore $\nabla^2\phi = 0$.

$\Rightarrow \nabla\cdot(\nabla\phi) = 0$

$\Rightarrow \nabla\phi$ is solenoidal. Also $\nabla\times(\nabla\phi) = \vec{0}$

$\Rightarrow \nabla\phi$ is irrotational.

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