PROBLEMS ON THE DIVERGENCE:

PROBLEM 19:

(i) $\nabla\cdot\vec{r} = 3$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\nabla\cdot\vec{r} = \left(\partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$= \partial x/\partial x + \partial y/\partial y + \partial z/\partial z$

$\nabla\cdot\vec{r} = 1 + 1 + 1 = 3$

(ii) $\nabla\cdot\left[f(r)\,\vec{r}\right] = 3f(r) + rf'(r)$

$\nabla\cdot\left[f(r)\vec{r}\right] = f(r)\,(\nabla\cdot\vec{r}) + \vec{r}\cdot\left[\nabla f(r)\right]$

$= f(r)\,(3) + \vec{r}\cdot\left[\frac{f'(r)\,\vec{r}}{r}\right]$ $\quad \because \nabla\cdot\vec{r} = 3$

$= 3f(r) + \frac{f'(r)}{r}\,(\vec{r}\cdot\vec{r})$ $\quad \nabla f(r) = \frac{f'(r)\vec{r}}{r}$

$= 3f(r) + \frac{f'(r)}{r}\,(r^2)$ $\quad \because \vec{r}\cdot\vec{r} = r^2$

$\nabla\cdot\left[f(r)\,\vec{r}\right] = 3f(r) + rf'(r).$

(iii) $\nabla\cdot\left[r^n\vec{r}\right] = (n+3)\,r^n$

$\nabla\cdot\left[r^n\vec{r}\right] = r^n\,(\nabla\cdot\vec{r}) + \vec{r}\cdot(\nabla r^n)$

$= r^n(3) + \vec{r}\cdot(n r^{n-2}\vec{r})$ $\quad \because \nabla\cdot\vec{r} = 3$

$= 3r^n + n r^{n-2}(\vec{r}\cdot\vec{r})$ $\quad \nabla(r^n) = n r^{n-2}\vec{r}$

$= 3r^n + n r^{n-2}\cdot r^2$

$= 3r^n + n r^n$

$\nabla\cdot\left[r^n\vec{r}\right] = (n+3)\,r^n$

(iv) $\nabla\cdot(\vec{r}/r^3) = 0$

$$\nabla\cdot(\vec{r}/r^3) = \nabla\cdot(\vec{r}\,r^{-3}) = \nabla\cdot(r^{-3}\vec{r})$$

$$= r^{-3}(\nabla\cdot\vec{r}) + \vec{r}\cdot(\nabla r^{-3}) \qquad \because \nabla\cdot\vec{r} = 3$$

$$= r^{-3}(3) + \vec{r}\cdot(-3r^{-3-2}\vec{r}) \qquad \because \nabla r^n = n r^{n-2}\vec{r}$$

$$= 3r^{-3} + (-3r^{-5})(\vec{r}\cdot\vec{r})$$

$$= 3r^{-3} - 3r^{-5}\cdot r^2$$

$$= 3r^{-3} - 3r^{-3}$$

$$= (3-3)r^{-3}$$

$$\nabla\cdot(\vec{r}/r^3) = 0$$

PROBLEM 20:

(i) $\nabla\cdot[\vec{r}\,\nabla(1/r^3)] = 3/r^4$

$$\nabla(1/r^3) = \nabla(r^{-3}) = -3r^{-3-2}\vec{r} = -3r^{-5}\vec{r} \qquad \because \nabla(r^n) = \ldots$$

$$\nabla\cdot[r\,\nabla(1/r^3)] = \nabla\cdot[r(-3r^{-5}\vec{r})].$$

$$= \nabla\cdot[-3r^{-4}\vec{r}]$$

$$= -3\,[\nabla\cdot(r^{-4}\vec{r})]$$

$$= -3[r^{-4}(\nabla\cdot\vec{r}) + \vec{r}\cdot(\nabla r^{-4})]$$

$$= -3[r^{-4}(3) + \vec{r}\cdot(-4r^{-6}\vec{r})]$$

$$= -3[3r^{-4} + (-4r^{-6})\,\vec{r}\cdot\vec{r}]$$

$$= -3[3r^{-4} - 4r^{-6}\cdot r^2]$$

$$= -3[3r^{-4} - 4r^{-4}] = -3[-r^{-4}]$$

$$= 3/r^4$$

(ii) $\nabla\left[\nabla\cdot\left(\vec{r}/r\right)\right] = -2\vec{r}/r^3$

$$\nabla\cdot(\vec{r}/r) = \nabla\cdot(r^{-1}\vec{r})$$

$$= r^{-1}(\nabla\cdot\vec{r}) + \vec{r}\cdot(\nabla r^{-1})$$

$$= r^{-1}(3) + \vec{r}\cdot(-1\, r^{-3}\vec{r}) \qquad \because \nabla\cdot\vec{r} = 3, \quad \nabla r^n = n r^{n-2}\vec{r}$$

$$= 3r^{-1} - r^{-3}(\vec{r}\cdot\vec{r})$$

$$= 3r^{-1} - r^{-3}r^2$$

$$\nabla\cdot(\vec{r}/r) = 3r^{-1} - r^{-1}$$

$$\nabla\left[\nabla\cdot(\vec{r}/r)\right] = \nabla(3r^{-1} - r^{-1})$$

$$= \nabla(2r^{-1})$$

$$= 2(\nabla r^{-1})$$

$$= 2\left[-1\, r^{-3}\vec{r}\right].$$

$$\nabla\left[\nabla\cdot(\vec{r}/r)\right] = -2\vec{r}/r^3$$

PROBLEM 21 :

If vector field

$a = ?$ $\quad \vec{v}$ is solenoidal outgoing flux of Body is

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Since $\vec{v}$ is solenoidal.

$$\Rightarrow \nabla\cdot\vec{v} = 0$$

$$\Rightarrow \left(\partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}\right)\cdot\left((x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)\hat{k}\right) = 0$$

$$\partial/\partial x(x+3y) + \partial/\partial y(y-2x) + \partial/\partial z(x+az) = 0$$

$$1 + 1 + a = 0 \Rightarrow \boxed{a = -2}$$

PROBLEM 22 :

$f(r) = ?$ $\quad f(r)\vec{r} =$ Solenoidal

Since $f(r)\vec{r}$ is solenoidal

$$\Rightarrow \nabla\cdot(f(r)\vec{r}) = 0$$

$\nabla \cdot [f(r)\,\vec{r}] = 0$

$f(r)\,(\nabla \cdot \vec{r}) + \vec{r} \cdot [\nabla f(r)] = 0$ $\quad \because \nabla \cdot \vec{r} = 3$

$3f(r) + \vec{r} \cdot \left[\frac{f'(r)\,\vec{r}}{r}\right] = 0$ $\quad \because \nabla f(r) = \frac{f'(r)\vec{r}}{r}$

$3f(r) + \frac{f'(r)}{r}(\vec{r} \cdot \vec{r}) = 0$

$3f(r) + \frac{f'(r)}{r} \cdot r^2 = 0$

$3f(r) + r f'(r) = 0$

$r f'(r) = -3 f(r)$

$\frac{f'(r)}{f(r)} = -3/r$

Integrating.

$\log[f(r)] = -3 \log r + \log C$

$\log[f(r)] = \log r^{-3} + \log C$

$\log[f(r)] = \log(C/r^3)$

$f(r) = C/r^3 \quad C = \text{constant}$

PROBLEM 23:

$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$

$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi)$

$= \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi)$ $\quad \because \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

$= \phi(\nabla \cdot \nabla \psi) + \cancel{(\nabla \psi) \cdot (\nabla \phi)} - \psi(\nabla \cdot \nabla \phi) - \cancel{(\nabla \psi) \cdot (\nabla \phi)}$

$= \phi(\nabla^2 \psi) - \psi(\nabla^2 \phi)$ $\quad \because \nabla \cdot \nabla \phi = \nabla^2 \phi$

$= \phi \nabla^2 \psi - \psi \nabla^2 \phi$

PROBLEM 24:

(i) $\nabla^2 (f(r)) = \frac{2}{r} f'(r) + f''(r)$

$$\nabla^2(f(r)) = \nabla\cdot[\nabla f(r)]$$

$$= \nabla\cdot\left[\frac{f'(r)\vec{r}}{r}\right] \qquad \because \nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

$$= \frac{f'(r)}{r}(\nabla\cdot\vec{r}) + \vec{r}\cdot\left[\nabla\frac{f'(r)}{r}\right]$$

$$= \frac{3f'(r)}{r} + \vec{r}\cdot\left[\frac{1}{r}f''(r) + f'\,\nabla(r^{-1})\right]$$

$$= 3\frac{f'(r)}{r} + r\left(\frac{1}{r}f''(r) + f'\,-1\cdot r^{-\cancel{n}-2}\cdot\vec{r}\right)$$

$$= 3\frac{f'(r)}{r} + f''(r) + \vec{r}f'\left(-\frac{1}{r^3}\right)\vec{r}$$

$$= 3\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= 2\frac{f'(r)}{r} + f''(r).$$