CURL OF A VECTOR FUNCTION:

Let $\vec{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the curl or rotation of $\vec{A}$, written as $\nabla \times \vec{A}$ or Curl $\vec{A}$ is defined by

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$ where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Note:

If $\vec{A}$ is a constant vector, then $\nabla \times \vec{A} = \vec{0}$.

If $\nabla \times \vec{A} = \vec{0}$ in some R, then $\vec{A}$ is called an irrotational vector point function in that region.

EXAMPLE 6:

$\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$ $\quad \nabla \times \vec{A} = ?$ at $P(1, -1, 1)$.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$\cancel{\nabla \times \vec{A} = \hat{i}(-4xyz - 0) - \hat{j}(0 - 3xz^2) + \hat{k}(-4xyz - 0)}$$

$$\cancel{\nabla \times \vec{A} = -4xyz\,\hat{i} + 3xz^2\,\hat{j} - 4xyz\,\hat{k}}$$

$$\cancel{|\nabla \times \vec{A}|_P = -4(1)(-1)(1)\hat{i} + 3(1)(1)^2\hat{j} - 4(1)(-1)(1)\hat{k}}$$

$$\cancel{= 4\hat{i} + 3\hat{j} + 4\hat{k}}$$

$$\nabla \times \vec{A} = \hat{i}(2z^4 + 2x^2y) - \hat{j}(0 - 3xz^2) + \hat{k}(-4xyz - 0)$$

$$\nabla \times \vec{A} = \hat{i}(2z^4 + 2x^2y) + \hat{j}(3xz^2) - \hat{k}(4xyz)$$

$$\nabla \times \vec{A} = (2z^4 + 2x^2 y)\hat{i} + (3xz^2)\hat{j} - (4xyz)\hat{k}$$

$$\nabla \times \vec{A}\Big|_p = [2(1) + 2(1)(-1)]\hat{i} + [3(1)(1)]\hat{j} - [4(1)(-1)(1)]\hat{k}$$

$$= 0\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\nabla \times \vec{A}\Big|_p = 3\hat{j} + 4\hat{k}$$

PROPERTIES OF THE CURL:

THEOREM 7: $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ $\quad \vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

(i) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

$$\nabla \times (\vec{A} + \vec{B}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left((A_1+B_1)\hat{i} + (A_2+B_2)\hat{j} + (A_3+B_3)\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1+B_1 & A_2+B_2 & A_3+B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= (\nabla \times \vec{A}) + (\nabla \times \vec{B})$$

$$\Rightarrow \nabla \times (\vec{A} + \vec{B}) = (\nabla \times \vec{A}) + (\nabla \times \vec{B})$$

(ii) $\nabla \times (\phi\vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$

$$\nabla \times \phi\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\hat{i} - \left[\frac{\partial}{\partial x}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_1)\right]\hat{j}$$
$$+ \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]\hat{k}$$

$$= \left[\phi \frac{\partial A_3}{\partial y} + A_3 \frac{\partial \phi}{\partial y} - \phi \frac{\partial A_2}{\partial z} - A_2 \frac{\partial \phi}{\partial z}\right]\hat{i}$$
$$+ \left[-\phi \frac{\partial A_3}{\partial x} - A_3 \frac{\partial \phi}{\partial x} + \phi \frac{\partial A_1}{\partial z} + A_1 \frac{\partial \phi}{\partial z}\right]\hat{j}$$
$$+ \left[\phi \frac{\partial A_2}{\partial x} + A_2 \frac{\partial \phi}{\partial x} - \phi \frac{\partial A_1}{\partial y} - A_1 \frac{\partial \phi}{\partial y}\right]\hat{k}$$

$$= \left\{\phi\left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right]\hat{i} - \phi\left[\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right]\hat{j} + \phi\left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]\hat{k}\right\}$$
$$+ \left[A_3 \frac{\partial \phi}{\partial y} - A_2 \frac{\partial \phi}{\partial z}\right]\hat{i} - \left[A_3 \frac{\partial \phi}{\partial x} - A_1 \frac{\partial \phi}{\partial z}\right]\hat{j} + \left[A_2 \frac{\partial \phi}{\partial x} - A_1 \frac{\partial \phi}{\partial y}\right]\hat{k}$$

$$= \phi \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \partial\phi/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi\,(\nabla \times \vec{A}) + (\nabla\phi) \times \vec{A}$$

$\Rightarrow \quad \nabla \times (\phi\vec{A}) = \phi\,(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$. Hence proved.

(iii) $\nabla \times (\nabla\phi) = \vec{0}$ (curl grad $\phi = \vec{0}$)

$$\nabla \times \nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\hat{i} + \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\hat{k}$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

$\Rightarrow \quad \nabla \times (\nabla\phi) = \vec{0}$.

(iv) $\nabla \cdot (\nabla \times \vec{A}) = \vec{0}$ (div curl $\vec{A} = \vec{0}$)

$$\nabla \cdot (\nabla \times \vec{A}) = \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right] \hat{i} + \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right] \hat{j} + \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right.\right.$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} \right.$$

$$= \cancel{\frac{\partial^2 A_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 A_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 A_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 A_1}{\partial z \partial y}}$$

$$= 0$$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{A}) = 0$$