UNIT # 04

GRADIANT DIVERGENCE AND CURL

Introduction:

In this chapter, we will discuss about partial derivatives, differential operators Like Gradient of a scalar ,Directional derivative , curl and divergence of a vector .

Partial Derivative:

Let $\vec{F}$ *be a vector function of independent scalar variable* x, y, z *as*

$$\vec{F} = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\,\hat{k}$$

Then 1st 0rder partial derivatives w .r . t x, y, z *are define as*

$$\frac{\partial\vec{F}}{\partial x} = \frac{\partial}{\partial x}F_1(x)\,\hat{i} + \frac{\partial}{\partial x}F_2(x)\hat{j} + \frac{\partial}{\partial x}F_3(x)\,\hat{k} \qquad (y, z \text{ behave as a constant})$$

$$\frac{\partial\vec{F}}{\partial y} = \frac{\partial}{\partial y}F_1(y)\,\hat{i} + \frac{\partial}{\partial y}F_2(y)\hat{j} + \frac{\partial}{\partial y}F_3(y)\,\hat{k} \qquad (x, z \text{ behave as a constant})$$

$$\frac{\partial\vec{F}}{\partial z} = \frac{\partial}{\partial z}F_1(z)\,\hat{i} + \frac{\partial}{\partial z}F_2(z)\hat{j} + \frac{\partial}{\partial z}F_3(z)\,\hat{k} \qquad (x, y \text{ behave as a constant})$$

Higher order partial derivatives of $\vec{F}$ *w .r . t* x, y, z *are define in a similar way.*

The vector Differential Operator Del $(\vec{\nabla})$ **:**

A vector $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ *is called Differential Operator Del* $(\vec{\nabla})$ *.*

Gradient of a scalar :

Let $\varphi\,(x, y, z)$ *is a scalar function in a space. Then Gradient of a scalar is define as ;*

$$\overrightarrow{\text{Grad}\,\varphi} = \vec{\nabla}\,\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

Properties of Gradient :

If φ *and* Ψ *are scalar function and c is constant then*

(i) $\vec{\nabla}(c\,\varphi) = c\,\vec{\nabla}\varphi$

Proof: We know that $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Then $\vec{\nabla}(c\,\varphi) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(c\varphi) = c\frac{\partial\varphi}{\partial x}\hat{\imath} + c\frac{\partial\varphi}{\partial y}\hat{\jmath} + c\frac{\partial\varphi}{\partial z}\hat{k} = c\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) = c\,\vec{\nabla}\varphi$

(ii) $\vec{\nabla}(\varphi + \Psi) = \vec{\nabla}\varphi + \vec{\nabla}\Psi$

Proof: *We know that* $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

Then $\vec{\nabla}(\varphi + \Psi) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(\varphi + \Psi) = \frac{\partial}{\partial x}(\varphi + \Psi)\hat{\imath} + \frac{\partial}{\partial y}(\varphi + \Psi)\hat{\jmath} + \frac{\partial}{\partial z}(\varphi + \Psi)\hat{k}$

$$= \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) + \left(\frac{\partial\Psi}{\partial x}\hat{\imath} + \frac{\partial\Psi}{\partial y}\hat{\jmath} + \frac{\partial\Psi}{\partial z}\hat{k}\right) = \vec{\nabla}\varphi + \vec{\nabla}\Psi$$

(iii) $\vec{\nabla}(\varphi\Psi) = \varphi\vec{\nabla}\Psi + \Psi\vec{\nabla}\varphi$

Proof: *We know that* $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

Then $\vec{\nabla}(\varphi\Psi) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(\varphi\Psi) = \frac{\partial}{\partial x}(\varphi\Psi)\hat{\imath} + \frac{\partial}{\partial y}(\varphi\Psi)\hat{\jmath} + \frac{\partial}{\partial z}(\varphi\Psi)\hat{k}$

$$= \left[\varphi\frac{\partial\Psi}{\partial x} + \Psi\frac{\partial\varphi}{\partial x}\right]\hat{\imath} + \left[\varphi\frac{\partial\Psi}{\partial y} + \Psi\frac{\partial\varphi}{\partial y}\right]\hat{\jmath} + \left[\varphi\frac{\partial\Psi}{\partial z} + \Psi\frac{\partial\varphi}{\partial z}\right]\hat{k}$$

$$= \varphi\left(\frac{\partial\Psi}{\partial x}\hat{\imath} + \frac{\partial\Psi}{\partial y}\hat{\jmath} + \frac{\partial\Psi}{\partial z}\hat{k}\right) + \Psi\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) = \varphi\vec{\nabla}\Psi + \Psi\vec{\nabla}\varphi$$

(iv) $\vec{\nabla}\left(\frac{\varphi}{\Psi}\right) = \frac{\Psi\vec{\nabla}\varphi - \varphi\vec{\nabla}\Psi}{\Psi^2}$

Proof: *Let*

$$\vec{\nabla}\left(\frac{\varphi}{\Psi}\right) = \vec{\nabla}\left(\varphi\frac{1}{\Psi}\right) = \varphi\vec{\nabla}\left(\frac{1}{\Psi}\right) + \frac{1}{\Psi}\vec{\nabla}\varphi = \varphi\left(\frac{-1}{\Psi^2}\right)\vec{\nabla}\Psi + \frac{1}{\Psi}\vec{\nabla}\varphi = \frac{-\varphi\vec{\nabla}\Psi + \Psi\vec{\nabla}\varphi}{\Psi^2} = \frac{\Psi\vec{\nabla}\varphi - \varphi\vec{\nabla}\Psi}{\Psi^2}$$

Laplacian Operator:

If $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ ***Then*** $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***is called Laplacian Operator.***

$$\therefore \left\{\nabla^2 = \vec{\nabla}.\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\}$$

Laplacian Equation:

If $f(x,y,z)$ ***is function then Laplacian Equation is written as*** $\nabla^2 f = 0$ ***Or*** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

Theorem: Prove that the gradient is a vector perpendicular to the level surface. $\varphi\,(x, y, z) = c$

Proof: ***Let*** $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***be a position vector of any point P on the given surface. Then***

$d\vec{r} = dx\,\hat{i} + dy\hat{j} + dz\,\hat{k}$ ***is a tangent vector to surface at point P*** (x, y, z).

We have to prove $\overrightarrow{\text{Grad}\,\varphi} \perp d\vec{r}$

Now as $\varphi\,(x, y, z) = c$

Then $d\,\varphi = 0$

By using calculus $d\,\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = 0$

$$\left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right).\left(d\,x\,\hat{i} + dy\hat{j} + dz\,\hat{k}\right) = 0$$

$$\vec{\nabla}\varphi\,.\,d\vec{r} = 0$$

$$\overrightarrow{\text{Grad}\,\varphi}\,.\,d\vec{r} = 0$$

This show that $\overrightarrow{\text{Grad}\,\varphi} \perp d\vec{r}$

Hence , Show that the gradient is a vector perpendicular to level surface at point P(x, y, z).

Theorem: Prove that the gradient of a scalar function $\varphi(x, y, z) = c$ ***is a directional derivative of*** φ ***perpendicular to the level surface at point P.***

Proof: Let P & Q be the two neighboring points in a region of space.

Consider the level surfaces $\varphi\,(x, y, z) = c$ ***&*** $\varphi\,(x, y, z) = c + \delta c$ ***through P & Q respectively. Let the normal to the level surface through P intersect the level surface through Q at point P. Let*** $\hat{s}$ ***&*** $\hat{r}$ ***unit vectors along*** $\overrightarrow{PQ}$ ***&*** $\overrightarrow{PR}$ ***.***

We have to prove $\frac{d\varphi}{ds} = \overrightarrow{\text{Grad}\,\varphi}\,.\,\hat{u}$

Let $\overrightarrow{PR} = \delta\vec{r}$ ***&*** $\overrightarrow{PQ} = \delta\vec{s}$ ***then*** $\frac{\overrightarrow{PR}}{\overrightarrow{PQ}} = \frac{\delta r}{\delta s} = \cos\theta$

Since $\frac{\delta\varphi}{\delta s} = \frac{\delta\varphi}{\delta r}\,.\frac{\delta r}{\delta s} = \frac{\delta\varphi}{\delta r}\,\cos\theta$

Applying limit when P $\rightarrow$ Q ***then*** $\delta r \rightarrow 0$

$$\lim_{\delta r \to 0}\frac{\delta\varphi}{\delta s} = \lim_{\delta r \to 0}\frac{\delta\varphi}{\delta r}\,\cos\theta$$

$$\frac{d\varphi}{ds}=\frac{d\varphi}{dr}\cos\theta=\frac{d\varphi}{dr}|\hat{s}||\hat{r}|\cos\theta=\frac{d\varphi}{dr}(\hat{s}\,.\,\hat{r})=\hat{s}\,.\,\hat{r}\frac{d\varphi}{dr}=\frac{d\varphi}{ds}=\overrightarrow{\text{Grad}\,\varphi}\,.\,\hat{s}$$

Here $\overrightarrow{\text{Grad}\,\varphi}=\hat{r}\frac{d\varphi}{dr}$ ***. It is clear that*** $\overrightarrow{\text{Grad}\,\varphi}$ ***lies in the directional of normal to the level surface*** φ

Type equation here.***and measure the rate of change of*** φ ***in that direction.***

$$\frac{d\varphi}{ds}=\frac{\partial\varphi}{\partial x}\frac{dx}{ds}+\frac{\partial\varphi}{\partial y}\frac{dy}{ds}+\frac{\partial\varphi}{\partial z}\frac{dz}{ds}=\left(\frac{\partial\varphi}{\partial x}\hat{\imath}+\frac{\partial\varphi}{\partial y}\hat{\jmath}+\frac{\partial\varphi}{\partial z}\hat{k}\right).\left(\frac{dx}{ds}\hat{\imath}+\frac{dy}{ds}\hat{\jmath}+\frac{dz}{ds}\hat{k}\right)=\vec{\nabla}\varphi\,.\frac{dr}{ds}$$

Let $\frac{dr}{ds}=\hat{u}$

$$\frac{d\varphi}{ds}=\vec{\nabla}\varphi\,.\,\hat{u}\qquad\Longrightarrow\qquad\frac{d\varphi}{ds}=\overrightarrow{\text{Grad}\,\varphi}\,.\,\hat{u}$$

Hence proved that the gradient of a scalar function $\varphi(x,y,z)=c$ ***is a directional derivative of*** φ ***perpendicular to the level surface at point P.***

Example#01: If $\varphi=x^2z+e^{y/x}$***. Find*** $\vec{\nabla}\varphi$ & $|\vec{\nabla}\varphi|$ ***at*** $(1,0,-2)$***.***

Solution: Given function $\varphi=x^2z+e^{y/x}$

We know that $\vec{\nabla}\varphi=\frac{\partial\varphi}{\partial x}\hat{\imath}+\frac{\partial\varphi}{\partial y}\hat{\jmath}+\frac{\partial\varphi}{\partial z}\hat{k}=\frac{\partial}{\partial x}(x^2z+e^{y/x})\hat{\imath}+\frac{\partial}{\partial y}(x^2z+e^{y/x})\hat{\jmath}+\frac{\partial}{\partial z}(x^2z+e^{y/x})\hat{k}$

$$\vec{\nabla}\varphi=\left(2xz+e^{y/x}.\frac{-y}{x^2}\right)\hat{\imath}+\left(e^{y/x}.\frac{1}{x}\right)\hat{\jmath}+(x^2)\hat{k}$$

At $(1,0,-2)$***:*** $\vec{\nabla}\varphi=\left(2(1)(-2)+e^{0/1}.\frac{-0}{1^2}\right)\hat{\imath}+\left(e^{0/1}.\frac{1}{1}\right)\hat{\jmath}+(1^2)\hat{k}=-4\hat{\imath}+\hat{\jmath}+\hat{k}$

Now $|\vec{\nabla}\varphi|=\sqrt{(-4)^2+(1)^2+(1)^2}=\sqrt{16+1+1}=\sqrt{18}=3\sqrt{2}$

Example#02: Prove that $\vec{\nabla}\varphi(r)=\frac{\varphi'(r)\vec{r}}{r}$ ***use above result to evaluate the following.***

(i) $\vec{\nabla}r^n$ ***(ii)*** $\vec{\nabla}\ln r$ ***(iii)*** $\nabla^2\left(\frac{1}{r}\right)$

Solution: ***Let*** $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ ***then*** $r^2=x^2+y^2+z^2$ -----***(i)***

$$\vec{\nabla}\varphi(r)=\frac{\partial\varphi(r)}{\partial x}\hat{\imath}+\frac{\partial\varphi(r)}{\partial y}\hat{\jmath}+\frac{\partial\varphi(r)}{\partial z}\hat{k}=\left[\varphi'(r)\frac{\partial r}{\partial x}\right]\hat{\imath}+\left[\varphi'(r)\frac{\partial r}{\partial y}\right]\hat{\jmath}+\left[\varphi'(r)\frac{\partial r}{\partial z}\right]\hat{k}$$

$$=\left[\varphi'(r)\frac{x}{r}\right]\hat{\imath}+\left[\varphi'(r)\frac{y}{r}\right]\hat{\jmath}+\left[\varphi'(r)\frac{z}{r}\right]\hat{k}\;\therefore\left\{\begin{array}{c}\text{From(i) Differentiate w.r.t }x\\ 2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}\text{ Similarly }\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}\end{array}\right\}$$

$$=\frac{\varphi'(r)\,[x\hat{\imath}+y\hat{\jmath}+z\hat{k}]}{r}$$

$$\vec{\nabla}\varphi(r)=\frac{\varphi'(r)\vec{r}}{r}$$

Hence proved.

(i) $\vec{\nabla}\, r^n$

Solution: ***Let*** $\varphi(r) = r^n$ ***then*** $\varphi'(r) = nr^{n-1}$

Using given equation. $\vec{\nabla}\varphi(r) = \dfrac{\varphi'(r)\vec{r}}{r} = \dfrac{(nr^{n-1})\vec{r}}{r} \implies \vec{\nabla}\, r^n = nr^{n-2}\ \vec{r}$

(ii) $\vec{\nabla}\ \ln r$

Solution: ***Let*** $\varphi(r) = \ln r$ ***then*** $\varphi'(r) = \frac{1}{r}$

Using given equation. $\vec{\nabla}\varphi(r) = \dfrac{\varphi'(r)\vec{r}}{r} = \dfrac{\left(\frac{1}{r}\right)\vec{r}}{r} \implies \vec{\nabla}\ln r = \frac{1}{r^2}\ \vec{r}$

(iii) $\nabla^2\left(\frac{1}{r}\right)$

Solution: Let $\varphi(r) = \frac{1}{r} = r^{-1}$ ***then*** $\varphi'(r) = (-1)r^{-1-1} = -r^{-2}$

Using given equation

$$\vec{\nabla}\varphi(r) = \frac{\varphi'(r)\vec{r}}{r} = \frac{(-r^{-2})\vec{r}}{r} \Rightarrow \vec{\nabla}\left(\frac{1}{r}\right) = -r^{-3}\,\vec{r} = -r^{-3}\,(x\hat{\imath} + y\hat{\jmath} + z\,\hat{k}) = -r^{-3}\,x\,\hat{\imath} - r^{-3}y\hat{\jmath} - r^{-3}z\,\hat{k}$$

Now

$$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla}.\vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(-r^{-3}\,x\,\hat{\imath} - r^{-3}y\hat{\jmath} - r^{-3}z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(-r^{-3}x) + \frac{\partial}{\partial y}(-r^{-3}y) + \frac{\partial}{\partial z}(-r^{-3}z) = -\left[\frac{\partial}{\partial x}(r^{-3}x) + \frac{\partial}{\partial y}(r^{-3}y) + \frac{\partial}{\partial z}(r^{-3}z)\right]$$

$$= -\left[\left(-3r^{-4}\frac{\partial r}{\partial x}.x + r^{-3}.1\right) + \left(-3r^{-4}\frac{\partial r}{\partial y}.y + r^{-3}.1\right) + \left(-3r^{-4}\frac{\partial r}{\partial z}.z + r^{-3}.1\right)\right]$$

$$= -\left[-3r^{-4}\frac{\partial r}{\partial x}.x + r^{-3} - 3r^{-4}\frac{\partial r}{\partial y}.y + r^{-3} + -3r^{-4}\frac{\partial r}{\partial z}.z + r^{-3}\right]$$

$$= -\left[-3r^{-4}\left\{\frac{\partial r}{\partial x}.x + \frac{\partial r}{\partial y}.y + \frac{\partial r}{\partial z}.z\right\} + 3r^{-3}\right]$$

$$= -\left[-3r^{-4}\left\{\left(\frac{x}{r}\right).x + \left(\frac{y}{r}\right).y + \left(\frac{z}{r}\right).z\right\} + 3r^{-3}\right] \quad \left\{\begin{array}{c}\text{From(i) Differentiate w.r.t } x \\ 2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\end{array}\right\}$$

$$= -\left[-3r^{-4}\left\{\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right\} + 3r^{-3}\right] = -\left[-3r^{-4}\left\{\frac{x^2+y^2+z^2}{r}\right\} + 3r^{-3}\right]$$

$$= -\left[-3r^{-4}\left\{\frac{r^2}{r}\right\} + 3r^{-3}\right] = -\left[-3r^{-4}.r + 3r^{-3}\right] = -\left[-3r^{-3} + 3r^{-3}\right] = -[0]$$

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

Example#03:If φ is a function of u and u is a function of x, y, z then show that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial u}\vec{\nabla}u$

Solution: We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

By using chain rule of differentiation

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x} \quad ; \quad \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y} \quad \& \quad \frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial z}$$

Then $\vec{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}\right)\hat{\imath} + \left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial z}\right)\hat{k} = \frac{\partial\varphi}{\partial u}\left(\frac{\partial u}{\partial x}\hat{\imath} + \frac{\partial u}{\partial y}\hat{\jmath} + \frac{\partial u}{\partial z}\hat{k}\right)$

$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial u}\vec{\nabla}u$ ***Hence proved.***

Example#04: Find the scalar function φ such that (i) $\vec{\nabla}\varphi = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$ ***(ii)*** $\vec{\nabla}\varphi = 2r^4\vec{r}$

(i) $\vec{\nabla}\varphi = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

Solution: We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***then*** $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

Comparing coefficients of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = x \implies \varphi = \int x\,\partial x \implies \varphi_1 = \frac{x^2}{2} + c_1(y,z) \text{ --------}\textit{(i)}$$

$$\frac{\partial\varphi}{\partial y} = 2y \implies \varphi = 2\int y\,\partial x \implies \varphi_2 = y^2 + c_2(x,z) \text{ --------}\textit{(ii)}$$

$$\frac{\partial\varphi}{\partial z} = z \implies \varphi = \int z\,\partial x \implies \varphi_3 = \frac{z^2}{2} + c_3(x,z)\text{--------}\textit{(iii)}$$

Adding (i) ,(ii) & (iii) :

$$\varphi_1 + \varphi_2 + \varphi_3 = \frac{x^2}{2} + y^2 + \frac{z^2}{2} + c_1(y,z) + c_2(x,z) + c_3(x,z)$$

Hence $\varphi = \left[\frac{x^2}{2} + y^2 + \frac{z^2}{2}\right] + c$

(ii) $\vec{\nabla}\boldsymbol{\varphi} = \boldsymbol{2}\,\mathbf{r}^4\,\vec{\mathbf{r}}$

Solution: Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ -----*(i)*

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***then***

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = 2r^4\vec{r} = 2r^4\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (x^2+y^2+z^2)^2.\,2x\hat{\imath} + (x^2+y^2+z^2)^2.\,2y\hat{\jmath} + (x^2+y^2+z^2)^2.\,2z\hat{k}$$

Comparing coefficients of $\hat{i}$ **,** $\hat{j}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = (x^2+y^2+z^2)^2.2x \Rightarrow \varphi = \int (x^2+y^2+z^2)^2.2x\,\partial x \Rightarrow \varphi = \frac{(x^2+y^2+z^2)^3}{3} + c_1(y,z) \text{-----}(i)$$

$$\frac{\partial\varphi}{\partial y} = (x^2+y^2+z^2)^2.2y \Rightarrow \varphi = \int (x^2+y^2+z^2)^2.2y\,\partial x \Rightarrow \varphi = \frac{(x^2+y^2+z^2)^3}{3} + c_2(x,z) \text{-----}(ii)$$

$$\frac{\partial\varphi}{\partial z} = (x^2+y^2+z^2)^2.2z \Rightarrow \varphi = \int (x^2+y^2+z^2)^2.2z\,\partial x \Rightarrow \varphi = \frac{(x^2+y^2+z^2)^3}{3} + c_3(x,y) \text{----}(iii)$$

From (i) ,(ii) & (iii) $\Rightarrow \varphi = \frac{(x^2+y^2+z^2)^3}{3} + c$

Example#05: If $\varphi = r^2\, e^{-r}$ ***.Then show that*** $\vec{\nabla}\varphi = (2-r)\, e^{-r}\, \vec{r}$.

Solution: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ***then*** $r^2 = x^2+y^2+z^2$ -----*(i)*

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$

$$\vec{\nabla}\varphi = \frac{\partial}{\partial x}(r^2 e^{-r})\hat{i} + \frac{\partial}{\partial y}(r^2 e^{-r})\hat{j} + \frac{\partial}{\partial z}(r^2 e^{-r})\hat{k}$$

$$\vec{\nabla}\varphi = \left[2r\frac{\partial r}{\partial x} e^{-r} + r^2(-e^{-r})\frac{\partial r}{\partial x}\right]\hat{i} + \left[2r\frac{\partial r}{\partial y} e^{-r} + r^2(-e^{-r})\frac{\partial r}{\partial y}\right]\hat{j} + \left[2r\frac{\partial r}{\partial z} e^{-r} + r^2(-e^{-r})\frac{\partial r}{\partial z}\right]\hat{k}$$

$$\vec{\nabla}\varphi = [2-r\,]re^{-r}\frac{\partial r}{\partial x}\,\hat{i} + [2-r\,]re^{-r}\frac{\partial r}{\partial y}\,\hat{j} + [2-r\,]re^{-r}\frac{\partial r}{\partial z}\,\hat{k}$$

$$\vec{\nabla}\varphi = (2-r)re^{-r}\left[\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right] \quad \therefore \left\{\begin{array}{c}\text{From(i) Differentiate w.r.t } x \\ 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{array}\right\}$$

$$\vec{\nabla}\varphi = (2-r)re^{-r}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$

$$\vec{\nabla}\varphi = (2-r)e^{-r}\left[x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right]$$

$$\vec{\nabla}\varphi = (2-r)\, e^{-r}\, \vec{r}$$

Hence proved.

Example #06: If $\vec{\nabla}\varphi = \frac{\vec{r}}{r^5}$ ***Then show that*** $\varphi(r) = \frac{1}{3}\left(1 - \frac{1}{r^5}\right)$ ***at*** $\varphi(1) = 0$.

Solution: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ***then*** $r = (x^2+y^2+z^2)^{1/2}$ -----*(i)*

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***then***

$$\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\vec{r}}{r^5} = r^{-5}\,\vec{r} \Rightarrow \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = r^{-5}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (x^2+y^2+z^2)^{-5/2}.x\,\hat{i} + (x^2+y^2+z^2)^{-5/2}.y\,\hat{j} + (x^2+y^2+z^2)^{-5/2}.z\,\hat{k}$$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = (x^2+y^2+z^2)^{-5/2}.x \Rightarrow \varphi = \frac{1}{2}\int (x^2+y^2+z^2)^{-5/2}.2x\,\partial x \Rightarrow \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_1(y,z) \text{------}(i)$$

$$\frac{\partial\varphi}{\partial y} = (x^2+y^2+z^2)^{-5/2}.y \Rightarrow \varphi = \frac{1}{2}\int (x^2+y^2+z^2)^{-5/2}.2y\,\partial x \Rightarrow \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_2(x,z) \text{-----}(ii)$$

$$\frac{\partial\varphi}{\partial z} = (x^2+y^2+z^2)^{-5/2}.z \Rightarrow \varphi = \frac{1}{2}\int (x^2+y^2+z^2)^{-5/2}.2z\,\partial x \Rightarrow \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_3(x,y) \text{-------}(iii)$$

From (i) ,(ii) & (iii): $\varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} \Rightarrow \varphi = -\frac{1}{3}\{(x^2+y^2+z^2)^{1/2}\}^{-3} + c \Rightarrow \varphi = -\frac{1}{3}r^{-3} + c$

Hence $\varphi(r) = -\frac{1}{3r^3} + c$ ------------***(a)***

At $\varphi(1) = 0 \Rightarrow -\frac{1}{3(1)^3} + c = 0 \Rightarrow -\frac{1}{3} + c = 0 \Rightarrow c = \frac{1}{3}$

Hence equation (a) will become

$$\varphi(r) = -\frac{1}{3r^3} + \frac{1}{3}$$

$$\Rightarrow \varphi(r) = \frac{1}{3}\left(1 - \frac{1}{r^5}\right)$$

Hence proved.

Example# 07: Show that $\vec{\nabla} r^{n+2} = (n+2)\, r^n\, \vec{r}$ **.**

Solution: Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ ***then*** $r^2 = x^2+y^2+z^2$ ------***(i)***

Now

$$\vec{\nabla} r^{n+2} = \frac{\partial}{\partial x} r^{n+2}\,\hat{\imath} + \frac{\partial}{\partial y} r^{n+2}\hat{\jmath} + \frac{\partial}{\partial z} r^{n+2}\,\hat{k}$$

$$\vec{\nabla} r^{n+2} = \left[(n+2)r^{n+2-1}\frac{\partial r}{\partial x}\right]\hat{\imath} + \left[(n+2)r^{n+2-1}\frac{\partial r}{\partial y}\right]\hat{\jmath} + \left[(n+2)r^{n+2-1}\frac{\partial r}{\partial y}\right]\hat{k}$$

$$\vec{\nabla} r^{n+2} = \left[(n+2)r^{n+1}\frac{\partial r}{\partial x}\right]\hat{\imath} + \left[(n+2)r^{n+1}\frac{\partial r}{\partial y}\right]\hat{\jmath} + \left[(n+2)r^{n+1}\frac{\partial r}{\partial y}\right]\hat{k}$$

$$\vec{\nabla} r^{n+2} = (n+2)r^{n+1}\left[\frac{\partial r}{\partial x}\,\hat{\imath} + \frac{\partial r}{\partial y}\,\hat{\jmath} + \frac{\partial r}{\partial z}\,\hat{k}\right] \quad \because \left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \;\; \frac{\partial r}{\partial z} = \frac{z}{r}\end{matrix}\right\}$$

$$\vec{\nabla} r^{n+2} = (n+2)r^{n+1}\left[\frac{x}{r}\,\hat{\imath} + \frac{y}{r}\,\hat{\jmath} + \frac{z}{r}\,\hat{k}\right]$$

$$\vec{\nabla} r^{n+2} = (n+2)r^{n+1-1}\left[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right]$$

$\vec{\nabla} r^{n+2} = (n+2)r^n\,\vec{r}$ ***Hence proved.***

Example#08: Find a unit vector perpendicular to the surface $\varphi = x^2 + y^2 - z$ ***at*** $(1,2,3)$.

Solution: Given function $\varphi = x^2 + y^2 - z$

We know that $\vec{\nabla}\varphi$ ***is perpendicular to the given surface. Therefore***

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(x^2 + y^2 - z)\,\hat{\imath} + \frac{\partial}{\partial y}(x^2 + y^2 - z)\hat{\jmath} + \frac{\partial}{\partial z}(x^2 + y^2 - z)\,\hat{k}$$

$$\vec{\nabla}\varphi = 2x\,\hat{\imath} + 2y\hat{\jmath} - \hat{k}$$

At $(1,2,3)$:

$$\vec{\nabla}\varphi = 2(1)\,\hat{\imath} + 2(2)\hat{\jmath} - 3\,\hat{k} = 2\,\hat{\imath} + 4\,\hat{\jmath} - \hat{k}$$

Now

Unit vector of $\vec{\nabla}\varphi = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{2\,\hat{\imath}+4\,\hat{\jmath}-\hat{k}}{\sqrt{2^2+4^2+(-1)^2}} = \frac{2\,\hat{\imath}+4\,\hat{\jmath}-\hat{k}}{\sqrt{4+16+1}} = \frac{2\,\hat{\imath}+4\,\hat{\jmath}-\hat{k}}{\sqrt{21}}$

Example#09:Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2$ ***at*** $(2,-1,2)$ ***in the direction of*** $2\,\hat{\imath} - 3\,\hat{\jmath} + 6\,\hat{k}$.

Solution: Given $\varphi = 4xz^3 - 3x^2y^2$

Then

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(4xz^3 - 3x^2y^2)\,\hat{\imath} + \frac{\partial}{\partial y}(4xz^3 - 3x^2y^2)\hat{\jmath} + \frac{\partial}{\partial z}(4xz^3 - 3x^2y^2)\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = (4z^3 - 6xy^2)\,\hat{\imath} + (-6x^2y)\hat{\jmath} + (12xz^2)\,\hat{k}$$

At ***P*** $(2,-1,2)$:

$$\overrightarrow{\text{grad}}\,\varphi = [4(2)^3 - 6(2)(-1)^2]\,\hat{\imath} + [-6(2)^2(-1)]\hat{\jmath} + [12(2)(2)^2]\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = [32 - 12]\,\hat{\imath} + [24]\hat{\jmath} + [96]\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = 20\,\hat{\imath} + 24\,\hat{\jmath} + 96\,\hat{k}$$

Let $\vec{u} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 6\,\hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{2\hat{\imath}-3\,\hat{\jmath}+6\,\hat{k}}{\sqrt{(2)^2+(-3)^2+(6)^2}} = \frac{2\,\hat{\imath}-3\,\hat{\jmath}+6\,\hat{k}}{\sqrt{4+9+36}} = \frac{2\,\hat{\imath}-3\,\hat{\jmath}+6\,\hat{k}}{\sqrt{49}} = \frac{2\,\hat{\imath}-3\,\hat{\jmath}+6\,\hat{k}}{7}$

Thus

Directional derivative of φ ***at Point P in the of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi \,.\, \hat{u} = (20\,\hat{\imath} + 24\,\hat{\jmath} + 96\,\hat{k}) \,.\, \frac{(2\,\hat{\imath}-3\,\hat{\jmath}+6\,\hat{k})}{7}$

$$= \frac{40-72+576}{7}$$

$$= \frac{544}{7}$$

Example #10: Find the Laplacian equation if f if $f(x,y,z) = x^2yz + xy^2z + xyz^2$

Solution: Given function $f(x,y,z) = x^2yz + xy^2z + xyz^2$

We know that Laplacian Equation is $\nabla^2 f = 0$ ***Or*** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

$$\frac{\partial^2}{\partial x^2}(x^2yz + xy^2z + xyz^2) + \frac{\partial^2}{\partial y^2}(x^2yz + xy^2z + xyz^2) + \frac{\partial^2}{\partial z^2}(x^2yz + xy^2z + xyz^2) = 0$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2yz + xy^2z + xyz^2)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(x^2yz + xy^2z + xyz^2)\right] + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2yz + xy^2z + xyz^2)\right] = 0$$

$$\frac{\partial}{\partial x}[2xyz + y^2z + yz^2] + \frac{\partial}{\partial y}[x^2z + 2xyz + xz^2] + \frac{\partial}{\partial x}[x^2y + xy^2 + 2xyz] = 0$$

$$2yz + 2xz + 2xy = 0 \qquad \textit{or} \qquad yz + xz + xy = 0$$

This is required equation .

Exercise# 4.1

Q#01: Find $\vec{\nabla}\varphi$. ***(i)*** $\varphi = \sin x \cosh y$ ***(ii)*** $\varphi = yz + zx + xy + xyz$

(iii) $\varphi = e^{xyz}$ ***at*** $(1,0,1)$ (iv) $\varphi = \tan(x^2 + y^2 + z^2)$ ***at (1,1,1)***

(i) $\boldsymbol{\varphi = \sin x \cosh y}$

Solution: ***Given function*** $\varphi = \sin x \cosh y$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(\sin x \cosh y)\,\hat{\imath} + \frac{\partial}{\partial y}(\sin x \cosh y)\hat{\jmath} + \frac{\partial}{\partial z}(\sin x \cosh y)\,\hat{k}$

$$\vec{\nabla}\varphi = \sin x \cosh y\,\hat{\imath} + \sin x \cosh y\,\hat{\jmath} + \sin x \cosh y\,\hat{k}$$

$(\mathbf{ii})\boldsymbol{\varphi = yz + zx + xy + xyz}$

Solution : Given function $\varphi = yz + zx + xy + xyz$

We know that

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\,x$$

$$= \frac{\partial}{\partial x}(yz + zx + xy + xyz)\,\hat{\imath} + \frac{\partial}{\partial y}(yz + zx + xy + xyz)\hat{\jmath} + \frac{\partial}{\partial z}(yz + zx + xy + xyz)\,\hat{k}$$

$$\vec{\nabla}\varphi = (z + y + yz)\,\hat{\imath} + (z + x + xz)\hat{\jmath} + (y + x + xy)\,\hat{k}$$

$(\mathbf{iii})\boldsymbol{\varphi = e^{xyz}}$ ***at*** $(\mathbf{1},\mathbf{0},\mathbf{1})$

Solution : Given function $\varphi = e^{xyz}$

We know that

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(e^{xyz})\,\hat{\imath} + \frac{\partial}{\partial y}(e^{xyz})\hat{\jmath} + \frac{\partial}{\partial z}(e^{xyz})\,\hat{k}$$

$$\vec{\nabla}\varphi = yz\,e^{xyz}\,\hat{\imath} + zxe^{xyz}\hat{\jmath} + xye^{xyz}\,\hat{k}$$

$$\vec{\nabla}\varphi = e^{xyz}\left[yz\,\hat{\imath} + xz\hat{\jmath} + xy\,\hat{k}\right]$$

At $(1,0,1)$ ***:***

$$\vec{\nabla}\varphi = e^{0}\left[(0)(1)\,\hat{\imath} + (1)(1)\hat{\jmath} + (1)(0)\,\hat{k}\right] = 0\,\hat{\imath} + 1\,\hat{\jmath} + 0\,\hat{k}$$

(iv) $\boldsymbol{\varphi = \tan(x^2 + y^2 + z^2)}$ ***at (1,1,1)***

Solution : Given function $\varphi = \tan(x^2 + y^2 + z^2)$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***x***

$$\vec{\nabla}\varphi = \frac{\partial}{\partial x}\tan(x^2 + y^2 + z^2)\,\hat{\imath} + \frac{\partial}{\partial y}\tan(x^2 + y^2 + z^2)\,\hat{\jmath} + \frac{\partial}{\partial z}\tan(x^2 + y^2 + z^2)\,\hat{k}$$

$$\vec{\nabla}\varphi = 2x\sec^2(x^2 + y^2 + z^2)\,\hat{\imath} + 2y\sec^2(x^2 + y^2 + z^2)\hat{\jmath} + 2z\sec^2(x^2 + y^2 + z^2)\,\hat{k}$$

$$\vec{\nabla}\varphi = 2\sec^2(x^2 + y^2 + z^2)\left[x\hat{\imath} + y\hat{\jmath} + z\,\hat{k}\right]$$

At $(1,1,1)$ ***:*** $\vec{\nabla}\varphi = 2\sec^2(1^2 + 1^2 + 1^2)\left[1\,\hat{\imath} + 1\hat{\jmath} + 1\,\hat{k}\right] = 2\sec^2(3)\left[1\,\hat{\imath} + 1\hat{\jmath} + 1\,\hat{k}\right]$

Q#02: Find $\vec{\nabla}\varphi$ ***.Where*** $\varphi = (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$

Solution: Given function $\varphi = (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

$$= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}\,\hat{\imath} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}\hat{\jmath} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}\,\hat{k}$$

$$\vec{\nabla}\varphi = \left[2xe^{\sqrt{x^2+y^2+z^2}} + (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}.\frac{-2x}{2\sqrt{x^2+y^2+z^2}}\right]\hat{\imath}$$

$$+ \left[2ye^{\sqrt{x^2+y^2+z^2}} + (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}.\frac{-2y}{2\sqrt{x^2+y^2+z^2}}\right]\hat{\jmath}$$

$$+ \left[2ze^{\sqrt{x^2+y^2+z^2}} + (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}.\frac{-2z}{2\sqrt{x^2+y^2+z^2}}\right]\hat{k}$$

$$\vec{\nabla}\varphi = \left[2xe^{\sqrt{x^2+y^2+z^2}} - x\sqrt{x^2 + y^2 + z^2}\,e^{-\sqrt{x^2+y^2+z^2}}\right]\hat{\imath} + \left[2ye^{\sqrt{x^2+y^2+z^2}} - y\sqrt{x^2 + y^2 + z^2}\,e^{-\sqrt{x^2+y^2+z^2}}\right]\hat{\jmath}$$

$$+ \left[2ze^{\sqrt{x^2+y^2+z^2}} - z\sqrt{x^2 + y^2 + z^2}\,e^{-\sqrt{x^2+y^2+z^2}}\right]\hat{k}$$

$$\vec{\nabla}\varphi = xe^{-\sqrt{x^2+y^2+z^2}}\left[2 - \sqrt{x^2 + y^2 + z^2}\right]\hat{\imath} + ye^{\sqrt{x^2+y^2+z^2}}\left[2 - \sqrt{x^2 + y^2 + z^2}\right]\hat{\jmath}$$

$$+ze^{-\sqrt{x^2+y^2+z^2}}\left[2 - \sqrt{x^2 + y^2 + z^2}\right]\hat{k}$$

$$\vec{\nabla}\varphi = e^{-\sqrt{x^2+y^2+z^2}}\left(2 - \sqrt{x^2 + y^2 + z^2}\right)\left[x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}\right]$$

$\therefore$ ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***Then*** $r = \sqrt{x^2 + y^2 + z^2}$ ***&*** $r^2 = x^2 + y^2 + z^2$ ***Thus***

$$\vec{\nabla}\varphi = e^{-r}(2 - r)\,\vec{r}$$

Q#03: If $\varphi = 2xz^4 - x^2y$ ***. Find*** $\vec{\nabla}\varphi$ ***&*** $|\vec{\nabla}\varphi|$ at $(2,-2,1)$

Solution: Given function $\varphi = 2xz^4 - x^2y$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(2xz^4 - x^2y)\hat{\imath} + \frac{\partial}{\partial y}(2xz^4 - x^2y)\hat{\jmath} + \frac{\partial}{\partial z}(2xz^4 - x^2y)\hat{k}$

$$\vec{\nabla}\varphi = (2z^4 - 2xy)\hat{\imath} + (-x^2)\hat{\jmath} + (8xz^3)\hat{k}$$

At $(2,-2,1)$: $\vec{\nabla}\varphi = [2(1)^4 - 2(2)(-2)]\hat{\imath} + [-(2)^2]\hat{\jmath} + [8(2)(1)^3]\hat{k} = [2+8]\hat{\imath} + [-4]\hat{\jmath} + [16]\hat{k}$

$$\vec{\nabla}\varphi = 10\hat{\imath} - 4\hat{\jmath} + 16\hat{k}$$

Now $|\vec{\nabla}\varphi| = \sqrt{(10)^2 + (-4)^2 + (16)^2} = \sqrt{100 + 16 + 256} = \sqrt{372} = 2\sqrt{93}$

Q#04: Find the Laplacian equation if $f(x,y,z) = yz\cos x + xz\cos y + xy\cos z$

Solution: Given function $f(x,y,z) = yz\cos x + xz\cos y + xy\cos z$

We know that Laplacian Equation is $\nabla^2 f = 0$ ***Or*** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

$$\frac{\partial^2}{\partial x^2}(yz\cos x + xz\cos y + xy\cos z) + \frac{\partial^2}{\partial y^2}(yz\cos x + xz\cos y + xy\cos z) + \frac{\partial^2}{\partial z^2}(yz\cos x + xz\cos y + xy\cos z) = 0$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(yz\cos x + xz\cos y + xy\cos z)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(yz\cos x + xz\cos y + xy\cos z)\right] + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(yz\cos x + xz\cos y + xy\cos z)\right] = 0$$

$$\frac{\partial}{\partial x}[-yz\sin x + z\cos y + y\cos z] + \frac{\partial}{\partial y}[z\cos x - xz\sin y + x\cos z] + \frac{\partial}{\partial x}[y\cos x + x\cos y - xy\sin z] = 0$$

$$-yz\cos x + 0 + 0 + 0 - xz\cos y + 0 + 0 + 0 - xy\cos z = 0$$

$-yz\cos x - xz\cos y - xy\cos z = 0$ ***Or*** $yz\cos x + xz\cos y + xy\cos z = 0$ ***This is required equation.***

Q#05 : Find the scalar function φ ***such that (i)*** $\vec{\nabla}\varphi = x\hat{\imath} + y\hat{\jmath}$ ***(ii)*** $\vec{\nabla}\varphi = 3x\hat{\imath} - 2y\hat{\jmath} + z\hat{k}$

(i) $\vec{\nabla}\varphi = x\hat{\imath} + y\hat{\jmath}$

Solution: We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***then*** $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = x\hat{\imath} + y\hat{\jmath} + 0\hat{k}$

Comparing coefficients of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$

$\frac{\partial\varphi}{\partial x} = x \Rightarrow \varphi = \int x\,\partial x \Rightarrow \varphi_1 = \frac{x^2}{2} + c_1(y,z)$ ---------***(i)***

$\frac{\partial\varphi}{\partial y} = y \Rightarrow \varphi = \int y\,\partial x \Rightarrow \varphi_2 = \frac{y^2}{2} + c_2(x,z)$ --------***(ii)***

$\frac{\partial\varphi}{\partial z} = 0 \Rightarrow \varphi = \int 0\,\partial x \Rightarrow \varphi_2 = c_3(x,y)$----------------***(iii)***

Adding (i) ,(ii) & (iii) : $\varphi_1 + \varphi_2 + \varphi_3 = \frac{x^2}{2} + \frac{y^2}{2} + c_1(y,z) + c_2(x,z) + c_3(x,y) \Rightarrow \varphi = \left[\frac{x^2}{2} + \frac{y^2}{2}\right] + c$

(ii) $\vec{\nabla}\varphi = 3x\,\hat{\imath} - 2y\hat{\jmath} + z\,\hat{k}$

Solution: We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ ***then*** $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = x\,\hat{\imath} + y\hat{\jmath} + 0\,\hat{k}$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = 3x \quad \Rightarrow \varphi = 3\int x\,\partial x \quad \Rightarrow \varphi_1 = \frac{3\,x^2}{2} + c_1(y,z) \text{ ----------------}(i)$$

$$\frac{\partial\varphi}{\partial y} = -2y \quad \Rightarrow \varphi = -2\int y\,\partial x \Rightarrow \varphi_2 = -y^2 + c_2(x,z) \text{ --------------}(ii)$$

$$\frac{\partial\varphi}{\partial z} = z \quad \Rightarrow \varphi = \int z\,\partial x \quad \Rightarrow \varphi_3 = \frac{z^2}{2} + c_3(x,y) \text{----------------}(iii)$$

Adding (i) ,(ii) & (iii) : $\varphi_1 + \varphi_2 + \varphi_3 = \frac{3\,x^2}{2} - y^2 + \frac{z^2}{2} + c_1(y,z) + c_2(x,z) + c_3(x,y)$

Hence $\varphi = \left[\frac{3x^2}{2} - y^2 + \frac{z^2}{2}\right] + c$

Q#06 : Find the scalar function φ ***such that*** $\vec{F} = \vec{\nabla}\varphi$ ***where***

(i) $\vec{F} = x\hat{\imath} + 2y\hat{\jmath} + z\,\hat{k}$ ***(ii)*** $\vec{F} = \frac{x\,\hat{\imath} + y\hat{\jmath}}{x^2 + y^2}$ ***(iii)*** $\vec{F} = e^x \sin y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}$

(iv) $\vec{F} = \frac{\vec{r}}{r^5}$ ***at*** $\varphi(1) = 0$ ***(v)*** $\vec{F} = (y^2 - 2xyz^3)\,\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2y\,z^2)\,\hat{k}$

(i) $\vec{F} = x\hat{\imath} + 2y\hat{\jmath} + z\,\hat{k}$

Solution: ***Given*** $\vec{F} = x\hat{\imath} + 2y\hat{\jmath} + z\,\hat{k}$ ***such that*** $\vec{F} = \vec{\nabla}\varphi$ ***The*** $\vec{\nabla}\varphi = x\hat{\imath} + 2y\hat{\jmath} + z\,\hat{k}$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

Then $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = x\hat{\imath} + 2y\hat{\jmath} + z\,\hat{k}$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = x \quad \Rightarrow \varphi = \int x\,\partial x \Rightarrow \varphi_1 = \frac{x^2}{2} + c_1(y,z) \text{ --------}(i)$$

$$\frac{\partial\varphi}{\partial y} = 2y \quad \Rightarrow \varphi = 2\int y\,\partial y \Rightarrow \varphi_2 = y^2 + c_2(x,z) \text{ --------}(ii)$$

$$\frac{\partial\varphi}{\partial z} = z \quad \Rightarrow \varphi = \int z\,\partial z \Rightarrow \varphi_3 = \frac{z^2}{2} + c_3(x,y) \text{--------}(iii)$$

Adding (i) ,(ii) & (iii) : $\varphi_1 + \varphi_2 + \varphi_3 = \frac{x^2}{2} + y^2 + \frac{z^2}{2} + c_1(y,z) + c_2(x,z) + c_3(x,y)$

Hence $\varphi = \left[\frac{x^2}{2} + y^2 + \frac{z^2}{2}\right] + c$

(ii) $\vec{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2}$

Solution: ***Given*** $\vec{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2}$ ***such that*** $\vec{F} = \vec{\nabla}\varphi$

Then $\vec{\nabla}\varphi = \frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2} = \frac{x}{x^2+y^2}\hat{\imath} + \frac{y}{x^2+y^2}\hat{\jmath}$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

Then $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{x}{x^2+y^2}\hat{\imath} + \frac{y}{x^2+y^2}\hat{\jmath} + 0\hat{k}$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$

$$\frac{\partial\varphi}{\partial x} = \frac{x}{x^2+y^2} \quad \Rightarrow \varphi = \frac{1}{2}\int\frac{2x}{x^2+y^2}\partial x \quad \Rightarrow \varphi = \frac{1}{2}\ln(x^2+y^2) + c_1(y,z) \text{ --------}\textbf{\textit{(i)}}$$

$$\frac{\partial\varphi}{\partial y} = \frac{y}{x^2+y^2} \quad \Rightarrow \varphi = \frac{1}{2}\int\frac{2y}{x^2+y^2}\partial y \quad \Rightarrow \varphi = \frac{1}{2}\ln(x^2+y^2) + c_2(x,z) \text{ -------}\textbf{\textit{(ii)}}$$

From (i) & (ii)

$$\Rightarrow \varphi = \frac{1}{2}\ln(x^2+y^2) + c$$

(iii) $\vec{\mathbf{F}} = \mathbf{e^x \sin y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}}$

Solution: ***Given*** $\vec{F} = e^x \sin y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}$ ***such that*** $\vec{F} = \vec{\nabla}\varphi$

Then $\vec{\nabla}\varphi = e^x \sin y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}$

We know that $\vec{\nabla}\varphi = e^x \sin y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}$

Then $\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = e^x \sin y\,\hat{\imath} + e^x \cos y\,\hat{\jmath} + 0\,\hat{k}$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$

$$\frac{\partial\varphi}{\partial x} = e^x \sin y \quad \Rightarrow \varphi = \sin y\int e^x\,\partial x \quad \Rightarrow \varphi = e^x \sin y + c_1(y,z) \text{ --------}\textbf{\textit{(i)}}$$

$$\frac{\partial\varphi}{\partial y} = e^x \cos y \quad \Rightarrow \varphi = e^x\int \cos y\,\partial y \quad \Rightarrow \varphi = e^x \sin y + c_2(x,z) \text{ -------}\textbf{\textit{(ii)}}$$

From (i) & (ii)

$$\Rightarrow \varphi = e^x \sin y + c$$

(iv) $\vec{F} = \frac{\vec{r}}{r^5}$ *at* $\varphi(1) = 0$. *(Example #06)*

Solution: Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ ***then*** $r = (x^2 + y^2 + z^2)^{1/2}$ -----*(i)*

Given $\vec{F} = \frac{\vec{r}}{r^5}$ ***such that*** $\vec{F} = \vec{\nabla}\varphi$ ***Then*** $\vec{\nabla}\varphi = \frac{\vec{r}}{r^5}$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\vec{r}}{r^5} = r^{-5}\,\vec{r} \implies \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = r^{-5}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (x^2 + y^2 + z^2)^{-5/2}.x\hat{\imath} + (x^2 + y^2 + z^2)^{-5/2}.y\hat{\jmath} + (x^2 + y^2 + z^2)^{-5/2}.z\hat{k}$$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = (x^2 + y^2 + z^2)^{-5/2}.x \implies \varphi = \frac{1}{2}\int(x^2 + y^2 + z^2)^{-5/2}.2x\,\partial x \implies \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_1(y,z) \text{-------}(i)$$

$$\frac{\partial\varphi}{\partial y} = (x^2 + y^2 + z^2)^{-5/2}.y \implies \varphi = \frac{1}{2}\int(x^2 + y^2 + z^2)^{-5/2}.2y\,\partial x \implies \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_2(x,z) \text{-----}(ii)$$

$$\frac{\partial\varphi}{\partial z} = (x^2 + y^2 + z^2)^{-5/2}.z \implies \varphi = \frac{1}{2}\int(x^2 + y^2 + z^2)^{-5/2}.2z\,\partial x \implies \varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c_3(x,z) \text{-------}(iii)$$

From (i) ,(ii) & (iii): $\varphi = \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2}}{-3/2} + c$

$$\varphi = -\frac{1}{3}\{(x^2 + y^2 + z^2)^{1/2}\}^{-3} + c$$

$$\implies \varphi = -\frac{1}{3}r^{-3} + c$$

Hence $\varphi(r) = -\frac{1}{3r^3} + c$ ----------- --*(a)*

At $\varphi(1) = 0 \implies -\frac{1}{3(1)^3} + c = 0 \implies -\frac{1}{3} + c = 0 \implies c = \frac{1}{3}$

Hence equation (a) will become

$$\varphi(r) = -\frac{1}{3r^3} + \frac{1}{3} \implies \varphi(r) = \frac{1}{3}\left(1 - \frac{1}{r^5}\right)$$

(v) $\vec{F} = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2y\,z^2)\,\hat{k}$

Solution: ***Given*** $\vec{F} = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2y\,z^2)\,\hat{k}$ ***such that*** $\vec{F} = \vec{\nabla}\varphi$

Then $\vec{\nabla}\varphi = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2y\,z^2)\,\hat{k}$

We know that $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$

Then $\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2y\,z^2)\,\hat{k}$

Comparing coefficients of $\hat{i}$ **,** $\hat{j}$ & $\hat{k}$

$\frac{\partial\varphi}{\partial x} = (y^2 - 2xyz^3) \implies \varphi = \int (y^2 - 2xyz^3)\,\partial x \implies \varphi_1 = xy^2 - \frac{2yz^3x^2}{2} + c_1(y,z)$ ----------***(i)***

$\frac{\partial\varphi}{\partial y} = (3 + 2xy - x^2z^3) \implies \varphi = \int (3 + 2xy - x^2z^3)\,\partial y \implies \varphi_2 = 3y + \frac{2x\,y^2}{2} - x^2z^3y + c_2(x,z)$ ---***(ii)***

$\frac{\partial\varphi}{\partial z} = (6z^3 - 3x^2y\,z^2) \implies \varphi = \int (6z^3 - 3x^2y\,z^2)\,\partial z \implies \varphi_3 = \frac{6z^4}{4} - \frac{3x^2y\,z^3}{3} + c_3(x,y)$--------***(iii)***

Adding (i) ,(ii) & (iii)

$$\varphi_1 + \varphi_2 + \varphi_3 = x\,y^2 - \frac{2yz^3x^2}{2} + 3y + \frac{2x\,y^2}{2} - x^2z^3 + \frac{6z^4}{4} - \frac{3x^2y\,z^3}{3} + c_1(y,z) + c_2(x,z) + c_3(x,z)$$

$$\varphi = \left[xy^2 - x^2\,yz^3 + 3y + x\,y^2 - x^2yz^3 + \frac{3z^4}{2} - x^2y\,z^3\right] + c$$

$$\varphi = \left[-3x^2\,yz^3 + 3y + 2x\,y^2 + \frac{3z^4}{2}\right] + c$$

Q#07: Evaluate the directional derivative of $\varphi = x^2 - y^2 + 2z^2$ ***at*** $(1,2,3)$ ***in the direction of*** $\overrightarrow{PQ}$ ***where Q has coordinates*** $(5,0,4)$

Solution: Given $\varphi = x^2 - y^2 + 2z^2$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(x^2 - y^2 + 2z^2)\,\hat{i} + \frac{\partial}{\partial y}(x^2 - y^2 + 2z^2)\,\hat{j} + \frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = (2x)\,\hat{i} + (-2y)\,\hat{j} + (4z)\,\hat{k}$$

At ***P*** $(1,2,3)$: $\overrightarrow{\text{grad}}\,\varphi = [2(1)]\,\hat{i} + [-2(2)]\hat{j} + [4(3)]\,\hat{k}$

$$\overrightarrow{\text{grad}}\,\varphi = 2\,\hat{i} - 4\hat{j} + 12\,\hat{k}$$

Let $\vec{u} = \overrightarrow{PQ} = Q(5,0,4) - P(1,2,3) = (5-1)\,\hat{i} + (0-2)\hat{j} + (4-3)\,\hat{k} = 4\,\hat{i} - 2\hat{j} + 1\,\hat{k}$

Then $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{4\,\hat{i} - 2\hat{j} + 1\,\hat{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}} = \frac{4\,\hat{i} - 2\hat{j} + 1\,\hat{k}}{\sqrt{16+4+1}} = \frac{4\,\hat{i} - 2\hat{j} + 1\,\hat{k}}{\sqrt{21}}$

Thus

Directional derivative of φ *at Point P in the of* $\overrightarrow{PQ} = \overrightarrow{\text{grad}}\,\varphi \cdot \hat{u}$

$$= (2\,\hat{i} - 4\hat{j} + 12\,\hat{k}) \cdot \frac{4\,\hat{i} - 2\hat{j} + 1\,\hat{k}}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

Q#08: Find the directional derivative of φ at P in the direction of $\vec{u}$ where

(i) $\varphi = x + 2y - z$ at $P(1,4,0)$ ***and*** $\vec{u} = \hat{j} - \hat{k}$

(ii) $\varphi = x^2 + y^2 + z^2$ at $P(2,0,3)$ ***and*** $\vec{u} = 2\hat{i} - \hat{j}$

(iii) $\varphi = e^{2x-y+z}$ at $P(1,1,1)$ ***and*** $\vec{u} = -3\hat{i} + 5\hat{j} + 6\hat{k}$

(i) $\boldsymbol{\varphi = x + 2y - z}$ **at** $\mathbf{P(1,4,0)}$ ***and*** $\vec{\mathbf{u}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Solution: Given $\varphi = x + 2y - z$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(x + 2y - z)\hat{i} + \frac{\partial}{\partial y}(x + 2y - z)\hat{j} + \frac{\partial}{\partial z}(x + 2y - z)\hat{k}$$

$$= 1\hat{i} + 2\hat{j} - 1\hat{k}$$

At ***P*** (1,4,0): $\overrightarrow{\text{grad}}\,\varphi = 1\hat{i} + 2\hat{j} - 1\hat{k}$

Let $\vec{u} = \hat{j} - \hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\hat{j} - \hat{k}}{\sqrt{(0)^2 + (1)^2 + (-1)^2}} = \frac{\hat{j} - \hat{k}}{\sqrt{0+1+1}} = \frac{\hat{j} - \hat{k}}{\sqrt{2}}$

Thus

Directional derivative of φ ***at Point P in the direction of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi \cdot \hat{u} = (1\hat{i} + 2\hat{j} - 1\hat{k}) \cdot \frac{\hat{j} - \hat{k}}{\sqrt{2}} = \frac{0+2+1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

(ii) $\boldsymbol{\varphi = x^2 + y^2 + z^2}$ **at** $\mathbf{P(2,0,3)}$ ***and*** $\vec{\mathbf{u}} = \mathbf{2\hat{i} - \hat{j}}$

Solution: Given $\varphi = x^2 + y^2 + z^2$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)\hat{i} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)\hat{j} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

At ***P*** (2,0,3): $\overrightarrow{\text{grad}}\,\varphi = [2(2)]\hat{i} + [2(0)]\hat{j} + [2(3)]\hat{k} = 4\hat{i} + 0\hat{j} + 6\hat{k}$

Let $\vec{u} = 2\hat{i} - \hat{j}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{2\hat{i} - \hat{j}}{\sqrt{(2)^2 + (-1)^2 + (0)^2}} = \frac{2\hat{i} - \hat{j}}{\sqrt{4+1+0}} = \frac{2\hat{i} - \hat{j}}{\sqrt{5}}$

Thus

Directional derivative of φ *at Point P in the direction of* $\vec{u} = \overrightarrow{\text{grad}}\,\varphi \cdot \hat{u}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{2\hat{i} - \hat{j}}{\sqrt{5}} = \frac{4+4+0}{\sqrt{5}}$$

$$= \frac{8}{\sqrt{5}}$$

(iii) $\boldsymbol{\varphi = e^{2x-y+z}}$ **at** $\mathbf{P(1,1,1)}$ ***and*** $\vec{\mathbf{u}} = \mathbf{-3\,\hat{i} + 5\hat{j} + 6\,\hat{k}}$

Solution: Given $\varphi = e^{2x-y+z}$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(e^{2x-y+z})\,\hat{i} + \frac{\partial}{\partial y}(e^{2x-y+z})\hat{j} + \frac{\partial}{\partial z}(e^{2x-y+z})\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = (2e^{2x-y+z})\,\hat{i} + (-e^{2x-y+z})\hat{j} + (e^{2x-y+z})\,\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = e^{2x-y+z}\,[2\,\hat{i} + \hat{j} + \hat{k}\,]$$

At ***P*** (2,0,3): $\overrightarrow{\text{grad}}\,\varphi = e^{2(1)-(1)+(1)}\,[2\,\hat{i} + \hat{j} + \hat{k}\,] = e^2\,[2\,\hat{i} + \hat{j} + \hat{k}]$

Let $\vec{u} = -3\,\hat{i} + 5\hat{j} + 6\,\hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{-3\,\hat{i}+5\hat{j}+6\,\hat{k}}{\sqrt{(-3)^2+(5)^2+(6)^2}} = \frac{-3\,\hat{i}+5\hat{j}+6\,\hat{k}}{\sqrt{9+25+36}} = \frac{-3\,\hat{i}+5\hat{j}+6\,\hat{k}}{\sqrt{70}}$

Thus Directional derivative of φ ***at Point P in the direction of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi\,.\,\hat{u}$

$$= e^2\,[2\,\hat{i} + \hat{j} + \hat{k}].\,\frac{-3\,\hat{i}+5\hat{j}+6\,\hat{k}}{\sqrt{70}}$$

$$= \frac{e^2[-6+5+6]}{\sqrt{70}}$$

$$= \frac{5e^2}{\sqrt{70}}$$

Q#09: Find the directional derivative of the function

(i) $\varphi = xy^2 + yz^2$ at $(2,-1,1)$ ***in the direction of*** $\hat{i} + 2\hat{j} + 2\hat{k}$

(ii) $\varphi = xyz$ at $(1,1,1)$ ***in the direction of*** $\hat{i} + \hat{j} + \hat{k}$

(iii) $\varphi = 4xz^3 - 3xyz^2$ at $(2,-1,1)$ ***along z-axis.***

(i) $\boldsymbol{\varphi = xy^2 + yz^2}$ **at** $\mathbf{(2,-1,1)}$ ***in the direction of*** $\mathbf{\hat{i} + 2\hat{j} + 2\hat{k}}$

Solution: Given $\varphi = xy^2 + yz^2$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(xy^2 + yz^2)\,\hat{i} + \frac{\partial}{\partial y}(xy^2 + yz^2)\hat{j} + \frac{\partial}{\partial z}(xy^2 + yz^2)\,\hat{k}$$

$\overrightarrow{\text{grad}}\,\varphi = (y^2)\,\hat{i} + (2xy + z^2)\hat{j} + (2yz)\,\hat{k}$

At ***P*** $(2,-1,1)$: $\overrightarrow{\text{grad}}\,\varphi = [(-1)^2]\,\hat{i} + [2(2)(-1) + (1)^2]\,\hat{j} + [2(-1)(1)]\,\hat{k} = 1\,\hat{i} - 3\hat{j} - 2\hat{k}$ ***Let*** $\vec{u} = \hat{i} + 2\hat{j} + 2\hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(2)^2+(2)^2}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{9}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$

Thus Directional derivative of φ ***at Point P in the direction of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi\,.\,\hat{u}$

$$= (1\,\hat{i} - 3\hat{j} - 2\hat{k})\,.\,\frac{\hat{i}+2\hat{j}+2\hat{k}}{3} = \frac{1-6-4}{3} = \frac{-9}{3} = -3$$

(ii) $\boldsymbol{\varphi = xyz}$ **at** $(\mathbf{1},\mathbf{1},\mathbf{1})$ ***in the direction of*** $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Solution: Given $\varphi = xyz$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(xyz)\hat{\imath} + \frac{\partial}{\partial y}(xyz)\hat{\jmath} + \frac{\partial}{\partial z}(xyz)\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = yz\,\hat{\imath} + xz\hat{\jmath} + xy\,\hat{k}$$

At ***P*** $(1,1,1)$: $\overrightarrow{\text{grad}}\,\varphi = [(1)(1)]\,\hat{\imath} + [(1)(1)]\,\hat{\jmath} + [(1)(1)]\,\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$

Let $\vec{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{(1)^2+(1)^2+(1)^2}} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{1+1+1}} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$

Thus ***Directional derivative of*** φ ***at Point P in the direction of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi \cdot \hat{u} = (\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(iii) $\boldsymbol{\varphi = 4xz^3 - 3xyz^2}$ **at** $(\mathbf{2},-\mathbf{1},\mathbf{1})$ ***along z-axis.***

Solution: Given $\varphi = 4xz^3 - 3xyz^2$ ***Then***

$$\overrightarrow{\text{grad}}\,\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(4xz^3 - 3xyz^2)\hat{\imath} + \frac{\partial}{\partial y}(4xz^3 - 3xyz^2)\hat{\jmath} + \frac{\partial}{\partial z}(4xz^3 - 3xyz^2)\hat{k}$$

$$\overrightarrow{\text{grad}}\,\varphi = (4z^3 - 3yz^2)\hat{\imath} + (-3xz^2)\hat{\jmath} + (12xz^2 - 6xyz)\hat{k}$$

At ***P*** $(2,-1,1)$:

$$\overrightarrow{\text{grad}}\,\varphi = [4(1)^3 - 3(-1)(1)^2]\,\hat{\imath} + [-3(2)(1)^2]\,\hat{\jmath} + [12(2)(1)^2 - 6(2)(-1)(1)]\,\hat{k}$$

$$= [4+3]\,\hat{\imath} + [-6]\,\hat{\jmath} + [24+12]\,\hat{k} = 12\,\hat{\imath} - 6\hat{\jmath} + 36\hat{k}$$

Let $\vec{u} = \hat{k}$ ***(along z-axis)*** ***Then*** $\hat{u} = \hat{k}$

Thus

Directional derivative of φ ***at Point P in the direction of*** $\vec{u} = \overrightarrow{\text{grad}}\,\varphi \cdot \hat{u} = (12\,\hat{\imath} - 6\hat{\jmath} + 36\hat{k}) \cdot \hat{k} = 36$

Q#10: Prove that

(i) $\vec{\nabla}\varphi^{n} = n\,\varphi^{n-1}\,\vec{\nabla}\varphi^{n}$ ***(ii)*** $\nabla^2(\varphi\Psi) = \Psi\nabla^2\varphi + 2\,\vec{\nabla}\varphi\,.\vec{\nabla}\Psi + \varphi\nabla^2\Psi$ ***(iii)*** $\nabla^2 r^n = n(n+1)\ r^{n-2}$

(i) $\vec{\nabla}\varphi^{n} = n\,\varphi^{n-1}\,\vec{\nabla}\varphi^{n}$

Solution: We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$------***(i)***

Then $\vec{\nabla}\varphi^{n} = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right]\varphi^{n} = \frac{\partial}{\partial x}\,\varphi^{n}\,\hat{\imath} + \frac{\partial}{\partial y}\varphi^{n}\hat{\jmath} + \frac{\partial}{\partial z}\varphi^{n}\,\hat{k}$

$$\vec{\nabla}\varphi^{n} = \left[n\,\varphi^{n-1}\ \frac{\partial\varphi}{\partial x}\right]\hat{\imath} + \left[n\,\varphi^{n-1}\ \frac{\partial\varphi}{\partial y}\right]\hat{\jmath} + \left[n\,\varphi^{n-1}\ \frac{\partial\varphi}{\partial y}\right]\hat{k}$$

$$\vec{\nabla}\varphi^{n} = n\,\varphi^{n-1}\left[\frac{\partial\varphi}{\partial x}\,\hat{\imath} + \frac{\partial\varphi}{\partial y}\,\hat{\jmath} + \frac{\partial\varphi}{\partial z}\,\hat{k}\right]$$

$$\vec{\nabla}\varphi^{n} = n\,\varphi^{n-1}\left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right]\varphi$$

$\vec{\nabla}\varphi^{n} = n\,\varphi^{n-1}\ \ \vec{\nabla}\varphi^{n}$ $\quad\therefore$ From(i) ***Hence proved.***

(ii) $\nabla^2(\varphi\Psi) = \Psi\nabla^2\varphi + 2\,\vec{\nabla}\varphi\,.\vec{\nabla}\Psi + \varphi\nabla^2\Psi$

Solution: We know that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***then***

$$\nabla^2(\varphi\Psi) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right](\varphi\Psi) = \frac{\partial^2}{\partial x^2}(\varphi\Psi) + \frac{\partial^2}{\partial y^2}(\varphi\Psi) + \frac{\partial^2}{\partial z^2}\,(\varphi\Psi)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\varphi\Psi)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(\varphi\Psi)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\,(\varphi\Psi)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial\varphi}{\partial x}\,\Psi + \varphi\frac{\partial\Psi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial\varphi}{\partial y}\,\Psi + \varphi\frac{\partial\Psi}{\partial y}\right] + \frac{\partial}{\partial z}\left[\frac{\partial\varphi}{\partial z}\,\Psi + \varphi\frac{\partial\Psi}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial\varphi}{\partial x}\,\Psi\right] + \frac{\partial}{\partial x}\left[\varphi\frac{\partial\Psi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial\varphi}{\partial y}\,\Psi\right] + \frac{\partial}{\partial y}\left[\varphi\frac{\partial\Psi}{\partial y}\right] + \frac{\partial}{\partial z}\left[\frac{\partial\varphi}{\partial z}\,\Psi\right] + + \frac{\partial}{\partial z}\left[\varphi\frac{\partial\Psi}{\partial z}\right]$$

$$= \left[\frac{\partial^2\varphi}{\partial x^2}\Psi + \frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x}\right] + \left[\frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x} + \varphi\frac{\partial^2\Psi}{\partial x^2}\right] + \left[\frac{\partial^2\varphi}{\partial y^2}\Psi + \frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y}\right] + \left[\frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y} + \varphi\frac{\partial^2\Psi}{\partial y^2}\right] + \left[\frac{\partial^2\varphi}{\partial z^2}\Psi + \frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z}\right] + \left[\frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z} + \varphi\frac{\partial^2\Psi}{\partial z^2}\right]$$

$$= \frac{\partial^2\varphi}{\partial x^2}\,\Psi + \frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x} + \frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x} + \varphi\,\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\,\Psi + \frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y} + \frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y} + \varphi\,\frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\,\Psi + \frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z} + \frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z} + \varphi\frac{\partial^2\Psi}{\partial z^2}$$

$$= \left[\frac{\partial^2\varphi}{\partial x^2}\,\Psi + \frac{\partial^2\varphi}{\partial y^2}\,\Psi + \frac{\partial^2\varphi}{\partial z^2}\,\Psi\right] + \left[2\,\frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x} + 2\,\frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y} + + 2\,\frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z}\right] + \left[\varphi\,\frac{\partial^2\Psi}{\partial x^2} + \varphi\,\frac{\partial^2\Psi}{\partial y^2} + \varphi\,\frac{\partial^2\Psi}{\partial z^2}\right]$$

$$= \Psi\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right] + 2\left[\frac{\partial\varphi}{\partial x}.\frac{\partial\Psi}{\partial x} + \frac{\partial\varphi}{\partial y}.\frac{\partial\Psi}{\partial y} + + \frac{\partial\varphi}{\partial z}.\frac{\partial\Psi}{\partial z}\right] + \varphi\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right]$$

$$= \Psi\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\varphi + 2\left[\left(\frac{\partial\varphi}{\partial x}\,\hat{\imath} + \frac{\partial\varphi}{\partial y}\,\hat{\jmath} + \frac{\partial\varphi}{\partial z}\,\hat{k}\right).\left(\frac{\partial\Psi}{\partial x}\,\hat{\imath} + \frac{\partial\Psi}{\partial y}\hat{\jmath} + \frac{\partial\Psi}{\partial z}\hat{k}\right)\right] + \varphi\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right]$$

$\nabla^2(\varphi\Psi) = \Psi\nabla^2\varphi + 2\,\vec{\nabla}\varphi\,.\vec{\nabla}\Psi + \varphi\nabla^2\Psi$ ***Hence proved.***

(iii) $\nabla^2 r^n = n(n+1)\ r^{n-2}$

Solution: We know that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***then***

$$\nabla^2 r^n = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] r^n = \frac{\partial^2}{\partial x^2} r^n + \frac{\partial^2}{\partial y^2} r^n + \frac{\partial^2}{\partial z^2} r^n = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} r^n\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} r^n\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} r^n\right]$$

$$= \frac{\partial}{\partial x}\left[n r^{n-1}\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial y}\left[n r^{n-1}\frac{\partial r}{\partial y}\right] + \frac{\partial}{\partial z}\left[n r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= n\left\{\frac{\partial}{\partial x}\left(r^{n-1}\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(r^{n-1}\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial z}\left(r^{n-1}\frac{\partial r}{\partial z}\right)\right\}$$

$$= n\left[\left\{(n-1)r^{n-2}\frac{\partial r}{\partial x}.\frac{\partial r}{\partial x} + r^{n-1}\frac{\partial^2 r}{\partial x^2}\right\} + \left\{(n-1)r^{n-2}\frac{\partial r}{\partial y}.\frac{\partial r}{\partial y} + r^{n-1}\frac{\partial^2 r}{\partial y^2}\right\} + \left\{(n-1)r^{n-2}\frac{\partial r}{\partial z}.\frac{\partial r}{\partial z} + r^{n-1}\frac{\partial^2 r}{\partial z^2}\right\}\right]$$

$$= n\left[(n-1)r^{n-2}\left(\frac{\partial r}{\partial x}\right)^2 + r^{n-1}\frac{\partial^2 r}{\partial x^2} + (n-1)r^{n-2}\left(\frac{\partial r}{\partial y}\right)^2 + r^{n-1}\frac{\partial^2 r}{\partial y^2} + (n-1)r^{n-2}\left(\frac{\partial r}{\partial z}\right)^2 + r^{n-1}\frac{\partial^2 r}{\partial z^2}\right]$$

$$= n\left[(n-1)r^{n-2}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right\} + r^{n-1}\left\{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right\}\right]$$

$$= n\left[(n-1)r^{n-2}\left\{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2\right\} + r^{n-1}\left\{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right\}\right] \text{---------------(a)}$$

Let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

$\therefore$ From(i) ***Differentiate w. r. t x*** $2r\frac{\partial r}{\partial x} = 2x \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ ***Similarly*** $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Again differentiate w. r. t x $\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\left(\frac{x}{r}\right)}{r^2} = \frac{\frac{r^2 - x^2}{r}}{r^2} = \frac{x^2+y^2+z^2-x^2}{r^3} \Longrightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{r^3}$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{r^3}$ & $\frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{r^3}$

$$\nabla^2 r^n = n\left[(n-1)r^{n-2}\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right\} + r^{n-1}\left\{\frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{x^2+y^2}{r^3}\right\}\right]$$

$$\nabla^2 r^n = n\left[(n-1)r^{n-2}\left\{\frac{x^2+y^2+z^2}{r^2}\right\} + r^{n-1}\left\{\frac{y^2+z^2+x^2+z^2+x^2+y^2}{r^3}\right\}\right]$$

$$\nabla^2 r^n = n\left[(n-1)r^{n-2}\left\{\frac{r^2}{r^2}\right\} + r^{n-1}\left\{\frac{2(x^2+y^2+z^2)}{r^3}\right\}\right]$$

$$\nabla^2 r^n = n\left[(n-1)r^{n-2}(1) + r^{n-1}\left\{\frac{2r^2}{r^3}\right\}\right]$$

$$\nabla^2 r^n = n\left[(n-1)r^{n-2} + r^{n-1}\left\{\frac{2}{r}\right\}\right]$$

$$\nabla^2 r^n = n[(n-1)r^{n-2} + 2r^{n-2}] = n[(n-1+2)r^{n-2}]$$

$\nabla^2 r^n = n(n+1)\ r^{n-2}$ ***Hence proved.***

Q#11: Prove that (i) $\vec{\nabla}\,r^3 = 3r\,\vec{r}$ (ii) $\vec{\nabla}\,e^{r^2} = 2e^{r^2}\,\vec{r}$

(i) $\vec{\nabla}\,r^3 = 3r\,\vec{r}$

Solution: ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla} r^3 = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right] r^3 = \frac{\partial}{\partial x} r^3\,\hat{\imath} + \frac{\partial}{\partial y} r^3\,\hat{\jmath} + \frac{\partial}{\partial z} r^3\,\hat{k}$

$= \left[3\,r^{3-1}\,\frac{\partial r}{\partial x}\right]\hat{\imath} + \left[3\,r^{3-1}\,\frac{\partial r}{\partial y}\right]\hat{\jmath} + \left[3\,r^{3-1}\,\frac{\partial r}{\partial y}\right]\hat{k}$

$= \left[3\,r^2\,\frac{x}{r}\right]\hat{\imath} + \left[3\,r^{3-1}\,\frac{y}{r}\right]\hat{\jmath} + \left[3\,r^2\,\frac{z}{r}\right]\hat{k} \;\therefore \left\{\begin{array}{l}\text{From(i) Differentiate w.r.t } x\\ 2r\,\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r}\ \text{Similarly}\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{array}\right\}$

$= \left[3r\,x\,\hat{\imath} + 3r\,y\,\hat{\jmath} + 3r\,z\,\hat{k}\right]$

$= 3\,r\left[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right]$

$\vec{\nabla} r^3 = 3r\,\vec{r}$ $\qquad\therefore$ From(i)

Hence proved.

(ii) $\vec{\nabla}\,e^{r^2} = 2e^{r^2}\,\vec{r}$

Solution: ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla} e^{r^2} = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right] e^{r^2} = \frac{\partial}{\partial x} e^{r^2}\,\hat{\imath} + \frac{\partial}{\partial y} e^{r^2}\,\hat{\jmath} + \frac{\partial}{\partial z} e^{r^2}\,\hat{k}$

$= \left[e^{r^2}.2r^{2-1}\frac{\partial r}{\partial x}\right]\hat{\imath} + \left[e^{r^2}.2\,r^{2-1}\frac{\partial r}{\partial y}\right]\hat{\jmath} + \left[e^{r^2}.2\,r^{2-1}\frac{\partial r}{\partial y}\right]\hat{k}$

$= \left[2re^{r^2}\,\frac{x}{r}\right]\hat{\imath} + \left[2re^{r^2}\,\frac{y}{r}\right]\hat{\jmath} + \left[2r\,e^{r^2}\,\frac{z}{r}\right]\hat{k} \;\therefore \left\{\begin{array}{l}\text{From(i) Differentiate w.r.t } x\\ 2r\,\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r}\ \text{Similarly}\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{array}\right\}$

$= \left[2e^{r^2}\,x\,\hat{\imath} + 2e^{r^2}\,y\,\hat{\jmath} + 2e^{r^2}\,z\,\hat{k}\right]$

$= 2\,e^{r^2}\left[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right]$

$\vec{\nabla} e^{r^2} = 2e^{r^2}\,\vec{r}$ $\qquad\therefore$ From(i)

Hence proved.

Q#12: Prove that ***(i)*** $\vec{\nabla} r = \hat{r}$ ***(ii)*** $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

(i) $\vec{\boldsymbol{\nabla}}\mathbf{r} = \hat{\mathbf{r}}$

Solution: ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ***------(i)***

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla} r = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right] r = \frac{\partial r}{\partial x}\,\hat{\imath} + \frac{\partial r}{\partial y}\,\hat{\jmath} + \frac{\partial r}{\partial z}\,\hat{k}$

$= \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\,\hat{k}$ $\quad \therefore \left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{matrix}\right\}$

$= \frac{x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}{r} = \frac{\vec{r}}{r}$

$\vec{\nabla} r^3 = \hat{r}$ $\quad \therefore\ \hat{r} = \frac{\vec{r}}{r}$

Hence proved.

(ii) $\vec{\boldsymbol{\nabla}}\left(\frac{\mathbf{1}}{\mathbf{r}}\right) = -\frac{\vec{\mathbf{r}}}{\mathbf{r}^3}$

Solution: ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ***------(i)***

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla}\left(\frac{1}{r}\right) = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right](r^{-1}) = \frac{\partial}{\partial x}(r^{-1})\,\hat{\imath} + \frac{\partial}{\partial y}(r^{-1})\hat{\jmath} + \frac{\partial}{\partial z}(r^{-1})\,\hat{k}$

$= (-1)r^{-1-1}.\frac{\partial r}{\partial x}\,\hat{\imath} + (-1)r^{-1-1}.\frac{\partial r}{\partial y}\,\hat{\jmath} + (-1)r^{-1-1}.\frac{\partial r}{\partial z}\,\hat{k}$

$= -r^{-2}.\frac{x}{r}\,\hat{\imath} - r^{-2}.\frac{y}{r}\,\hat{\jmath} - r^{-2}.\frac{z}{r}\,\hat{k}$ $\quad \therefore \left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{matrix}\right\}$

$= -\frac{1}{r^2}\left[\frac{x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}{r}\right] = -\left[\frac{x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}{r^3}\right]$

$\vec{\nabla} r^3 = -\frac{\vec{r}}{r^3}$ ***Hence proved.***

Q#13: Let $\vec{a}$ ***be a constant vector show that*** $\vec{\nabla}(\vec{a}.\vec{r}) = \vec{a}$ ***where*** $\vec{r}$ ***is a position vector.***

Solution: ***Let*** $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$ ***&*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$

Then $\vec{a}.\vec{r} = (a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}).(x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}) = a_1\,x + a_2\,y + a_3\,z$

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla}(\vec{a}.\vec{r}) = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right](\vec{a}.\vec{r}) = \frac{\partial}{\partial x}(\vec{a}.\vec{r})\,\hat{\imath} + \frac{\partial}{\partial y}(\vec{a}.\vec{r})\hat{\jmath} + \frac{\partial}{\partial z}(\vec{a}.\vec{r})\,\hat{k}$

$= \frac{\partial}{\partial x}(a_1\,x + a_2\,y + a_3\,z)\,\hat{\imath} + \frac{\partial}{\partial y}(a_1\,x + a_2\,y + a_3\,z)\hat{\jmath} + \frac{\partial}{\partial z}(a_1\,x + a_2\,y + a_3\,z)\,\hat{k}$

$\vec{\nabla}(\vec{a}.\vec{r}) = a_1\,\hat{\imath} + a_2\hat{\jmath} + a_3\,\hat{k}$

$\vec{\nabla}(\vec{a}.\vec{r}) = \vec{a}$ ***Hence proved.***

Q#14: Find $\overrightarrow{\text{grad}}\,f(r)$ ***where*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$

Solution: ***Given that*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

Then $\overrightarrow{\text{grad}}\,f(r) = \vec{\nabla}f(r) = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right] f(r)$ $\quad \therefore \vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

$= \frac{\partial}{\partial x}f(r)\,\hat{\imath} + \frac{\partial}{\partial y}f(r)\hat{\jmath} + \frac{\partial}{\partial z}f(r)\,\hat{k}$

$= f'(r)\frac{\partial r}{\partial y}\hat{\imath} + f'(r)\frac{\partial r}{\partial y}\,\hat{\jmath} + f'(r)\frac{\partial r}{\partial z}\,\hat{k}$

$= f'(r)\left[\frac{x}{r}\,\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\,\hat{k}\right] \therefore \left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \;\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\end{matrix}\right\}$

$= \left[\frac{x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}{r}\right] f'(r)$

$\overrightarrow{\text{grad}}\,f(r) = \frac{\vec{r}}{r}\,f'(r)$

Q#15: If $\varphi = 2z - x^3y$ and $\vec{a} = 2x^2\,\hat{\imath} - 3yz\hat{\jmath} + xz^2\,\hat{k}$. Find $\vec{a}\,.\,\vec{\nabla}\varphi$ & $\vec{a} \times \vec{\nabla}\varphi$ at $(1,-1,1)$

Solution: : Given that If $\varphi = 2z - x^3y$ and $\vec{a} = 2x^2\,\hat{\imath} - 3yz\hat{\jmath} + xz^2\,\hat{k}$

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla}\varphi = \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right]\varphi = \frac{\partial\varphi}{\partial x}\,\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\,\hat{k}$

$$= \frac{\partial}{\partial x}(2z - x^3y)\,\hat{\imath} + \frac{\partial}{\partial y}(2z - x^3y)\hat{\jmath} + \frac{\partial}{\partial z}(2z - x^3y)\,\hat{k}$$

$$\vec{\nabla}\varphi = -3x^2y\,\hat{\imath} - x^3\,\hat{\jmath} + 2\,\hat{k}$$

Now $\vec{a}\,.\,\vec{\nabla}\varphi = \left(2x^2\,\hat{\imath} - 3yz\hat{\jmath} + xz^2\,\hat{k}\right).\left(-3x^2y\,\hat{\imath} - x^3\,\hat{\jmath} + 2\,\hat{k}\right)$

$$= (2x^2)(-3x^2y) + (-3yz)(-x^3) + (xz^2)(2)$$

$$\vec{a}\,.\,\vec{\nabla}\varphi = -6x^4y + 3x^3yz + 2xz^2$$

At $1,-1,1)$:

$$\vec{a}\,.\,\vec{\nabla}\varphi = -6(1)^4(-1) + 3(1)^3(-1)(1) + 2(1)(1)^2 = 6 - 3 + 2 = 5$$

Now $\vec{a} \times \vec{\nabla}\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -3yz & xz^2 \\ -x^3 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2x^2 & xz^2 \\ -3x^2y & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2x^2 & -3yz \\ -3x^2y & -x^3 \end{vmatrix}$

$$\vec{a} \times \vec{\nabla}\varphi = \hat{\imath}\,(-6yz + x^4z^2) - \hat{\jmath}\,(4x^2 + 3x^3yz^2) + \hat{k}(-2x^5 - 9x^2y^2z)$$

At $1,-1,1)$:

$$\vec{a} \times \vec{\nabla}\varphi = \hat{\imath}\,[-6(-1)(1) + (1)^4(1)^2] - \hat{\jmath}\,[4(1)^2 + 3(1)^3(-1)(1)^2] + \hat{k}[-2(1)^5 - 9(1)^2(-1)^2(1)]$$

$$= \hat{\imath}\,[6 + 1] - \hat{\jmath}\,[4 - 3] + \hat{k}[-2 - 9]$$

$$\vec{a} \times \vec{\nabla}\varphi = 7\hat{\imath} - \hat{\jmath} - 11\hat{k}$$

Q#16: If $\varphi = x^n + y^n + z^n$. Show that $\vec{r}\,.\,\vec{\nabla}\varphi = n\,\varphi$.

Solution: ***Given that*** $\varphi = x^n + y^n + z^n$ ***Let*** $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

We know that $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla}\varphi = \left[\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right](x^n + y^n + z^n)$

$\vec{\nabla}\varphi = \frac{\partial}{\partial x}(x^n + y^n + z^n)\,\hat{i} + \frac{\partial}{\partial y}(x^n + y^n + z^n)\hat{j} + \frac{\partial}{\partial z}(x^n + y^n + z^n)\,\hat{k}$

$\vec{\nabla}\varphi = [n\,x^{n-1}]\,\hat{i} + [n\,y^{n-1}]\hat{j} + [n\,z^{n-1}]\,\hat{k}$

Now $\vec{r}\,.\,\vec{\nabla}\varphi = (x\,\hat{i} + y\hat{j} + z\,\hat{k}).([n\,x^{n-1}]\,\hat{i} + [n\,y^{n-1}]\hat{j} + [n\,z^{n-1}]\,\hat{k})$

$= x.\,n\,x^{n-1} + y.\,n\,y^{n-1} + z\,.\,n\,z^{n-1} = \boldsymbol{n}\,[x^n + y^n + z^n]$

$\vec{r}\,.\,\vec{\nabla}\varphi = n\,\varphi$ ***Hence proved.***

Q#17: If $\varphi = 3x^2y$ & $\psi = xz^2 - zy$. Evaluate $\vec{\nabla}(\vec{\nabla}\varphi\,.\,\vec{\nabla}\psi)$.

Solution: ***Given that*** $\varphi = 3x^2y$ & $\psi = xz^2 - zy$ ***We know that*** $\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{\nabla}\varphi = \left[\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right](3x^2y) = \frac{\partial}{\partial x}(3x^2y)\,\hat{i} + \frac{\partial}{\partial y}(3x^2y)\hat{j} + \frac{\partial}{\partial z}(3x^2y)\,\hat{k}$

$\vec{\nabla}\varphi = 6xy\,\hat{i} + 3x^2\hat{j} + 0\,\hat{k}$

& $\vec{\nabla}\psi = \left[\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right](xz^2 - zy)$

$\vec{\nabla}\psi = \frac{\partial}{\partial x}(xz^2 - zy)\,\hat{i} + \frac{\partial}{\partial y}(xz^2 - zy)\hat{j} + \frac{\partial}{\partial z}(xz^2 - zy)\,\hat{k}$

$\vec{\nabla}\psi = z^2\,\hat{i} - z\hat{j} + (2xz - y)\,\hat{k}$

Now taking dot product of $\vec{\nabla}\varphi$ & $\vec{\nabla}\psi$.

$\vec{\nabla}\varphi\,.\,\vec{\nabla}\psi = [\,6xy\,\hat{i} + 3x^2\hat{j} + 0\,\hat{k}].[z^2\,\hat{i} - z\hat{j} + (2xz - y)\,\hat{k}] = \boldsymbol{6xy}z^2 - 3x^2z$

Now applying $\vec{\nabla}$ ***operator***

$\vec{\nabla}(\vec{\nabla}\varphi\,.\,\vec{\nabla}\psi) = \vec{\nabla}(6xyz^2 - 3x^2z) = \left[\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right](6xyz^2 - 3x^2z)$

$= \frac{\partial}{\partial x}(6xyz^2 - 3x^2z)\,\hat{i} + \frac{\partial}{\partial y}(6xyz^2 - 3x^2z)\,\hat{j} + \frac{\partial}{\partial z}(6xyz^2 - 3x^2z)\,\hat{k}$

$\vec{\nabla}(\vec{\nabla}\varphi\,.\,\vec{\nabla}\psi) = (6yz^2 - 6xz)\,\hat{i} + (6xz^2)\hat{j} + (12xyz - 3x^2)\,\hat{k}$

Q#18: Show that $\vec{\nabla}f(r)\times\vec{r}=0$.

Solution: ***Let*** $\vec{r}=x\,\hat{i}+y\hat{j}+z\,\hat{k}$ ***then*** $r^2=x^2+y^2+z^2$ ------***(i)***

Then $\vec{\nabla}f(r)=\left[\frac{\partial}{\partial x}\,\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\,\hat{k}\right]f(r)$ $\therefore\vec{\nabla}=\frac{\partial}{\partial x}\,\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\,\hat{k}$

$$=\frac{\partial}{\partial x}f(r)\,\hat{i}+\frac{\partial}{\partial y}f(r)\hat{j}+\frac{\partial}{\partial z}f(r)\,\hat{k}$$

$$=f'(r)\,\frac{\partial r}{\partial y}\hat{i}+f'(r)\frac{\partial r}{\partial y}\,\hat{j}+f'(r)\frac{\partial r}{\partial z}\,\hat{k}$$

$$=f'(r)\left[\frac{x}{r}\,\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\,\hat{k}\right]\quad\therefore\left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x\\ 2r\,\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}\ \text{Similarly}\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}\end{matrix}\right\}$$

$$=f'(r)\left[\frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{r}\right]$$

$$\vec{\nabla}f(r)=f'(r)\,\frac{\vec{r}}{r}$$

Now taking cross product with $\vec{r}$

$$\vec{\nabla}f(r)\times\vec{r}=f'(r)\,\frac{\vec{r}}{r}\times\vec{r}==\frac{f'(r)}{r}\,(\vec{r}\times\vec{r})=\frac{f'(r)}{r}\,(0)\qquad\therefore\ \vec{r}\times\vec{r}=0$$

$\vec{\nabla}f(r)\times\vec{r}=\boldsymbol{0}$ ***Hence proved.***

Q#19: Show that $(\vec{a}\,.\vec{\nabla})\vec{r}=\vec{a}$. Where $\vec{a}$ is a constant vector.

Solution: ***Let*** $\vec{a}=a_1\,\hat{i}+a_2\,\hat{j}+a_3\,\hat{k}$ ***&*** $\vec{r}=x\,\hat{i}+y\hat{j}+z\,\hat{k}$

We know that $\vec{\nabla}=\frac{\partial}{\partial x}\,\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\,\hat{k}$

Then $\vec{a}.\vec{\nabla}=(a_1\,\hat{i}+a_2\hat{j}+a_3\,\hat{k}).\left(\frac{\partial}{\partial x}\,\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\,\hat{k}\right)=a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}$

Now $(\vec{a}\,.\vec{\nabla})\vec{r}=\left[a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right](x\,\hat{i}+y\hat{j}+z\,\hat{k})$

$$=a_1\frac{\partial}{\partial x}(x\,\hat{i}+y\hat{j}+z\,\hat{k})+a_2\frac{\partial}{\partial y}(x\,\hat{i}+y\hat{j}+z\,\hat{k})+a_3\frac{\partial}{\partial z}(x\,\hat{i}+y\hat{j}+z\,\hat{k})$$

$$(\vec{a}\,.\vec{\nabla})\vec{r}=a_1\,\hat{i}+a_2\hat{j}+a_3\,\hat{k}$$

$$(\vec{a}\,.\vec{\nabla})\vec{r}=\vec{a}$$

Hence proved.

Divergence of a Vector:

Le $\vec{F}$ (x, y, z) ***is a vector .Then Divergence of a vector*** $\vec{F}$ ***is defined as;*** ***Div*** $\vec{F} = \vec{\nabla} . \vec{F}$.

Solenoid Vector:

A vector $\vec{F}$ ***is said to be Solenoid, if*** ***Div*** $\vec{F} = 0$.

Properties of the Divergence:

If $\vec{a}$ ***&*** $\vec{b}$ ***are two vector &*** φ ***is a scalar function then***

(i) ***Div*** $(\vec{a} + \vec{b}) = \vec{\nabla} . (\vec{a} + \vec{b}) = \vec{\nabla} . \vec{a} + \vec{\nabla} . \vec{b}$

(ii) ***Div*** $(\varphi \vec{a}) = \vec{\nabla} . (\varphi \vec{a}) = \varphi (\vec{\nabla} . \vec{a}) + (\vec{\nabla} \varphi) . \vec{a}$

Curl of a Vector:

Le $\vec{F}$ (x, y, z) ***is a vector .Then Curl of a vector*** $\vec{F}$ ***is defined as;*** ***Curl*** $\vec{F} = \vec{\nabla} \times \vec{F}$.

Irrotational Vector:

A vector $\vec{F}$ ***is said to be Irrotational, if*** ***Curl*** $\vec{F} = 0$.

Properties of the Curl:

If $\vec{a}$ ***&*** $\vec{b}$ ***are two vector &*** φ ***is a scalar function then***

(i) ***Curl*** $(\vec{a} + \vec{b}) = \vec{\nabla} \times (\vec{a} + \vec{b}) = \vec{\nabla} \times \vec{a} + \vec{\nabla} \times \vec{b}$

(ii) ***Curl*** $(\varphi \vec{a}) = \vec{\nabla} \times (\varphi \vec{a}) = \varphi (\vec{\nabla} \times \vec{a}) + (\vec{\nabla} \varphi) \times \vec{a}$

(iii) ***Curl*** $(\overrightarrow{\text{grad}\,\varphi})$ = ***Curl*** $(\vec{\nabla} \varphi) = \vec{\nabla} \times (\vec{\nabla} \varphi) = \mathbf{0}$

(iv) ***Curl*** $(\text{Div}\, \vec{a})$ = ***Curl*** $(\vec{\nabla} . \vec{a}) = \vec{\nabla} \times (\vec{\nabla} . \vec{a}) = \mathbf{0}$

Theorems: If $\vec{F}$ **&** $\vec{G}$ ***are two vector functions. Then prove that***

(i) $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = (\vec{\nabla} . \vec{F}) \vec{\nabla} - \nabla^2 \vec{F}$

Prove that $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = (\vec{\nabla} . \vec{F}) \vec{\nabla} - \nabla^2 \vec{F}$

Proof: ***We know that***

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c}) \vec{b} - (\vec{a} . \vec{b}) \vec{c}$$

Then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = (\vec{\nabla} . \vec{F}) \vec{\nabla} - (\vec{\nabla} . \vec{\nabla}) \vec{F}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = (\vec{\nabla} . \vec{F}) \vec{\nabla} - \nabla^2 \vec{F}$$

Here $\vec{\nabla} . \vec{\nabla} = \nabla^2$

Example#01: Find the divergence of $\vec{F}$ where $\vec{F} = \frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

Solution: Given $\vec{F} = \frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \frac{x\,\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \frac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \frac{z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

We know that ***Div*** $\vec{F} = \vec{\nabla}.\vec{F}$

$$= \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(\frac{x\,\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \frac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \frac{z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{\partial}{\partial x}\,\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\,\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\,\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}}(1) - x.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{\frac{3}{2}}(1) - y.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2y)}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{\frac{3}{2}}(1) - x.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2z)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[x^2+y^2+z^2-3x^2]}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[x^2+y^2+z^2-3y^2]}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[x^2+y^2+z^2-3z^2]}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}\left[x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2\right]$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}[0]$$

Hence Div $\vec{F}$ ***= 0***

Example#02: If $\vec{F} = 2yz\,\hat{i} + x^2y\,\hat{j} + xz^2\,\hat{k}$ ***;*** $\vec{G} = x^2\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}$ ***and*** $\varphi = 2x^2yz^3$

Find ***(i)*** $(\vec{F}.\vec{\nabla})\,\varphi$ ***(ii)*** $(\vec{F}\times\vec{\nabla})\varphi$ ***(ii)*** $\vec{F}\times\vec{\nabla}\varphi$ ***(iv)*** $(\vec{\nabla}\times\vec{F})\times\vec{G}$

Solution: Given $\vec{F} = 2yz\,\hat{i} + x^2y\,\hat{j} + xz^2\,\hat{k}$ ***;*** $\vec{G} = x^2\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}$ ***and*** $\varphi = 2x^2yz^3$

(i) $(\vec{F}.\vec{\nabla})\,\varphi$

$$(\vec{F}.\vec{\nabla})\,\varphi = \left[\left(2yz\hat{i} + x^2y\,\hat{j} + xz^2\,\hat{k}\right).\left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right)\right](2x^2yz^3)$$

$$= \left(2yz\frac{\partial}{\partial x} + x^2y\,\frac{\partial}{\partial y} + xz^2\,\frac{\partial}{\partial z}\right)(2x^2yz^3)$$

$$= 2yz\frac{\partial}{\partial x}\,(2x^2yz^3) + x^2y\,\frac{\partial}{\partial y}\,(2x^2yz^3) + xz^2\,\frac{\partial}{\partial z}\,(2x^2yz^3)$$

$$= 2yz(4xyz^3) + x^2y\,(2x^2z^3) + xz^2\,(6x^2yz^2)$$

$$(\vec{F}.\vec{\nabla})\,\varphi = 8xy^2z^4 + 2x^4yz^3 + 6x^3yz^4)$$

(ii) $(\vec{F}\times\vec{\nabla})\varphi$

$$(\vec{F}\times\vec{\nabla})\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2yz & x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\varphi = \left[\hat{\imath}\begin{vmatrix} x^2y & xz^2 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2yz & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z}\end{vmatrix} + \hat{k}\begin{vmatrix} 2yz & x^2y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{vmatrix}\right]\varphi$$

$$= \hat{\imath}\left[x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}\right]\varphi - \hat{\jmath}\left[2yz\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial x}\right]\varphi + \hat{k}\left[2yz\frac{\partial}{\partial y} - x^2y\frac{\partial}{\partial x}\right]\varphi$$

$$= \hat{\imath}\left[x^2y\frac{\partial}{\partial z}\varphi - xz^2\frac{\partial}{\partial y}\varphi\right] - \hat{\jmath}\left[2yz\frac{\partial}{\partial z}\varphi - xz^2\frac{\partial}{\partial x}\varphi\right] + \hat{k}\left[2yz\frac{\partial}{\partial y}\varphi - x^2y\frac{\partial}{\partial x}\varphi\right]$$

$$= \hat{\imath}\left[x^2y\frac{\partial}{\partial z}(2x^2yz^3) - xz^2\frac{\partial}{\partial y}(2x^2yz^3)\right] - \hat{\jmath}\left[2yz\frac{\partial}{\partial z}(2x^2yz^3) - xz^2\frac{\partial}{\partial x}(2x^2yz^3)\right]$$

$$+ \hat{k}\left[2yz\frac{\partial}{\partial y}(2x^2yz^3) - x^2y\frac{\partial}{\partial x}(2x^2yz^3)\right]$$

$$= \hat{\imath}[x^2y(6x^2yz^2) - xz^2(2x^2z^3)] - \hat{\jmath}\,[2yz(6x^2yz^2) - xz^2(4xyz^3)] + \hat{k}[2yz(2x^2z^3) - x^2y(4xyz^3)]$$

$$(\vec{F}\times\vec{\nabla})\varphi = \hat{\imath}\,[6x^4y^2z^4 - 2x^3z^5] - \hat{\jmath}\,[12x^2y^2z^3 - 4x^2yz^5] + \hat{k}[2x^2yz^4 - 4x^3y^2z^3]$$

(iii) $\vec{F}\times\vec{\nabla}\varphi$

$$\vec{F}\times\vec{\nabla}\varphi = [2yz\,\hat{\imath} + x^2y\,\hat{\jmath} + xz^2\,\hat{k}] \times \left[\frac{\partial}{\partial x}\varphi\,\hat{\imath} + \frac{\partial}{\partial y}\varphi\hat{\jmath} + \frac{\partial}{\partial z}\varphi\,\hat{k}\right]$$

$$= [2yz\,\hat{\imath} + x^2y\,\hat{\jmath} + xz^2\,\hat{k}] \times \left[\frac{\partial}{\partial x}(2x^2yz^3)\,\hat{\imath} + \frac{\partial}{\partial y}(2x^2yz^3)\hat{\jmath} + \frac{\partial}{\partial z}(2x^2yz^3)\,\hat{k}\right]$$

$$= [2yz\,\hat{\imath} + x^2y\,\hat{\jmath} + xz^2\,\hat{k}] \times [4xyz^3 + 2x^2z^3\hat{\jmath} + 6x^2yz^2\,\hat{k}] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2yz & x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2\end{vmatrix}$$

$$= \left[\hat{\imath}\begin{vmatrix} x^2y & xz^2 \\ 2x^2z^3 & 6x^2yz^2\end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2yz & xz^2 \\ 4xyz^3 & 6x^2yz^2\end{vmatrix} + \hat{k}\begin{vmatrix} 2yz & x^2y \\ 4xyz^3 & 2x^2z^3\end{vmatrix}\right]$$

$$= \hat{\imath}[x^2y(6x^2yz^2) - xz^2(2x^2z^3)] - \hat{\jmath}\,[2yz(6x^2yz^2) - xz^2(4xyz^3)] + \hat{k}[2yz(2x^2z^3) - x^2y(4xyz^3)]$$

$$= \hat{\imath}\,[6x^4y^2z^4 - 2x^3z^5] - \hat{\jmath}\,[12x^2y^2z^3 - 4x^2yz^5] + \hat{k}[2x^2yz^4 - 4x^3y^2z^3]$$

(iv) $(\vec{\nabla}\times\vec{F})\times\vec{G}$

$$(\vec{\nabla}\times\vec{F})\times\vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & x^2y & xz^2\end{vmatrix} \times\vec{G} = \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz^2\end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2yz & xz^2\end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2yz & x^2y\end{vmatrix}\right]\times\vec{G}$$

$$= \left\{\hat{\imath}\left[\frac{\partial}{\partial y}xz^2 - \frac{\partial}{\partial z}x^2y\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}xz^2 - \frac{\partial}{\partial z}2yz\right] + \hat{k}\left[\frac{\partial}{\partial x}x^2y - \frac{\partial}{\partial y}2yz\right]\right\}\times\vec{G}$$

$$= \{\hat{\imath}\,[0-0] - \hat{\jmath}\,[z^2 - 2y] + \hat{k}[2xy - 2z]\}\times\vec{G}$$

$$= \{0\,\hat{i} + [\,2y - z^2]\hat{j} + [2xy - 2z]\,\hat{k}\} \times \{x^2\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2y - z^2 & 2xy - 2z \\ x^2 & yz & xy \end{vmatrix} = \left[\hat{i}\begin{vmatrix} 2y - z^2 & 2xy - 2z \\ yz & xy \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & 2xy - 2z \\ x^2 & xy \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 2y - z^2 \\ x^2 & yz \end{vmatrix}\right]$$

$$= \left\{\begin{matrix} \hat{i}\,[(2y - z^2)xy - (2xy - 2z)yz] - \hat{j}\,[0(xy) - (2xy - 2z)x^2] \\ + \hat{k}[0(yz) - (2y - z^2)x^2] \end{matrix}\right\}$$

$$= \{\hat{i}\,[2xy^2 - xyz^2 - 2xy^2z + 2yz^2] - \hat{j}\,[0 - 2x^3y - 2x^2z] + \hat{k}[0 - 2x^2y + x^2z^2]\}$$

$$= \hat{i}\,[2xy^2 - xyz^2 - 2xy^2z + 2yz^2] - \hat{j}\,[2x^3y - 2x^2z] + \hat{k}[x^2z^2 - 2x^2y]$$

Example#03: If $\varphi = 2x^3y^2z^4$ ***, Find Div***$\left(\overrightarrow{\text{grad}\,\varphi}\right)$.

Solution: We know that

$$\overrightarrow{\text{grad}\,\varphi} = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\,\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\,\hat{k} = \frac{\partial}{\partial x}(2x^3y^2z^4)\,\hat{i} + \frac{\partial}{\partial y}(2x^3y^2z^4)\hat{j} + \frac{\partial}{\partial z}(2x^3y^2z^4)\,\hat{k}$$

$$\overrightarrow{\text{grad}\,\varphi} = 6x^2y^2z^4\,\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\,\hat{k}$$

Now

$$\textit{Div}\left(\overrightarrow{\text{grad}\,\varphi}\right) = \vec{\nabla}.\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(6x^2y^2z^4\,\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(6x^2yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$\textit{Div}\left(\overrightarrow{\text{grad}\,\varphi}\right) = 12xy^2z^4 + 12x^2yz^4 + 24x^3y^2z^2$$

Example#04:Show that Div $r^7\,\vec{r}$ = ***10*** r^7 .

Solution: Let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***Then*** ***r*** $= |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ ***or*** $r^2 = x^2 + y^2 + z^2$ ------------***(i)***

Now Div $r^7\,\vec{r} = \vec{\nabla}.(r^7\,\vec{r}) = \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(r^7\,[x\,\hat{i} + y\hat{j} + z\,\hat{k}]\right)$

$$= \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(r^7\,x\,\hat{i} + r^7y\hat{j} + r^7z\,\hat{k}\right) = \frac{\partial}{\partial x}(r^7x) + \frac{\partial}{\partial y}(r^7y) + \frac{\partial}{\partial z}(r^7z)$$

$$= \left[r^7(1) + x.7r^6\,\frac{\partial r}{\partial x}\right] + \left[r^7(1) + y.7r^6\,\frac{\partial r}{\partial y}\right] + \left[r^7(1) + z.7r^6\,\frac{\partial r}{\partial z}\right]$$

$$= \left[r^7 + x.7r^6\,\frac{\partial r}{\partial x} + r^7 + y.7r^6\,\frac{\partial r}{\partial y} + r^7 + z.7r^6\,\frac{\partial r}{\partial z}\right]$$

$$= \left[3\,r^7 + 7r^6\left(x.\frac{\partial r}{\partial x} + y.\frac{\partial r}{\partial y} + z.\frac{\partial r}{\partial z}\right)\right]$$

$$= \left[3\,r^7 + 7r^6\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)\right] \qquad \therefore \left\{\begin{matrix} \text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r} \end{matrix}\right\}$$

$$= \left[3\,r^7 + 7r^6\left(\frac{x^2+y^2+z^2}{r}\right)\right]$$

$$= \left[3\,r^7 + 7r^6\left(\frac{r^2}{r}\right)\right]$$

$$= [3\,r^7 + 7r^6.r\,]$$

$$= [3\,r^7 + 7r^7\,]$$

$$Div\;\; r^7\,\vec{r} = 10\,r^7$$

Hence proved.

Example#05: Show that Div $\frac{\vec{r}}{r^3} = 0$ ***.***

Solution: Let $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ ***Then*** $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ ***or*** $r^2 = x^2 + y^2 + z^2$ ***----------(i)***

Now

$$\textbf{Div}\,\frac{\vec{r}}{r^3} = \textbf{Div}\;\; r^{-3}\,\vec{r} = \vec{\nabla}.(r^{-3}\,\vec{r}\,) = \left(\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(r^{-3}\left[x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}\right]\right)$$

$$= \left(\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(r^{-3}\,x\,\hat{\imath} + r^{-3}y\hat{\jmath} + r^{-3}z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(r^{-3}x) + \frac{\partial}{\partial y}(r^{-3}y) + \frac{\partial}{\partial z}(r^{-3}z)$$

$$= \left[r^{-3}(1) + x.(-3)r^{-4}\,\frac{\partial r}{\partial x}\right] + \left[r^{7}(1) + y.(-3)r^{-4}\,\frac{\partial r}{\partial y}\right] + \left[r^{7}(1) + z.(-3)r^{-4}\,\frac{\partial r}{\partial z}\right]$$

$$= \left[r^{-3} - x.3r^{-4}\,\frac{\partial r}{\partial x} + r^{-3} + y.3r^{-3}\,\frac{\partial r}{\partial y} + r^{-3} + z.3r^{-3}\,\frac{\partial r}{\partial z}\right]$$

$$= \left[3\,r^{-3} - 3r^{-4}\left(x.\frac{\partial r}{\partial x} + y.\frac{\partial r}{\partial y} + z.\frac{\partial r}{\partial z}\right)\right]$$

$$= \left[3\,r^{-3} - 3r^{-4}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)\right]$$

$$= \left[3\,r^{-3} - 3r^{-4}\left(\frac{x^2+y^2+z^2}{r}\right)\right]$$

$$= \left[3\,r^{-3} - 3r^{-4}\left(\frac{r^2}{r}\right)\right]$$

$$= [3\,r^{-3} - 3r^{-4}.r]$$

$$= [3\,r^{-3} - 3r^{3}\,]$$

$\textbf{Div}\,\frac{\vec{r}}{r^3} = 0$ ***Hence proved.***

Example#06: If $\vec{a} = xy\,\hat{i} - 2xz\hat{j} + 2y\,z\,\hat{k}$. ***Show that Curl***$(\text{curl}\,\vec{a}) = 3\,\hat{j}$.

Solution:

$$\text{curl}\,\vec{a} = \vec{\nabla} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2xz & 2yz \end{vmatrix} = \left[\hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2xz & 2yz \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xy & 2yz \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & -2xz \end{vmatrix} \right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}2yz - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}\,2yz - \frac{\partial}{\partial z}xy\right] + \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}xy\right]$$

$$= \hat{i}\,[2z - (-2x)] - \hat{j}\,[0 - 0] + \hat{k}[-2z - x]$$

$$\text{curl}\,\vec{a} = [2z + 2x]\,\hat{i} + 0\hat{j} + [-2z - x]\,\hat{k}$$

Now Curl$(\text{curl}\,\vec{a}) = \vec{\nabla} \times \text{curl}\,\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 2x & 0 & -2z - x \end{vmatrix}$

$$= \left[\hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -2z - x \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2z + 2x & -2z - x \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2z + 2x & 0 \end{vmatrix} \right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-2z - x) - \frac{\partial}{\partial z}0\right] - \hat{j}\left[\frac{\partial}{\partial x}\,(-2z - x) - \frac{\partial}{\partial z}(2z + 2x)\right] + \hat{k}\left[\frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}(2z + 2x)\right]$$

$$= \hat{i}[0] - \hat{j}\,[-2 - 1] + \hat{k}[0] = \boldsymbol{0}\,\hat{i} - \hat{j}\,[-3] + 0\,\hat{k}$$

Curl$(\text{curl}\,\vec{a}) = \boldsymbol{3}\,\hat{j}$ ***Hence proved.***

Example#08: If $\vec{v} = \vec{a} \times \vec{r}$ ***then prove that*** $\vec{a} = \frac{1}{2}\,\text{curl}\,\vec{v}$, where $\vec{a}$ is a constant vector.

Solution: Let $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ **&** $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Then $\vec{v} = \vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \left[\hat{i} \begin{vmatrix} a_2 & a_3 \\ y & z \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ x & z \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ x & y \end{vmatrix} \right]$

$$= \hat{i}\,[a_2 z - a_3 y] - \hat{j}\,[a_1 z - a_3 x] + \hat{k}[a_1 y - a_2 z]$$

$$\vec{v} = [a_2 z - a_3 y]\hat{i} + [a_3 x - a_1 z]\hat{j} + [a_1 y - a_2 z]\,\hat{k}$$

Now $\text{curl}\,\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 z \end{vmatrix}$

$$= \left[\hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3 x - a_1 z & a_1 y - a_2 z \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_1 y - a_2 z \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ a_2 z - a_3 y & a_3 x - a_1 z \end{vmatrix} \right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(a_1y - a_2z) - \frac{\partial}{\partial z}(a_3x - a_1z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(a_1y - a_2z) - \frac{\partial}{\partial z}(a_2z - a_3y)\right] +$$

$$\hat{k}\left[\frac{\partial}{\partial x}(a_3x - a_1z) - \frac{\partial}{\partial y}(a_2z - a_3y)\right]$$

$$= \hat{i}[a_1 + a_1] - \hat{j}[a_2 + a_2] + \hat{k}[a_3 + a_3]$$

$$= 2a_1\,\hat{i} + 2a_2\,\hat{j} + 2a_3\,\hat{k}$$

$$\text{curl}\,\vec{v} = 2(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k})$$

$$\text{curl}\,\vec{v} = 2\,\vec{a}$$

$$\Rightarrow \vec{a} = \frac{1}{2}\,\text{curl}\,\vec{v}$$

Exercise# 4.2

Q#01: Fin the divergence & curl of the vector functions.

(i) $\vec{F} = (x^2 + yz)\,\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\,\hat{k}$ ***(ii)*** $\vec{F} = (x-y)\,\hat{\imath} + (y-z)\hat{\jmath} + (z-x)\,\hat{k}$

(i) $\vec{F} = (x^2 + yz)\,\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\,\hat{k}$

Solution: We know that

$$\textbf{\textit{Div}}\,\vec{F} = \vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left[(x^2 + yz)\,\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + zx) + \frac{\partial}{\partial z}(z^2 + xy)$$

$$\textbf{\textit{Div}}\,\vec{F} = 2x + 2y + 2z$$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix} = \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + zx & z^2 + xy \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 + yz & z^2 + xy \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 + yz & y^2 + zx \end{vmatrix}\right]$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(z^2 + xy) - \frac{\partial}{\partial z}(y^2 + zx)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(z^2 + xy) - \frac{\partial}{\partial z}(x^2 + yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2 + zx) - \frac{\partial}{\partial y}(x^2 + yz)\right]$$

$$= \hat{\imath}[x - x] - \hat{\jmath}[y - y] + \hat{k}[z - z]$$

$$\text{curl}\,\vec{F} = 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$$

(ii) $\vec{\mathbf{F}} = (\mathbf{x} - \mathbf{y})\,\hat{\mathbf{i}} + (\mathbf{y} - \mathbf{z})\hat{\mathbf{j}} + (\mathbf{z} - \mathbf{x})\,\hat{\mathbf{k}}$

Solution: We know that

$$\textbf{\textit{Div}}\,\vec{F} = \vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left[(x-y)\,\hat{\imath} + (y-z)\hat{\jmath} + (z-x)\,\hat{k}\right] = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-z) + \frac{\partial}{\partial z}(z-x)$$

$$\textbf{\textit{Div}}\,\vec{F} = 1 + 1 + 1 = 3$$

$$\&\ \text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & y - z & z - x \end{vmatrix} = \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x - y & z - x \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x - y & y - z \end{vmatrix}\right]$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(z-x) - \frac{\partial}{\partial z}(y-z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(z-x) - \frac{\partial}{\partial z}(x-y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y-z) - \frac{\partial}{\partial y}(x-y)\right]$$

$$= \hat{\imath}[0 - (-1)] - \hat{\jmath}[(-1) - 0] + \hat{k}[0 - (-1)]$$

$$\text{curl}\,\vec{F} = 1\,\hat{\imath} + 1\,\hat{\jmath} + 1\,\hat{k}$$

Q#02: Find Div $\vec{F}$ ***&*** curl $\vec{F}$ ***where***

(i) $\vec{F}$ = ***grad(***$x^3 + y^3 + z^3 - 3xyz$) ***(ii)*** $\vec{F} = (x-y)\,\hat{\imath} + (y-z)\hat{\jmath} + (z-x)\,\hat{k}$

(i) $\vec{\mathbf{F}}$ = ***grad(***$\mathbf{x^3 + y^3 + z^3 - 3xyz}$)

Solution: Given $\vec{F} = \vec{\nabla}(x^3 + y^3 + z^3 - 3xyz) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(x^3 + y^3 + z^3 - 3xyz)$

$$\vec{F} = \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)\ \hat{\imath} + \frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)\,\hat{\jmath} + \frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)\,\hat{k}$$

$$\vec{F} = (3x^2 - 3yz)\ \hat{\imath} + (3y^2 - 3xz)\,\hat{\jmath} + (3z^2 - 3xy)\,\hat{k}$$

We know that

$$\textit{Div}\,\vec{F} = \vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left[(3x^2 - 3yz)\,\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$\textit{Div}\,\vec{F} = 6x + 6y + 6z$$

$$\&\ \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3z^2 - 3xy \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3x^2 - 3yz & 3y^2 - 3xz \end{vmatrix}\right]$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$

$$= \hat{\imath}[-3x - (-3x)] - \hat{\jmath}\,[-3y - (-3y)] + \hat{k}[-3z - (-3z)]$$

$$= \hat{\imath}[-3x + 3x] - \hat{\jmath}\,[-3y + 3y] + \hat{k}[-3z + 3z]$$

$$\Rightarrow \quad \text{curl}\,\vec{F} = 0\ \hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$$

(ii) $\overrightarrow{\mathbf{F}} = \mathbf{xyz\,\hat{\imath} + x^2y^2z\,\hat{\jmath} + yz^3\,\hat{k}}$

Solution: We know that

$$\textit{Div}\ \vec{F} = \vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left[\, xyz\,\hat{\imath} + x^2y^2z\,\hat{\jmath} + yz^3\,\hat{k}\right] = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(x^2y^2z) + \frac{\partial}{\partial z}(yz^3)$$

$$\textit{Div}\ \vec{F} = yz + 2x^2yz + 3yz^2$$

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x^2y^2z & yz^3 \end{vmatrix} = \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2z & yz^3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xyz & yz^3 \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xyz & x^2y^2z \end{vmatrix}\right]$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(yz^3) - \frac{\partial}{\partial z}(x^2y^2z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(yz^3) - \frac{\partial}{\partial z}(xyz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \hat{\imath}[z^3 - x^2y^2] - \hat{\jmath}\,[0 - xy] + \hat{k}[2xy^2z - xz]$$

$$\text{curl}\,\vec{F} = (z^3 - x^2y^2)\ \hat{\imath} + xy\,\hat{\jmath} + (2xy^2z - xz)\,\hat{k}$$

Q#03: Find m , so that the vector $(mxy - z^3)\ \hat{\imath} + (m-2)x^2\,\hat{\jmath} + (1-m)xz^2\,\hat{k}$ ***has its curl equal to zero .***

Solution: Let $\vec{F} = (mxy - z^3)\ \hat{\imath} + (m-2)x^2\,\hat{\jmath} + (1-m)xz^2\,\hat{k}$

Given condition : $\text{curl}\,\vec{F} = \boldsymbol{0} \Rightarrow \vec{\nabla}\times\vec{F} = \boldsymbol{0} \quad \Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ mxy - z^3 & (m-2)x^2 & (1-m)xz^2 \end{vmatrix} = \boldsymbol{0}$

$$\left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (m-2)x^2 & yz^3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ mxy - z^3 & yz^3 \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ mxy - z^3 & (m-2)x^2 \end{vmatrix}\right] = \boldsymbol{0}$$

$$\hat{\imath}\left[\frac{\partial}{\partial y}\big((1-m)xz^2\big) - \frac{\partial}{\partial z}\big((m-2)x^2\big)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}\big((1-m)xz^2\big) - \frac{\partial}{\partial z}(mxy - z^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}\big((m-2)x^2\big) - \frac{\partial}{\partial y}(mxy - z^3)\right] = \boldsymbol{0}$$

$$\hat{\imath}[0 - 0] - \hat{\jmath}\,[(1-m)z^2 - (-3z^2)] + \hat{k}[2(m-2)x - mx] = 0$$

$$0\ \hat{\imath} + [(1-m)z^2 + 3z^2]\,\hat{\jmath} + [2(m-2)x - mx]\,\hat{k} = \boldsymbol{0}$$

Putting coefficients of $\hat{k}$ ***is equal to zero.***

$$2(m-2)x - mx = 0 \Rightarrow 2mx - 4x - mx = 0 \Rightarrow \quad mx - 4x = 0 \Rightarrow mx = 4x$$

By using cancelation Property $\qquad m = 4$

Q#04: (i) Show that Div $r^{n-3}\,\vec{r} = n\, r^{n-3}$ ***(ii) show that*** $\nabla^2 r^{n-1} = n(n-1)\; r^{n-3}$

(i) ***Show that Div*** $r^{n-3}\,\vec{r} = n\, r^{n-3}$

Solution: Let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***Then*** $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ ***or*** $r^2 = x^2+y^2+z^2$ ------------***(i)***

Now Div $r^{n-3}\,\vec{r} = \vec{\nabla}.(r^{n-3}\,\vec{r}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(r^{n-3}\left[x\,\hat{i} + y\hat{j} + z\,\hat{k}\right]\right)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(r^{n-3}\,x\,\hat{i} + r^{n-3}y\hat{j} + r^{n-3}z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(r^{n-3}x) + \frac{\partial}{\partial y}(r^{n-3}y) + \frac{\partial}{\partial z}(r^{n-3}z)$$

$$= \left[r^{n-3}(1) + x.(n-3)r^{n-4}\frac{\partial r}{\partial x}\right] + \left[r^{n-3}(1) + y.(n-3)r^{n-4}\frac{\partial r}{\partial y}\right] + \left[r^{n-3}(1) + z.(n-3)r^{n-4}\frac{\partial r}{\partial z}\right]$$

$$= \left[r^{n-3} + x(n-3)r^{n-4}\frac{\partial r}{\partial x} + r^{n-3} + y(n-3)r^{n-4}\frac{\partial r}{\partial y} + r^{n-3} + z(n-3)r^{n-4}\frac{\partial r}{\partial z}\right]$$

$$= \left[3\,r^{n-3} + (n-3)r^{n-4}\left(x.\frac{\partial r}{\partial x} + y.\frac{\partial r}{\partial y} + z.\frac{\partial r}{\partial z}\right)\right] \therefore \left\{\begin{matrix}\text{From(i) Differentiate w.r.t } x \\ 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}\;\; \frac{\partial r}{\partial z} = \frac{z}{r}\end{matrix}\right\}$$

$$= \left[3\,r^{n-3} + (n-3)r^{n-4}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)\right]$$

$$= \left[3\,r^{n-3} + (n-3)r^{n-4}\left(\frac{x^2+y^2+z^2}{r}\right)\right]$$

$$= \left[3\,r^{n-3} + (n-3)r^{n-4}\left(\frac{r^2}{r}\right)\right]$$

$$= \left[3\,r^{n-3} + (n-3)r^{n-4}.r\right]$$

$$= \left[3\,r^{n-3} + 7r^{n-3}\right]$$

$$= \left[(3 + n - 3)r^{n-3}\right]$$

Div $r^{n-3}\,\vec{r} = n\,r^{n-3}$ ***Hence proved.***

(ii) $\boldsymbol{\nabla^2 r^{n-1} = n(n-1)\; r^{n-3}}$

Solution: We know that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***then***

$$\nabla^2 r^{n-1} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] r^{n-1} = \frac{\partial^2}{\partial x^2}r^{n-1} + \frac{\partial^2}{\partial y^2}r^{n-1} + \frac{\partial^2}{\partial z^2}r^{n-1}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}r^{n-1}\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}r^{n-1}\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}r^{n-1}\right]$$

$$= \frac{\partial}{\partial x}\left[(n-1)r^{n-2}\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial y}\left[(n-1)r^{n-2}\frac{\partial r}{\partial y}\right] + \frac{\partial}{\partial z}\left[(n-1)r^{n-2}\frac{\partial r}{\partial z}\right]$$

$$= (n-1)\left\{\frac{\partial}{\partial x}\left(r^{n-2}\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial y}\left(r^{n-2}\frac{\partial r}{\partial y}\right)+\frac{\partial}{\partial z}\left(r^{n-2}\frac{\partial r}{\partial z}\right)\right\}$$

$$= (n-1)\left[\left\{(n-2)r^{n-3}\frac{\partial r}{\partial x}.\frac{\partial r}{\partial x}+r^{n-2}\frac{\partial^2 r}{\partial x^2}\right\}+\left\{(n-2)r^{n-3}\frac{\partial r}{\partial y}.\frac{\partial r}{\partial y}+r^{n-1}\frac{\partial^2 r}{\partial y^2}\right\}+\left\{(n-2)r^{n-3}\frac{\partial r}{\partial z}.\frac{\partial r}{\partial z}+r^{n-2}\frac{\partial^2 r}{\partial z^2}\right\}\right]$$

$$= (n-1)\left[(n-2)r^{n-3}\left(\frac{\partial r}{\partial x}\right)^2+r^{n-2}\frac{\partial^2 r}{\partial x^2}+(n-2)r^{n-3}\left(\frac{\partial r}{\partial y}\right)^2+r^{n-2}\frac{\partial^2 r}{\partial y^2}+(n-2)r^{n-3}\left(\frac{\partial r}{\partial z}\right)^2+r^{n-2}\frac{\partial^2 r}{\partial z^2}\right]$$

$$= (n-1)\left[(n-2)r^{n-3}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2+\left(\frac{\partial r}{\partial z}\right)^2\right\}+r^{n-2}\left\{\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}\left\{\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2+\left(\frac{z}{r}\right)^2\right\}+r^{n-2}\left\{\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right\}\right] \text{---------------}(a)$$

Let $\vec{r} = x\,\hat{i}+y\hat{j}+z\,\hat{k}$ ***then*** $r^2 = x^2+y^2+z^2$ ------***(i)***

∴ From(i) ***Differentiate w. r. t x*** $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ ***Similarly*** $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Again differentiate w. r. t x $\frac{\partial^2 r}{\partial x^2} = \frac{r(1)-x\frac{\partial r}{\partial x}}{r^2} = \frac{r-x\left(\frac{x}{r}\right)}{r^2} = \frac{\frac{r^2-x^2}{r}}{r^2} = \frac{x^2+y^2+z^2-x^2}{r^3} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{r^3}$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{r^3}$ & $\frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{r^3}$

Putting values in Equation (a)

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}\left\{\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}\right\}+r^{n-2}\left\{\frac{y^2+z^2}{r^3}+\frac{x^2+z^2}{r^3}+\frac{x^2+y^2}{r^3}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}\left\{\frac{x^2+y^2+z^2}{r^2}\right\}+r^{n-2}\left\{\frac{y^2+z^2+x^2+z^2+x^2+y^2}{r^3}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}\left\{\frac{r^2}{r^2}\right\}+r^{n-2}\left\{\frac{2(x^2+y^2+z^2)}{r^3}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}(1)+r^{n-2}\left\{\frac{2r^2}{r^3}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)\left[(n-2)r^{n-3}+r^{n-2}\left\{\frac{2}{r}\right\}\right]$$

$$\nabla^2 r^{n-1} = (n-1)[(n-2)r^{n-3}+2r^{n-3}] = (n-1)[(n-2+2)r^{n-3}]$$

$$\nabla^2 r^{n-1} = n(n-1)\,r^{n-3}$$

Hence proved.

Q#05: If $\vec{a} = 2x^2\ \hat{\imath} - 3yz\,\hat{\jmath} + xz^2\,\hat{k}$ ***&*** $\varphi = 2z - x^3y$ ***. Find (i)*** $\vec{a}\,.\,\vec{\nabla}\varphi$ ***(ii)*** $\vec{a}\times\vec{\nabla}\varphi$ ***at point*** $(1,-1,1)$ ***.***

Solution: Given $\vec{a} = 2x^2\ \hat{\imath} - 3yz\,\hat{\jmath} + xz^2\,\hat{k}$ ***&*** $\varphi = 2z - x^3y$ ***then***

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \frac{\partial}{\partial x}(2z - x^3y)\hat{\imath} + \frac{\partial}{\partial y}(2z - x^3y)\hat{\jmath} + \frac{\partial}{\partial z}(2z - x^3y)\hat{k}$$

$$\vec{\nabla}\varphi = -3x^2y\,\hat{\imath} - x^3\hat{\jmath} + 2\hat{k}$$

(i) $$\vec{a}\,.\,\vec{\nabla}\varphi = \left(2x^2\ \hat{\imath} - 3yz\,\hat{\jmath} + xz^2\,\hat{k}\right).\left(-3x^2y\,\hat{\imath} - x^3\hat{\jmath} + 2\hat{k}\right)$$

$$= (2x^2)(-3x^2y) + (-3yz)(-x^3) + (xz^2)(2)$$

$$\vec{a}\,.\,\vec{\nabla}\varphi = -6x^4y + 3x^3yz + 2xz^2$$

At $(1,-1,1)$: $\vec{a}\,.\,\vec{\nabla}\varphi = -6(1)^4(-1) + 3(1)^3(-1)(1) + 2(1)(1)^2 = 6 - 3 + 2 = 5$

(ii) $$\vec{a}\times\vec{\nabla}\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix} = \left[\hat{\imath}\begin{vmatrix} -3yz & xz^2 \\ -x^3 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2x^2 & xz^2 \\ -3x^2y & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2x^2 & -3yz \\ -3x^2y & -x^3 \end{vmatrix}\right]$$

$$= \hat{\imath}\,[-3yz(2) - xz^2(-x^3)] - \hat{\jmath}\,[2x^2\,(2) - xz^2(-3x^2y)] + \hat{k}[2x^2(-x^3) - (-3yz)(-3x^2y)]$$

$$\vec{a}\times\vec{\nabla}\varphi = \hat{\imath}[-6yz + x^4z^2] - \hat{\jmath}\,[4x^2 + 3x^3yz^2] + \hat{k}[-2x^5 - 9x^2y^2z]$$

At $(1,-1,1)$: $\vec{a}\times\vec{\nabla}\varphi = \hat{\imath}[-6(-1)(1) + (1)^4(1)^2] - \hat{\jmath}[4(1)^2 + 3(1)^3(-1)(1)^2] + \hat{k}[-2(1)^5 - 9((1)^2(-1)^2(1)]$

$\vec{a}\times\vec{\nabla}\varphi = \hat{\imath}[6 + 1] - \hat{\jmath}[4 - 3] + \hat{k}[-2 - 9]$ $\vec{a}\times\vec{\nabla}\varphi = 7\hat{\imath} - \hat{\jmath} - 11\hat{k}$

Q#06: If $\vec{a} = (x + y + z)\ \hat{\imath} + \hat{\jmath} - (x + y)\hat{k}$ ***. Show that*** $\vec{a}\,.\,$***curl*** $\vec{a} = 0.$

Solution: Given $\vec{a} = (x + y + z)\ \hat{\imath} + \hat{\jmath} - (x + y)\hat{k}$

Now Curl $$\vec{a} = \vec{\nabla}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & 1 & -(x+y) \end{vmatrix} = \left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & -(x+y) \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x+y+z & -(x+y) \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x+y+z & 1 \end{vmatrix}\right]$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(-(x+y)) - \frac{\partial}{\partial z}(1)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(-(x+y)) - \frac{\partial}{\partial z}(x+y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+z)\right]$$

$$= \hat{\imath}[-1 - 0] - \hat{\jmath}\,[-1 - 1] + \hat{k}[0 - 1]$$

$$\text{curl}\,\vec{a} = -\ \hat{\imath} + 2\,\hat{\jmath} - \hat{k}$$

Now $\vec{a}\,.\,$***curl*** $\vec{a} = \left[(x+y+z)\ \hat{\imath} + \hat{\jmath} - (x+y)\hat{k}\right].\left[-\ \hat{\imath} + 2\,\hat{\jmath} - \hat{k}\right] = (x+y+z)\ (-1) + (1)(2) - (x+y)(-1)$

$$= -x - y - z + 2 + x + y$$

$\vec{a}\,.\,$***curl*** $\vec{a} =$ ***0***

Q#07: If $\vec{u} = \frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{\sqrt{x^2+y^2+z^2}}$ ***. Show that (i)*** $\vec{\nabla}.\vec{u} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ ***(ii)*** $\vec{\nabla}\times\vec{u} = 0$

(i) $\boldsymbol{\vec{\nabla}.\vec{u} = \frac{2}{\sqrt{x^2+y^2+z^2}}}$

Solution: Given $\vec{u} = \frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\,\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$\vec{\nabla}.\vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{x\,\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\,\hat{k}}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial z}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\sqrt{x^2+y^2+z^2}\,(1) - x.\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x)}{\left(\sqrt{x^2+y^2+z^2}\right)^2} + \frac{\sqrt{x^2+y^2+z^2}\,(1) - y.\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2y)}{\left(\sqrt{x^2+y^2+z^2}\right)^2} + \frac{\sqrt{x^2+y^2+z^2}\,(1) - x.\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2z)}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - \frac{y^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{\frac{[x^2+y^2+z^2-x^2]}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} + \frac{\frac{[x^2+y^2+z^2-y^2]}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} + \frac{\frac{[x^2+y^2+z^2-z^2]}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} + \frac{x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$= \frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$\vec{\nabla}.\vec{u} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ ***Hence proved.***

(ii) $\boldsymbol{\vec{\nabla}\times\vec{u} = 0}$

Solution: Given $\vec{u} = \frac{x\,\hat{i}+y\hat{j}+z\,\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\,\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$\vec{\nabla}\times\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \quad \therefore \left\{\frac{1}{\sqrt{x^2+y^2+z^2}} \text{ common from } R_3\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \left[\hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \left\{\hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \{\hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0]\} = \frac{1}{\sqrt{x^2+y^2+z^2}} \{0\hat{i} + 0\hat{j} + 0\,\hat{k}\} = \frac{1}{\sqrt{x^2+y^2+z^2}} \{0\}$$

$\vec{\nabla}\times\vec{u} = 0$ ***Hence proved.***

Q#08:If $\varphi = x^2 + y^2 + z^2$ and $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***Then show that*** $\boldsymbol{Div}(\varphi\vec{r}) = 5\,\varphi$.

Solution: Given $\varphi = x^2 + y^2 + z^2$ and $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$

Div $(\varphi\vec{r}) = \vec{\nabla}.(\varphi\vec{r}) = \vec{\nabla}.\left[(x^2 + y^2 + z^2)\left(x\,\hat{i} + y\hat{j} + z\,\hat{k}\right)\right]$

$$= \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left[x(x^2 + y^2 + z^2)\,\hat{i} + y(x^2 + y^2 + z^2)\hat{j} + z(x^2 + y^2 + z^2)\,\hat{k}\right]$$

$$= \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right).\left[(x^3 + xy^2 + xz^2)\,\hat{i} + (yx^2 + y^3 + yz^2)\hat{j} + (zx^2 + zy^2 + z^3)\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^3 + xy^2 + xz^2) + \frac{\partial}{\partial y}(yx^2 + y^3 + yz^2) + \frac{\partial}{\partial z}(zx^2 + zy^2 + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5x^2 + 5y^2 + 5z^2$$

Div$(\varphi\vec{r}) = \boldsymbol{5}\,(x^2 + y^2 + z^2)$

Div$(\varphi\vec{r}) = \boldsymbol{5}\,\varphi$ ***Hence proved.***

Q#09:If $\vec{a}$ ***is a constant vector and*** $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***.Show that***

(i) $\vec{\nabla}(\vec{a}.\vec{r}) = \vec{a}$ ***(ii)*** $\vec{\nabla}.(\vec{a} \times \vec{r}) = 0$ (iii)Curl $[(\vec{a}.\vec{r})\vec{r}] = \vec{a} \times \vec{r}$ (iv) Div $[(\vec{a}.\vec{r})\vec{r}] = \boldsymbol{4}(\vec{a}.\vec{r})$

Solution: Given $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***& Let*** $\vec{a} = a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k}$

Then $\vec{a}.\vec{r} = \left(a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k}\right).\left(x\,\hat{i} + y\hat{j} + z\,\hat{k}\right) = a_1\,x + a_2 y + a_3\,z$

& $\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ x & y \end{vmatrix}$

$$\vec{a} \times \vec{r} = \hat{i}\,[a_2 z - a_3 y] - \hat{j}\,[a_1 z - a_3 x] + \hat{k}[a_1 y - a_2 x]$$

$$\vec{a} \times \vec{r} = \hat{i}\,[a_2 z - a_3 y] + \hat{j}\,[a_3 x - a_1 z] + \hat{k}[a_1 y - a_2 x]$$

(i) $\boldsymbol{\vec{\nabla}(\vec{a}.\vec{r}) = \vec{a}}$

Let $\vec{\nabla}(\vec{a}.\vec{r}) = \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right)(a_1\,x + a_2 y + a_3\,z)$

$$= \frac{\partial}{\partial x}(a_1\,x + a_2 y + a_3\,z)\;\hat{i} + \frac{\partial}{\partial y}(a_1\,x + a_2 y + a_3\,z)\;\hat{j} + \frac{\partial}{\partial z}(a_1\,x + a_2 y + a_3\,z)\;\hat{k}$$

$$= a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k}$$

$\vec{\nabla}(\vec{a}.\vec{r}) = \vec{a}$ ***Hence proved.***

(ii) $\vec{\nabla}.(\vec{a}\times\vec{r}\,) = 0$

Let $\vec{\nabla}.(\vec{a}\times\vec{r}\,) = \left(\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right).\left(\hat{\imath}\,[a_2z - a_3y] + \hat{\jmath}\,[a_3x - a_1z] + \hat{k}[a_1y - a_2x]\right)$

$$= \frac{\partial}{\partial x}[a_2z - a_3y] + \frac{\partial}{\partial y}[a_3x - a_1z] + \frac{\partial}{\partial z}[a_1y - a_2x]$$

$$= 0 + 0 + 0$$

$\vec{\nabla}.(\vec{a}\times\vec{r}\,) = 0$ ***Hence proved***

(iii)Curl $[(\vec{a}.\vec{r})\vec{r}] = \vec{a}\times\vec{r}$

Let $(\vec{a}.\vec{r})\vec{r} = (a_1\,x + a_2y + a_3\,z)(x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k})$

$$= x(a_1\,x + a_2y + a_3\,z)\,\hat{\imath} + y(a_1\,x + a_2y + a_3\,z)\hat{\jmath} + z(a_1\,x + a_2y + a_3\,z)\,\hat{k}$$

$$(\vec{a}.\vec{r})\vec{r} = (a_1\,x^2 + a_2xy + a_3x\,z)\,\hat{\imath} + (a_1\,xy + a_2y^2 + a_3\,yz)\hat{\jmath} + (a_1\,xz + a_2y\,z + a_3\,z^2)\,\hat{k}$$

Now

$$\text{Curl}\,[(\vec{a}.\vec{r})\vec{r}] = \vec{\nabla}\times[(\vec{a}.\vec{r})\vec{r}] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_1\,x^2 + a_2xy + a_3x\,z) & (a_1\,xy + a_2y^2 + a_3\,yz) & (a_1\,xz + a_2y\,z + a_3\,z^2) \end{vmatrix}$$

$$= \hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_1\,xy + a_2y^2 + a_3\,yz) & (a_1\,xz + a_2y\,z + a_3\,z^2) \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ (a_1\,x^2 + a_2xy + a_3x\,z) & (a_1\,xz + a_2y\,z + a_3\,z^2) \end{vmatrix}$$

$$+ \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ (a_1\,x^2 + a_2xy + a_3x\,z) & (a_1\,xy + a_2y^2 + a_3\,yz) \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(a_1\,xz + a_2y\,z + a_3\,z^2) - \frac{\partial}{\partial z}(a_1\,xy + a_2y^2 + a_3\,yz)\right]$$

$$-\hat{\jmath}\left[\frac{\partial}{\partial x}(a_1\,xz + a_2y\,z + a_3\,z^2) - \frac{\partial}{\partial z}(a_1\,x^2 + a_2xy + a_3x\,z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(a_1\,xy + a_2y^2 + a_3\,yz) - \frac{\partial}{\partial y}(a_1\,x^2 + a_2xy + a_3x\,z)\right]$$

$$= \hat{\imath}\,[a_2z - a_3y] + \hat{\jmath}\,[a_3x - a_1z] + \hat{k}[a_1y - a_2x]$$

Curl $[(\vec{a}.\vec{r})\vec{r}] = \vec{a}\times\vec{r}$ ***Hence proved.***

(iv) Div $[(\vec{a}\,.\,\vec{r})\vec{r}] = 4(\vec{a}\,.\,\vec{r}\,)$

Let $(\vec{a}\,.\,\vec{r})\vec{r} = (a_1\ x + a_2 y\ +\ a_3\, z)\big(x\,\hat{\imath} + y\hat{\jmath}\ +\ z\,\hat{k}\big)$

$$= x(a_1\ x + a_2 y\ +\ a_3\, z)\ \hat{\imath} + y(a_1\ x + a_2 y\ +\ a_3\, z)\hat{\jmath}\ +\ z(a_1\ x + a_2 y\ +\ a_3\, z)\,\hat{k}$$

$$(\vec{a}\,.\,\vec{r})\vec{r} = (a_1\ x^2 + a_2 xy\ +\ a_3 x\, z)\ \hat{\imath} + (a_1\ xy + a_2 y^2 +\ a_3\, yz)\hat{\jmath}\ +\ (a_1\ xz + a_2 y\, z +\ a_3\, z^2)\,\hat{k}$$

Now

$$\text{Div}\,[(\vec{a}\,.\,\vec{r})\vec{r}] = \vec{\nabla}\ .\ [(\vec{a}\,.\,\vec{r})\vec{r}]$$

$$= \left(\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\ +\ \frac{\partial}{\partial z}\,\hat{k}\ \right).\big[(a_1\ x^2 + a_2 xy\ +\ a_3 x\, z)\ \hat{\imath} + (a_1\ xy + a_2 y^2 +\ a_3\, yz)\hat{\jmath}\ +\ (a_1\ xz + a_2 y\, z +\ a_3\, z^2)\,\hat{k}\big]$$

$$= \frac{\partial}{\partial x}(a_1\ x^2 + a_2 xy\ +\ a_3 x\, z) + \frac{\partial}{\partial y}(a_1\ xy + a_2 y^2 +\ a_3\, yz)\ +\ \frac{\partial}{\partial z}\,(a_1\ xz + a_2 y\, z +\ a_3\, z^2)$$

$$=\ 2\,a_1\ x + a_2 y\ +\ a_3\, z + a_1\ x + 2a_2 y +\ a_3\, z + a_1\ x + a_2 y\ +\ 2a_3 z$$

$$= 4a_1\ x + 4a_2 y\ + 4\,a_3\, z$$

$$= 4(a_1\ x + a_2 y\ + a_3\, z)$$

$\text{Div}\,[(\vec{a}\,.\,\vec{r})\vec{r}] = 4(\vec{a}\,.\,\vec{r}\,)$ ***Hence proved.***

Q#10:If $\vec{a} = e^{xy}\ \hat{\imath} + \sin(xy)\,\hat{\jmath}\ + \cos\ (yz^2)\ \hat{k}$ ***then evaluate Curl*** $\vec{a}$.

Solution: Given $\vec{a} = e^{xy}\ \hat{\imath} + \sin(xy)\,\hat{\jmath}\ + \cos\ (yz^2)\ \hat{k}$

Then Curl $\vec{a}\ = \vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin(xy) & \cos\ (yz^2) \end{vmatrix} = \hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(xy) & \cos\ (yz^2) \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ e^{xy} & \cos\ (yz^2) \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^{xy} & \sin(xy) \end{vmatrix}$

$$= \hat{\imath}\ \left[\frac{\partial}{\partial y}\cos\ (yz^2) - \frac{\partial}{\partial z}\sin(xy)\right] - \hat{\jmath}\ \left[\frac{\partial}{\partial x}\cos\ (yz^2) - \frac{\partial}{\partial z}e^{xy}\right]\ + \hat{k}\left[\frac{\partial}{\partial x}\sin(xy) - \frac{\partial}{\partial y}\ e^{xy}\right]$$

$$= \hat{\imath}\ [-z^2\sin\ (yz^2) - 0] + \hat{\jmath}\ [0 - 0] +\ \hat{k}[y\cos(xy) - xe^{xy}]$$

$$= -z^2\sin\ (yz^2)\ \hat{\imath}\ + 0\hat{\jmath}\ + [y\cos(xy) - xe^{xy}]\ \hat{k}$$

Curl $\vec{a}\ = -z^2\sin\ (yz^2)\ \hat{\imath}\ + [y\cos(xy) - xe^{xy}]\ \hat{k}$

Q#11: Evaluate $\vec{\nabla}\cdot\left[r\,\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$

Solution: Let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ then $r^2 = x^2 + y^2 + z^2$ ***------(i)***

Now $\vec{\nabla}\cdot\left[r\,\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \vec{\nabla}\cdot\left[r\,\vec{\nabla}\,(r^{-3})\right]$

$\therefore \left\{ \begin{matrix} \text{From(i) Differentiate w.r.t } x \\ 2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r} \end{matrix} \right\}$

$$= \vec{\nabla}\cdot\left[r\left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right)(r^{-3})\right]$$

$$= \vec{\nabla}\cdot\left[r\left\{\frac{\partial}{\partial x}(r^{-3})\,\hat{i} + \frac{\partial}{\partial y}(r^{-3})\hat{j} + \frac{\partial}{\partial z}(r^{-3})\,\hat{k}\right\}\right]$$

$$= \vec{\nabla}\cdot\left[r\left\{(-3)\,r^{-4}\frac{\partial r}{\partial x}\,\hat{i} + (-3)\,r^{-4}\frac{\partial r}{\partial y}\,\hat{j} + (-3)\,r^{-4}\frac{\partial r}{\partial z}\,\hat{k}\right\}\right]$$

$$= \vec{\nabla}\cdot\left[r\left\{(-3)\,r^{-4}\frac{x}{r}\,\hat{i} + (-3)\,r^{-4}\frac{y}{r}\,\hat{j} + (-3)\,r^{-4}\frac{z}{r}\,\hat{k}\right\}\right]$$

$$= \vec{\nabla}\cdot\left[(-3)\,r^{-4}.x\,\hat{i} + (-3)\,r^{-4}.y\,\hat{j} + (-3)\,r^{-4}.z\,\hat{k}\right]$$

$$= \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right)\cdot\left[(-3)\,r^{-4}.x\,\hat{i} + (-3)\,r^{-4}.y\,\hat{j} + (-3)\,r^{-4}.z\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(-3)r^{-4}.x + \frac{\partial}{\partial y}(-3)r^{-4}.y + \frac{\partial}{\partial z}(-3)r^{-4}.z$$

$$= (-3)\left[\frac{\partial}{\partial x}(r^{-4}.x) + \frac{\partial}{\partial y}(r^{-4}.y) + \frac{\partial}{\partial z}(r^{-4}.z)\right]$$

$$= (-3)\left[\left\{r^{-4}\frac{\partial x}{\partial x} + (-4)r^{-5}\frac{\partial r}{\partial x}.x\right\} + \left\{r^{-4}\frac{\partial y}{\partial y} + (-4)r^{-5}\frac{\partial r}{\partial y}.y\right\} + \left\{r^{-4}\frac{\partial z}{\partial z} + (-4)r^{-5}\frac{\partial r}{\partial z}.z\right\}\right]$$

$$= (-3)\left[\left\{r^{-4} + (-4)r^{-5}.\frac{x}{r}.x\right\} + \left\{r^{-4} + (-4)r^{-5}.\frac{y}{r}.y\right\} + \left\{r^{-4} + (-4)r^{-5}.\frac{z}{r}.z\right\}\right]$$

$$= (-3)[\{r^{-4} + (-4)r^{-6}.x^2\} + \{r^{-4} + (-4)r^{-6}.y^2\} + \{r^{-4} + (-4)r^{-6}.z^2\}]$$

$$= (-3)[r^{-4} - 4r^{-6}.x^2 + r^{-4} - 4r^{-6}.y^2 + r^{-4} - 4r^{-6}.z^2]$$

$$= (-3)[3r^{-4} - 4r^{-6}(x^2 + y^2 + z^2)]$$

$$= (-3)[3r^{-4} - 4r^{-6}.r^2]$$

$$= (-3)[3r^{-4} - 4r^{-4}]$$

$$= (-3)[-r^{-4}]$$

$$= 3r^{-4}$$

Hence $\vec{\nabla}\cdot\left[r\,\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$

Q#12:If $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***and*** $\vec{a}$ ***is a constant vector then show that***

(i)Curl $\vec{r} = 0$ (ii) Curl $(r^{n}\vec{r}) = \mathbf{0}$ ***(iii) Curl*** $(\vec{a} \times \vec{r}) = 2\vec{a}$ ***(iv)*** $\vec{\nabla} \times \left(\frac{\vec{r}}{r^{2}}\right)$

Let $\vec{a} = a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k}$ & $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$

(i) ***Curl*** $\vec{r} = 0$

Solution: Let

$$\textbf{Curl}\ \vec{r} = \vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{j}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right]$$

$$= \hat{i}\,[0-0] + \hat{j}\,[0-0] + \hat{k}[0-0]$$

$$= 0\hat{i} + 0\hat{j} + 0\,\hat{k}$$

Curl $\vec{r}$ ***= 0***

(ii) **Curl** $(\mathbf{r}^{n}\vec{\mathbf{r}}) = \mathbf{0}$

Solution: $\therefore r^{n}\vec{r} = r^{n}(x\,\hat{i} + y\hat{j} + z\,\hat{k}) = r^{n}x\,\hat{i} + r^{n}y\hat{j} + r^{n}z\,\hat{k}$

Now $\text{Curl}\,(r^{n}\vec{r}) = \vec{\nabla} \times (r^{n}\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^{n}x & r^{n}y & r^{n}z \end{vmatrix} = r^{n}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$ $\therefore$ (r^{n} common from R_3)

$$= r^{n}\left[\hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}\right]$$

$$= r^{n}\left\{\hat{i}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{j}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right]\right\}$$

$$= r^{n}\{\hat{i}\,[0-0] + \hat{j}\,[0-0] + \hat{k}[0-0]\}$$

$$= r^{n}\{0\,\hat{i} + 0\hat{j} + 0\hat{k}\}$$

$$= r^{n}\{0\}$$

Curl $(r^{n}\vec{r}) = \mathbf{0}$

(iii) ***Curl*** $(\vec{a}\times\vec{r}) = 2\vec{a}$

Solution: $\therefore\ \vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ x & y \end{vmatrix}$

$$\vec{a}\times\vec{r} = \hat{i}\,[a_2z - a_3y] - \hat{j}\,[a_1z - a_3x] + \hat{k}[a_1y - a_2x] = \hat{i}\,[a_2z - a_3y] + \hat{j}\,[a_3x - a_1z] + \hat{k}[a_1y - a_2x]$$

Now Curl $(\vec{a}\times\vec{r}) = \vec{\nabla}\times(\vec{a}\times\vec{r}) = \vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3x - a_1z & a_1y - a_2x \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ a_2z - a_3y & a_1y - a_2x \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ a_2z - a_3y & a_3x - a_1z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(a_1y - a_2x) - \frac{\partial}{\partial z}(a_3x - a_1z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(a_1y - a_2x) - \frac{\partial}{\partial z}(a_2z - a_3y)\right] + \hat{k}$$

$$= \hat{i}\,[a_1 - (-a_1)] + \hat{j}\,[a_2 - (-a_2)] + \hat{k}[a_3 - (-a_3)]$$

$$= \hat{i}\,[a_1 + a_1] + \hat{j}\,[a_2 + a_2] + \hat{k}[a_3 + a_3]$$

$$= 2a_1\,\hat{i} + 2a_2\hat{j} + 2a_3\,\hat{k} = 2\left(a_1\,\hat{i} + a_2\hat{j} + a_3\,\hat{k}\right)$$

Curl $(\vec{a}\times\vec{r}) = 2\vec{a}$

(iv) $\vec{\nabla}\times\left(\frac{\vec{r}}{r^2}\right)$

Solution : let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$

$$\therefore r^{-2}\vec{r} = r^{-2}\left(x\,\hat{i} + y\hat{j} + z\hat{k}\right) = r^{-2}x\,\hat{i} + r^{-2}y\hat{j} + r^{-2}z\,\hat{k}$$

Now $\vec{\nabla}\times\left(\frac{\vec{r}}{r^2}\right) = \vec{\nabla}\times(r^{-2}\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^{-2}x & r^{-2}y & r^{-2}z \end{vmatrix} = r^{-2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$ $\therefore$ (r^{-2} common from R_3)

$$= r^{-2}\left[\hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}\right]$$

$$= r^{-2}\left\{\hat{i}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{j}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right]\right\}$$

$$= r^{-2}\left\{\hat{i}\,[0-0] + \hat{j}\,[0-0] + \hat{k}[0-0]\right\} = r^{-2}\left\{0\,\hat{i} + 0\hat{j} + 0\hat{k}\right\}$$

$$= r^{-2}\,\{0\}$$

$\vec{\nabla}\times\left(\frac{\vec{r}}{r^2}\right) = 0$ ***Hence proved.***

Q#13: Show that $\vec{F} = \frac{\vec{r}}{r^2}$ ***is an Irrotational vector also find*** φ ***, when*** $\vec{F} = -\vec{\nabla}\varphi$ ***such that*** $\varphi(a) = 0$. ***(a>0)***

Solution: : ***Let*** $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ & $r^2 = x^2 + y^2 + z^2$

$$\therefore r^{-2}\vec{r} = r^{-2}(x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}) = r^{-2}x\,\hat{\imath} + r^{-2}y\hat{\jmath} + r^{-2}z\,\hat{k}$$

For Irrotational vector, we have to prove Now Curl$\vec{F} = 0$

$$Curl\vec{F} = \vec{\nabla}\times\vec{F} = \vec{\nabla}\times\left(\frac{\vec{r}}{r^2}\right) = \vec{\nabla}\times(r^{-2}\vec{r}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^{-2}x & r^{-2}y & r^{-2}z \end{vmatrix} = r^{-2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \quad \therefore (r^{-2} \text{ common from } R_3)$$

$$= r^{-2}\left[\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}\right]$$

$$= r^{-2}\left\{\hat{\imath}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right]\right\}$$

$$= r^{-2}\{\hat{\imath}[0-0] + \hat{\jmath}[0-0] + \hat{k}[0-0]\} = r^{-2}\{0\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k}\} = r^{-2}\{0\}$$

Curl$\vec{F}$ ***= 0***

Hence prove that $\vec{F} = \frac{\vec{r}}{r^2}$ ***is an Irrotational vector .***

Now we have find φ ***for this given condition is*** $\vec{F} = -\vec{\nabla}\varphi$ ***Then*** $\vec{\nabla}\varphi = -\frac{\vec{r}}{r^2}$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = -\frac{x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}{x^2+y^2+z^2} = -\frac{x}{x^2+y^2+z^2}\hat{\imath} - \frac{y}{x^2+y^2+z^2}\hat{\jmath} - \frac{z}{x^2+y^2+z^2}\hat{k}$$

Comparing coefficients of $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$

$$\frac{\partial\varphi}{\partial x} = -\frac{x}{x^2+y^2+z^2} \Rightarrow \varphi = -\frac{1}{2}\int\frac{2x}{x^2+y^2+z^2}\partial x \Rightarrow \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c_1(y,z) \text{-------}(i)$$

$$\frac{\partial\varphi}{\partial y} = -\frac{y}{x^2+y^2+z^2} \Rightarrow \varphi = -\frac{1}{2}\int\frac{2y}{x^2+y^2+z^2}\partial x \Rightarrow \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c_2(x,z) \text{-----}(ii)$$

$$\frac{\partial\varphi}{\partial z} = -\frac{z}{x^2+y^2+z^2} \Rightarrow \varphi = -\frac{1}{2}\int\frac{2z}{x^2+y^2+z^2}\partial x \Rightarrow \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c_3(x,z) \text{-------}(iii)$$

From (i) ,(ii) & (iii): $\varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c$

$$\varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c \Rightarrow \varphi = -\frac{1}{2}\ln r^2 + c = -\frac{1}{2}.2\ln r + c \Rightarrow \varphi(r) = -\ln r + c \text{ -----}(a)$$

At $\varphi(a) = 0 \Rightarrow -\ln a + c = 0 \Rightarrow c = \ln a$

Hence equation (a) will become $\varphi(r) = \varphi(r) = -\ln r + \ln a \Rightarrow \varphi(r) = \ln\left(\frac{a}{r}\right)$

Q#14: Find a, b, c so that $\vec{F} = (x + 2y + a\,z)\,\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\,\hat{k}$ ***is Irrotational vector.***

Solution: Given $\vec{F} = (x + 2y + a\,z)\,\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\,\hat{k}$

By using Given condition that $\vec{F}$ is an irrotational vector therefore

$$\textbf{\textit{Curl}}\,\vec{F} = 0 \implies \vec{\nabla} \times \vec{F} = 0$$

$$\implies \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + a\,z) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} = 0$$

$$\hat{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ (x + 2y + a\,z) & (4x + cy + 2z) \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ (x + 2y + a\,z) & (bx - 3y - z) \end{vmatrix} = \mathbf{0}$$

$$\hat{\imath}\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + a\,z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + a\,z)\right] = 0$$

$$\hat{\imath}\,[c - (-1)] + \hat{\jmath}\,[4 - a] + \hat{k}[b - 2] = 0\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\hat{\imath}\,[c + 1] + \hat{\jmath}\,[4 - a] + \hat{k}[b - 2] = 0\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

Comparing coefficients of $\hat{\imath}\,,\hat{\jmath}$ ***&*** $\hat{k}$ **.**

$c + 1 = 0 \implies$ ***c = −1***

$4 - a = 0 \implies$ ***a= 4***

$b - 2 = 0 \implies$ ***b = 2***

Q#15: Prove that . $\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$

Solution: We know that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***then***

$$\nabla^2 f(r) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] f(r) = \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} f(r)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} f(r)\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{\partial r}{\partial y}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{\partial r}{\partial z}\right]$$

$$= \left[\left\{f''(r)\frac{\partial r}{\partial x}.\frac{\partial r}{\partial x} + f'(r)\frac{\partial^2 r}{\partial x^2}\right\} + \left\{f''(r)\frac{\partial r}{\partial y}.\frac{\partial r}{\partial y} + f'(r)\frac{\partial^2 r}{\partial y^2}\right\} + \left\{f''(r)\frac{\partial r}{\partial z}.\frac{\partial r}{\partial z} + f'(r)\frac{\partial^2 r}{\partial z^2}\right\}\right]$$

$$= \left[f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} + f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2} + f''(r)\left(\frac{\partial r}{\partial z}\right)^2 + f'(r)\frac{\partial^2 r}{\partial z^2}\right]$$

$$= \left[f''(r)\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right\} + f'(r)\left\{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right\}\right] \text{---------------}(a)$$

Let $\vec{r} = x\,\hat{i} + y\hat{j} + z\,\hat{k}$ ***then*** $r^2 = x^2 + y^2 + z^2$ ------***(i)***

$\therefore$ From(i) ***Differentiate w. r. t x*** $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ ***Similarly*** $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Again differentiate w. r. t x $\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\left(\frac{x}{r}\right)}{r^2} = \frac{\frac{r^2 - x^2}{r}}{r^2} = \frac{x^2+y^2+z^2-x^2}{r^3} \Longrightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{r^3}$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{r^3}$ & $\frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{r^3}$

Putting values in Equation (a)

$$\nabla^2 f(r) = f''(r)\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right\} + f'(r)\left\{\frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{x^2+y^2}{r^3}\right\}$$

$$\nabla^2 f(r) = f''(r)\left\{\frac{x^2+y^2+z^2}{r^2}\right\} + f'(r)\left\{\frac{y^2+z^2+x^2+z^2+x^2+y^2}{r^3}\right\}$$

$$\nabla^2 f(r) = f''(r)\left\{\frac{r^2}{r^2}\right\} + f'(r)\left\{\frac{2(x^2+y^2+z^2)}{r^3}\right\}$$

$$\nabla^2 f(r) = f''(r)(1) + f'(r)\left\{\frac{2r^2}{r^3}\right\}$$

$$\nabla^2 f(r) = f''(r) + \left\{\frac{2}{r}\right\} f'(r)$$

$$\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$$ ***Hence proved.***