UNIT # 01

VECTOR ALGEBRA

Introduction:

In this chapter, we will discuss about the basic concepts of vectors.

Scalars:

Scalars are physical quantities, which are described completely by its magnitude and units.

Examples: Mass, length, time, density, energy, work, temperature, charge etc.

Scalar can be added, subtracted and multiplied by the ordinary rule of algebra.

Vectors:

Vectors are the physical quantities which are described completely by its magnitude, unit and its direction.

Examples: Force, velocity, acceleration, momentum, torque, electric field, magnetic field etc.

Vectors are added, subtracted, multiplied by using vector algebra.

Representation of vector:

A vector quantity is represented by two ways.

1. Symbolically 2. Graphically

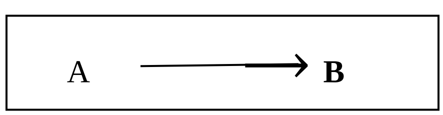
1. Symbolic Representation:

A vector quantity is represented by a bold letter such as F, a, d. or

It is represented by a bar or an arrow over their symbols. Such as $\bar{F}, \bar{a}, \bar{d}$ or $\vec{F}, \vec{a}, \vec{d}$.

2. Graphical Representation:

A vector can be represented by a line segment with an arrow head as shown in figure.



Let a line $\overrightarrow{AB}$ ***with arrow head at B represent a vector*** $\vec{v}$. ***The length of line AB gives the magnitude of vector*** $\vec{v}$ ***on a selected scale. While the direction of the line A to B gives the direction of vector*** $\vec{v}$.

Position vector:

A vector, whose initial point is origin O and whose terminal point is P, is called position vector of point P and it is written as $\overrightarrow{OP}$.

Vector representation in two and three dimensions coordinate system:

Let R be set of real numbers.

The Cartesian plane is define as R^2 ***= { (x,y) : x,y ∈ R} and it is written as*** $\overrightarrow{OP}$***= x i+ yj***

Similarly, in three dimension coordinate system. It is define as R^3 ***= { (x,y,z) : x,y,z ∈ R}***

And it is written as $\overrightarrow{OP}$ ***= x i+ yj + zk***

Magnitude (length or norm):

Magnitude (length or norm) of a vector $\overrightarrow{OP}$ ***is its absolute value and it is written as*** $|\overrightarrow{OP}|$.

As $|\overrightarrow{OP}|=\sqrt{x^2 + y^2 + Z^2}$

Null or zero vector:

A vector having zero magnitude is called Null or zero vectors.

Unit vector:

A vector having unit magnitude and having direction along the given vector is called unit vector. These are usually represented by $\hat{a}, \hat{b}, \hat{c}$ ***or*** $\hat{\imath}, \hat{\jmath}, \hat{k}$.

If we consider a vector $\vec{A}$***, then its unit vector can be written as*** $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

Direction cosines:

Let $\vec{A} = Ax\,\hat{\imath} + Ay\,\hat{\jmath} + Az\,\hat{k}$ ***&***

If a vector $\vec{A}$ ***makes angles*** α, β and γ ***with x,y and z-axis. Then Direction cosines are define as***

Cos $\alpha = \frac{A_x}{|\vec{A}|}$; ***Cos*** $\beta = \frac{A_y}{|\vec{A}|}$; ***Cos*** $\gamma = \frac{A_z}{|\vec{A}|}$

Vector addition:

A process in which two or more vectors can be added in the form of single vector is called vectors addition.

For vector addition, we use a graphical method called Head To Tail Rule.

Resultant vector:

It is the sum of two or more than two vectors called resultant vector.

Rectangular components:

The components of a vector perpendicular to each other are called rectangular components.

Collinear vectors:

Let $\vec{a}$ ***and*** $\vec{b}$ ***be the two vectors. They are said to be collinear if*** $\vec{a} = \lambda\vec{b}$ ***.where*** λ ***is a scalar number.***

(a) If $\lambda > 0$ ***then*** $\vec{a}$ ***and*** $\vec{b}$ ***are said to be parallel vectors.***

(b) If < 0 ***then*** $\vec{a}$ ***and*** $\vec{b}$ ***are said to be anti-parallel vectors.***

(c) If $\lambda = 0$ ***then*** $\vec{a}$ ***and*** $\vec{b}$ ***are said to be equal vectors. In this case*** $\vec{a} = \vec{b}$***.***

Free vectors:

A vector whose position is not fixed in the space is called free vector.

Example: displacement

Localized vector:

A vector which can't be shifted to parallel to itself and whose line of action is fixed is called localized vector (bounded vector).

Examples: Force and Momentum.

Parallel vectors:

If two or more than two vectors having same direction are called parallel vectors.

Let $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ ***&*** $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

They are said to be parallel if their directional component are proportional to each other as

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Perpendicular vector:

If two or more than two vectors making an angle of 90^0 ***with each other are called perpendicular vectors.***

Let $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ ***&*** $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

They are said to be perpendicular if the sum of product of their directional component is equal to zero.

$$a_1b_1 + a_2b_2 + a_3b_3 = 0$$

Properties of vectors addition:

(i) Commutative property:

If $\vec{a}$, $\vec{b}$ ***be the two vectors.*** ***Then*** $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ ***is called commutative property.***

(ii) Associative property:

If $\vec{a}$, $\vec{b}$ ***and*** $\vec{c}$ ***be the three vectors. Then*** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ ***is called associative property.***

(iii) Scalar multiplication with vectors:

Let $\vec{a}$ ***be a vector and be a scalar number then*** $\lambda\vec{a}$ ***is called Scalar multiplication with vector.***

If $\vec{a}$ ***and*** $\vec{b}$ ***be the vectors and*** λ ***and*** μ ***be the two scalar numbers then***

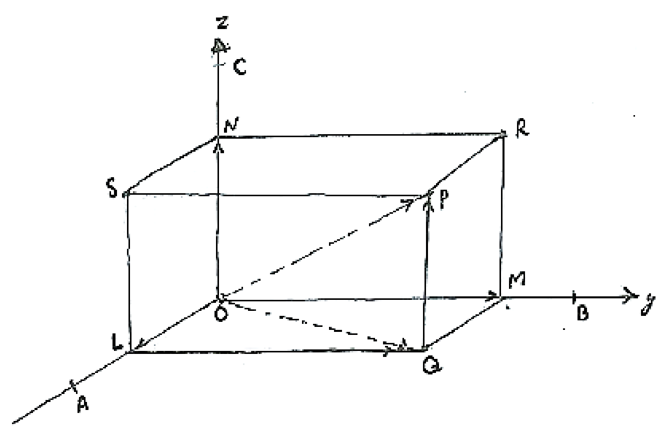
(a) $(\lambda+\mu)\vec{a}=\vec{a}+\mu\vec{a}$ ***(b)*** $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$

Theorem#01: If $\vec{a}$, $\vec{b}$ ***and*** $\vec{c}$ ***are three given non coplanar vectors ,then any vector*** $\vec{r}$ ***can be expressed uniquely as linear combination of*** $\vec{a}$, $\vec{b}$ ***and*** $\vec{c}$ ***i.e.*** $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$ ***where x, y and z are scalars.***

Proof: Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OC}=\vec{c}$ ***and*** $\overrightarrow{OP}=\vec{r}$ ***as shown in the figure.***

Let us complete the parallelepiped with $\overrightarrow{OP}$ ***as its diagonal whose edges*** $\overrightarrow{OL}$, $\overrightarrow{OM}$ ***and*** $\overrightarrow{ON}$ ***are along the vectors*** $\overrightarrow{OA}$, $\overrightarrow{OB}$ ***and*** $\overrightarrow{OC}$.

$\overrightarrow{OL}$ ***and*** $\overrightarrow{OA}$, $\overrightarrow{OM}$ ***and*** $\overrightarrow{OB}$, $\overrightarrow{ON}$ ***and*** $\overrightarrow{OC}$ ***are coplanar and parallel. Then there exist Three scalars x, y and z respectively.***



$\overrightarrow{OL}=x\ \overrightarrow{OA}$; $\overrightarrow{OM}=y\ \overrightarrow{OB}$ & $\overrightarrow{ON}=z\ \overrightarrow{OC}$

By using head to tail rule

In ΔABC

$\overrightarrow{OP}=\overrightarrow{OQ}+\overrightarrow{QP}=(\overrightarrow{OL}+\overrightarrow{LQ})+\overrightarrow{QP}$ $\because \overrightarrow{OQ}=\overrightarrow{OL}+\overrightarrow{LQ}$

$\overrightarrow{OP}=\overrightarrow{OL}+\overrightarrow{OM}+\overrightarrow{ON}$ ***but*** $\overrightarrow{LQ}=\overrightarrow{OM}$ ***and*** $\overrightarrow{QP}=\overrightarrow{ON}$

$\overrightarrow{OP}=x\ \overrightarrow{OA}+y\ \overrightarrow{OB}+z\ \overrightarrow{OC}$

$\vec{r}=x\ \vec{a}+y\ \vec{b}+z\ \vec{c}$ ------------------***(i)***

Uniqueness

Let $\vec{r} = x'\,\vec{a} + y'\,\vec{b} + z'\,\vec{c}$ ----------------***(ii)***

Comparing (i) and (ii)

$$x\,\vec{a} + y\,\vec{b} + z\,\vec{c} = x'\,\vec{a} + y'\,\vec{b} + z'\,\vec{c}$$

$$x\,\vec{a} + y\,\vec{b} + z\,\vec{c} - x'\,\vec{a} - y'\,\vec{b} - z'\,\vec{c} = \boldsymbol{0}$$

$$(x - x')\,\vec{a} + (y - y')\,\vec{b} + (z - z')\,\vec{c} = \boldsymbol{0}$$

Since, $\vec{a}$ **,** $\vec{b}$ ***and*** $\vec{c}$ ***are non coplanar Therefore***

$$x - x' = 0 \quad ; \quad y - y' = 0 \quad ; \quad z - z' = 0$$

$$x = x' \quad ; \quad y = y' \quad ; \quad z = z'$$

Hence, uniqueness proved.

Theorem#02: Find the position vector of a point which divides the join of two given points whose position vectors are $\vec{a}$ ***and*** $\vec{b}$ ***in the given ratio : μ.***

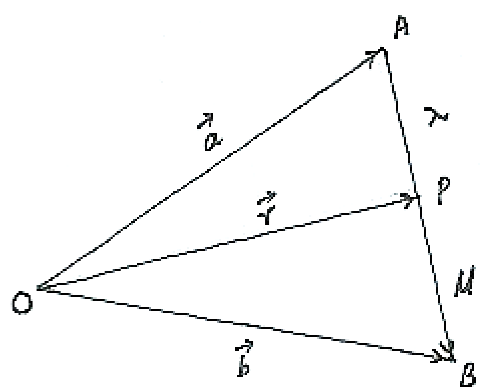
Proof: Let $\vec{a}$ ***and*** $\vec{b}$ ***be the position vector of point A and B referred to point O and let*** $\vec{r}$ ***be the position vector of point P which divide AB internally in ratio λ: μ.***

As $\overrightarrow{AP} : \overrightarrow{PB} = \ : \mu$ ***or*** $\dfrac{\overrightarrow{AP}}{\overrightarrow{PB}} = \dfrac{\lambda}{\mu} \Longrightarrow \mu\,\overrightarrow{AP} = \lambda\,\overrightarrow{PB}$ -------------***(i)***

Now $\overrightarrow{AP}$ = ***p.v's of P – p .v's of A*** = $\vec{r} - \vec{a}$

$\overrightarrow{PB}$ = ***p.v's of B – p .v's of P*** = $\vec{b} - \vec{r}$

Using values in equation (i)



$$\mu(\vec{r} - \vec{a}) = \lambda\,(\vec{b} - \vec{r})$$

$$\mu\vec{r} - \mu\,\vec{a} = \lambda\,\vec{b} - \lambda\,\vec{r}$$

$$\mu\,\vec{r} + \lambda\,\vec{r} = \mu\,\vec{a} + \lambda\,\vec{b}$$

$$(\mu + \lambda)\,\vec{r} = \mu\,\vec{a} + \lambda\,\vec{b}$$

$$\boxed{\vec{r} = \frac{\mu\,\vec{a} + \lambda\,\vec{b}}{\mu + \lambda}}$$

Special Case:

If $\lambda = \mu$ ***Then P is the mid-point of AB and its position vector*** $\vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$.

Example # 01- Find the sum of vectors $3\hat{i}+7\hat{j}-4\hat{k}$ ***;*** $\hat{i}-5\hat{j}-8\hat{k}$ ***and*** $6\hat{i}-2\hat{j}+12\hat{k}$

Also calculate the magnitude and direction cosines of each.

Solution: Let $\vec{a}=3\hat{i}+7\hat{j}-4\hat{k}$ ***;*** $\vec{b}=\hat{i}-5\hat{j}-8\hat{k}$ ***and*** $\vec{c}=6\hat{i}-2\hat{j}+12\hat{k}$

Let $\vec{r}$ ***be the sum of given vectors .***

$\vec{r}=\vec{a}+\vec{b}+\vec{c}=3\hat{i}+7\hat{j}-4\hat{k}+\hat{i}-5\hat{j}-8\hat{k}+6\hat{i}-2\hat{j}+12\hat{k}=10\hat{i}+0\hat{j}+0\hat{k}=10\hat{i}$

Magnitude of vector $\vec{a}$ ***,*** $\vec{b}$ ***and*** $\vec{c}$ ***are***

$$|\vec{A}|=\sqrt{(3)^2+(7)^2+(-4)^2}=\sqrt{9+49+16}=\sqrt{74}$$

$$|\vec{B}|=\sqrt{(1)^2+(-5)^2+(-8)^2}=\sqrt{1+25+64}=\sqrt{90}$$

$$|\vec{C}|=\sqrt{(6)^2+(-2)^2+(12)^2}=\sqrt{36+4+144}=\sqrt{184}$$

Direction cosines of vector $\vec{a}$ ***are*** $\frac{3}{\sqrt{74}}, \frac{7}{\sqrt{74}}, \frac{-4}{\sqrt{74}}$

Direction cosines of vector $\vec{b}$ ***are*** $\frac{1}{\sqrt{90}}, \frac{-5}{\sqrt{90}}, \frac{-8}{\sqrt{90}}$

Direction cosines of vector $\vec{c}$ ***are*** $\frac{6}{\sqrt{184}}, \frac{-2}{\sqrt{184}}, \frac{12}{\sqrt{184}}$

Example #02: Find the value of m .if the vector $5\hat{i}+4\hat{j}-3\hat{k}$ ***and*** $2\hat{i}+2\hat{j}-m\hat{k}$ ***have the same direction.***

Solution: Let $\vec{a}=5\hat{i}+4\hat{j}-3\hat{k}$ ***&*** $\vec{b}=2\hat{i}+2\hat{j}-m\hat{k}$

According to given condition $\hat{a}=\hat{b}$

$$\frac{\vec{a}}{a}=\frac{\vec{b}}{b}$$

$$\frac{5\hat{i}+4\hat{j}-3\hat{k}}{\sqrt{(5)^2+(4)^2+(-3)^2}}=\frac{2\hat{i}+2\hat{j}-m\hat{k}}{\sqrt{(2)^2+(2)^2+(-m)^2}} \Rightarrow \frac{5\hat{i}+4\hat{j}-3\hat{k}}{\sqrt{25+16+9}}=\frac{2\hat{i}+2\hat{j}-m\hat{k}}{\sqrt{4+4+m^2}}$$

$$\frac{5\hat{i}+4\hat{j}-3\hat{k}}{\sqrt{50}}=\frac{2\hat{i}+2\hat{j}-m\hat{k}}{\sqrt{8+m^2}}$$

$$\frac{5}{\sqrt{50}}\hat{i}+\frac{4}{\sqrt{50}}\hat{j}-\frac{3}{\sqrt{50}}\hat{k}=\frac{2}{\sqrt{8+m^2}}\hat{i}+\frac{2}{\sqrt{8+m^2}}\hat{j}-\frac{m}{\sqrt{8+m^2}}\hat{k}$$

Comparing coefficients of $\hat{k}$ ***unit vector***

$$-\frac{3}{\sqrt{50}}=-\frac{m}{\sqrt{8+m^2}}$$

$$\Rightarrow \frac{3}{\sqrt{50}}=\frac{m}{\sqrt{8+m^2}}$$

Taking square on both sides $\frac{9}{50}=\frac{m^2}{8+m^2}$

$$9(+m^2) = 50m^2$$

$$72 + 9m^2 = 50m^2$$

$$72 = 50m^2 - 9m^2$$

$$72 = 41\,m^2$$

$$\frac{72}{41} = m^2$$

Taking square root on both sides

$$m = \pm\sqrt{\frac{72}{41}} \quad \text{or} \quad m = \pm\frac{6\sqrt{2}}{\sqrt{41}}$$

Example# 03: ***The unit vector i , j ,k are represented respectively by the three edges $\overrightarrow{OA}$,$\overrightarrow{OB}$ and $\overrightarrow{OC}$ of a unit cube , write down the expression for the vector represented by the diagonals $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ of the cube , find the length of and direction cosines of these diagonals also.***

Solution: *Let a unit cube whose origin is at point O as shown in figure. Point of each corner of a cube are represented in the figure as O(0,0,0) ,P(1,1,1) , A(1,0,0), B(0,1,0) ,C (0,0,1),A′ (0,1,1) , B′ (1,0, 1) and C′ (1,1,0). Required diagonals of a unit cube are $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$. Then*

$\overrightarrow{AA'}$ = P.v's of A′ – P.v's of A = A′ (0,1,1) – A(1,0,0) $= -\hat{\imath} + \hat{\jmath} + \hat{k}$

$\overrightarrow{BB'}$ = P.v's of B′ – P.v's of B = B′ (1,0,1) – B(0,1,0) $= \hat{\imath} - \hat{\jmath} + \hat{k}$

$\overrightarrow{CC'}$ = P.v's of C′ – P.v's of C = C′ (1,1,0) – C(0,0,1) $= \hat{\imath} + \hat{\jmath} - \hat{k}$

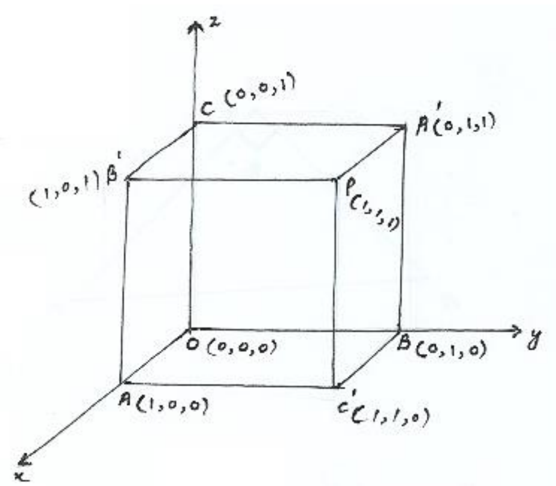
Lengths of above diagonals are $|\overrightarrow{AA'}| = |\overrightarrow{BB'}| = |\overrightarrow{CC'}| = \sqrt{1+1+1} = \sqrt{3}$

Now

Direction cosines of vector $\overrightarrow{AA'}$ *are* $\frac{-1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}$

Direction cosines of vector $\overrightarrow{BB'}$ *are* $\frac{1}{\sqrt{3}}, \quad \frac{-1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}$

Direction cosines of vector $\overrightarrow{CC'}$ *are* $\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{-1}{\sqrt{3}}$



Example#04: Given the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ ***and*** $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$ ***find the magnitude and direction cosines of*** ***(i)*** $\vec{a} - \vec{b}$ ***and*** ***(ii)*** $3\vec{a} - 2\vec{b}$ **.**

Solution: Given $\vec{a} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ ***and*** $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$

$$\vec{a} - \vec{b} = (3\hat{i} - 2\hat{j} + 4\hat{k}) - (2\hat{i} + \hat{j} + 3\hat{k}) = 3\hat{i} - 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} - 3\hat{k}$$

$$= \hat{i} - 3\hat{j} + \hat{k}$$

Magnitude: $|\vec{a} - \vec{b}| = \sqrt{(1)^2 + (-3)^2 + (1)^2} = \sqrt{1 + 9 + 1} = \sqrt{11}$

Direction Cosines: $\frac{1}{\sqrt{11}}$, $\frac{-3}{\sqrt{11}}$, $\frac{1}{\sqrt{11}}$

(ii) $3\vec{a} - 2\vec{b} = 3(3\hat{i} - 2\hat{j} + 4\hat{k}) - 2(2i + j + 3k) = 9\hat{i} - 6\hat{j} + 12\hat{k} - 4\hat{i} - 2\hat{j} - 6\hat{k}$

$$= 5\hat{i} - 8\hat{j} + 6\hat{k}$$

Magnitude: $|3\vec{a} - 2\vec{b}| = \sqrt{(5)^2 + (-8)^2 + (6)^2} = \sqrt{25 + 64 + 36} = \sqrt{125} = 5\sqrt{5}$

Direction Cosines: $\frac{5}{5\sqrt{5}}$, $\frac{-8}{5\sqrt{5}}$, $\frac{6}{5\sqrt{5}}$ ***Or*** $\frac{1}{\sqrt{5}}$, $\frac{-8}{5\sqrt{5}}$, $\frac{6}{5\sqrt{5}}$

Example#05: Prove that the points $-2\,a + 3\,b + 5\,c\,, a + 2b + 3c$ and $7a - c$ ***are collinear.***

Solution: Let $A(-4\,a + 6\,b + 10\,c), B(2a + 4b + 6c)$ and $C(14a - 2c)$

be three points. Take A be the initial point of B and C

Now $\overrightarrow{AB}$ = P. v's of B – P. v's of A = $(2a + 4b + 6c) - (-4\,a + 6\,b + 10\,c)$

$$= 2a + 4b + 6c + 4\,a - 6\,b - 10\,c$$

$$= 6a - 2b - 4c$$

$\overrightarrow{AC}$ = P. v's of C – P. v's of A = $(14a - 2\,c) - (-\,4\,a + 6\,b + 10\,c)$

$$= 14a - 2\,c + 4\,a - 6\,b - 10\,c$$

$$= 18a - 6b - 12\,c$$

$$\overrightarrow{AC} = 3(6a - 2\,b - 4c)$$

$$\overrightarrow{AC} = 3\,\overrightarrow{AB}$$

According to above condition , this shows that the given points are collinear .

EXERCISE: 1.1

Q#01: Find magnitude (length or norm) of vectors (i) $2\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}$ **(ii)** $\left(\frac{-3}{5}\right)\hat{\imath} - \left(\frac{-4}{5}\right)\hat{\jmath} + 6\hat{k}$

(i) $2\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}$

Solution: Let $\vec{r} = 2\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}$

Magnitude of $\vec{r} = |\vec{r}| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9}$ $\Rightarrow |\vec{r}| = 3$

(ii) $\left(\frac{-3}{5}\right)\hat{\imath} - \left(\frac{-4}{5}\right)\hat{\jmath} + 6\hat{k}$

Solution: Let $\vec{r} = \left(\frac{-3}{5}\right)\hat{\imath} - \left(\frac{-4}{5}\right)\hat{\jmath} + 6\hat{k} = \left(\frac{-3}{5}\right)\hat{\imath} + \left(\frac{4}{5}\right)\hat{\jmath} + 6\hat{k}$

Magnitude of $\vec{r} = |\vec{r}| = \sqrt{\left(\frac{-3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + (6)^2} = \sqrt{\left(\frac{9}{25}\right) + \left(\frac{16}{25}\right) + 4} = \sqrt{\frac{9+16+100}{25}} = \sqrt{\frac{125}{25}} \Rightarrow |\vec{r}| = \sqrt{5}$

Q#02: Given the points A (1,2,- 1) : B(-3, 1, 2) and C (0, -4, 3)

(i)Find $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{AC}, \overrightarrow{BA}, \overrightarrow{CB}, \overrightarrow{BA}$ **(ii) Prove that** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(i)Find $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{AC}, \overrightarrow{BA}, \overrightarrow{CB}, \overrightarrow{BA}$

Solution: $\therefore \overrightarrow{AB}$ = ***P.v's of B – P.v's of A***

$= B(-3,1,2) - A(1,2,-1)$

$= (-3-1)i + (1-2)j + (2+1)k$

$= -4\,\hat{\imath} - \hat{\jmath} + 3\hat{k}$

$\therefore \overrightarrow{BA}$ = ***P.v's of A – P.v's of B***

$= A(1,2,-1) - B(-3,1,2)$

$= (1+3)i + (2-1)j + (-1-2)$

$= 4\,\hat{\imath} + \hat{\jmath} - 3\hat{k}$

$\therefore \overrightarrow{BC}$ = ***P.v's of C – P.v's of B***

$= C(0,-4,3) - B(-3,1,2)$

$= (0+3)i + (-4-1)j + (3-2)k$

$= 3\,\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$\therefore \overrightarrow{CB}$ = ***P.v's of B – P.v's of***

$= B(-3,1,2) - C(0,-4,3)$

$= (-3-0)i + (1+4)j + (2-3)k$

$= -3\hat{\imath} + 5\hat{\jmath} - \hat{k}$

$\therefore \overrightarrow{AC}$ = ***P.v's of C – P.v's of A***

$= C(0,-4,3) - A(1,2,-1)$

$= (0-1)i + (-4-2)j + (3+1)k$

$= -\hat{\imath} - 6\hat{\jmath} + 4\hat{k}$

$\therefore \overrightarrow{CA}$ = ***P.v's of A – P.v's of C***

$= A(1,2,-1) - C(0,-4,3)$

$= (1-0)i + (2+4)j + (-1-3)k$

$= \hat{\imath} + 6\hat{\jmath} - 4\hat{k}$

(ii)Prove that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Solution: $\therefore$ $\overrightarrow{AB}$ = ***P.v's of B – P.v's of A*** $= B(-3,1,2) - A(1,2,-1)$

$= (1+3)i + (2+1)j + (-1-2)k$

$= -4\,\hat{i} - \hat{j} + 3\hat{k}$

$\therefore$ $\overrightarrow{BC}$ = ***P.v's of C – P.v's of B*** $= C\,(0,-4,3) - B(-3,1,2)$

$= (-3-0)i + (1+4)j + (2-3)k$

$= 3\,\hat{i} - 5\hat{j} + \hat{k}$

$\therefore$ $\overrightarrow{AC}$ = ***P.v's of C – P.v's of A*** $= C\,(0,-4,3) - A\,(1,2,-1)$

$= (0-1)i + (2+4)j + (-1-3)k$

$= -\hat{i} - 6\hat{j} + 4\hat{k}$

Now $\overrightarrow{AB}+\overrightarrow{BC} = -4\hat{i} - \hat{j} + 3\hat{k} + 3\hat{i} - 5\hat{j} + \hat{k} = -\hat{i} - 6\hat{j} + 4\hat{k}$

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ***Hence proved***

Q#03: Given $\overrightarrow{r_1} = 3\hat{i} - 2\hat{j} + \hat{k}$ ***;*** $\overrightarrow{r_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$ ***and*** $\overrightarrow{r_3} = -\hat{i} + 2\hat{j} + 2\hat{k}$ ***then find the magnitude of***

(a) $\overrightarrow{r_3}$ ***(b)*** $\overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3}$ ***(c)*** $2\overrightarrow{r_1} - 3\,\overrightarrow{r_2} - 5\,\overrightarrow{r_3}$

(a) $\overrightarrow{r_3}$

Solution: ***Let*** $\overrightarrow{r_3} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Magnitude of $\overrightarrow{r_3} = |\overrightarrow{r_3}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9}$ $\Rightarrow$ $|\overrightarrow{r_3}| = \mathbf{3}$

(b) $\overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3}$

Solution: Let $\vec{r} = \overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3} = 3\hat{i} - 2\hat{j} + \hat{k} + 2\hat{i} - 4\hat{j} - 3\hat{k} - \hat{i} + 2\hat{j} + 2\hat{k} = \mathbf{4}\,\hat{i} - 4\hat{j} + 0\hat{k}$

Magnitude of $\vec{r} = |\vec{r}| = \sqrt{(4)^2 + (4)^2 + (0)^2} = \sqrt{16+16+0} = \sqrt{32}$ $\Rightarrow$ $|\vec{r}| = \mathbf{4}\sqrt{2}$

(c) $2\overrightarrow{r_1} - 3\,\overrightarrow{r_2} - 5\,\overrightarrow{r_3}$

Solution: Let $\vec{r} = 2\overrightarrow{r_1} - 3\,\overrightarrow{r_2} - 5\,\overrightarrow{r_3} = 2(3\hat{i} - 2\hat{j} + \hat{k}) - 3(2\hat{i} - 4\hat{j} - 3\hat{k}) - 5(-\hat{i} + 2\hat{j} + 2\hat{k})$

$= 6\hat{i} - 4\hat{j} + 2\hat{k} - 6\hat{i} + 12\hat{j} + 9\hat{k} + 5\hat{i} - 10\hat{j} - 10\hat{k}$

$\vec{r} = 5\hat{i} - 2\hat{j} + \hat{k}$

Magnitude of $\vec{r} = |\vec{r}| = \sqrt{(5)^2 + (-2)^2 + (1)^2} = \sqrt{25+4+1}$ $\Rightarrow$ $|\vec{r}| = \sqrt{30}$

Q#04: if Given $\vec{r_1} = 2\hat{i} - \hat{j} + \hat{k}$ ***;*** $\vec{r_2} = \hat{i} + 3\hat{j} - 2\hat{k}$ ***and*** $\vec{r_3} = -2\hat{i} + \hat{j} - 3\hat{k}$ ***and*** $\vec{r_4} = 3\hat{i} + 2\hat{j} + 5\hat{k}$

Find scalar a, b, c such that $\vec{r_4} = a\,\vec{r_1} + b\,\vec{r_2} + c\,\vec{r_3}$

S0lution: Since given condition $\vec{r_4} = a\,\vec{r_1} + b\,\vec{r_2} + c\,\vec{r_3}$

Putting values $3i + 2j + 5k = a(2i - j + k) + b(i + 3j - 2k) + c(-2i + j - 3k)$

$$3i + 2j + 5k = 2a\,i - a\,j + a\,k + bi + 3b\,j - 2b\,k - 2c\,i + c\,j - 3ck$$

$$3i + 2j + 5k = (2a + b - 2c)i + (-a + 3b + c)j + (a - 2b - 3c)k$$

Comparing coefficients of i, j, k from both sides

$$2a + b - 2c = 3 \quad - - - - - - -(i)$$

$$-a + 3b + c = 2 \quad - - - - - - -(ii)$$

$$a - 2b - 3c = 5 \quad - - - - - -(iii)$$

Adding equation (ii) and (iii)

$$-a + 3b + c = 2$$

$$a - 2b - 3c = 5$$

$$b - 2c = 7 \quad - - - (iv)$$

Multiplying equation (ii) by 2 and adding in equation(i)

$$-2a + 6b + 2c = 4$$

$$2a + b - 2c = 3$$

$$7b = 7 \quad \Longrightarrow \boxed{b=1}$$

Putting b=1 in equation (iv) $1 - 2c = 7$

$$1 - 7 = 2c$$

$$-6 = 2c \quad \Longrightarrow -\frac{6}{2} = c \quad \Longrightarrow \boxed{c = -3}$$

Putting $b = 1$ ***and c=*** -3 ***in equation (i)***

$$2a + 1 - 2(-3) = 3$$

$$2a + 1 + 6 = 3$$

$$2a + 7 = 3$$

$$2a = 3 - 7$$

$$2a = -4 \Longrightarrow \boxed{a = -2}$$

Q#05: Find a unit vector parallel to the resultant of vectors $\vec{r_1} = 2i + 4j - 5k$; $\vec{r_2} = i + 2j + 3k$

Solution: let $\vec{r}$ ***be resultant of*** $\vec{r_1}$ ***&*** $\vec{r_2}$ ***. Then***

$$\vec{r} = \vec{r_1} + \vec{r_2} = 2i + 4j - 5k + i + 2j + 3k$$

$$\vec{r} = 3i + 6j - 2k$$

Let $\hat{r}$ ***be unit vector in the direction of resultant vector*** $\vec{r}$ ***. since***

$$\hat{r} = \frac{\vec{r}}{r} = \frac{3i + 6j - 2k}{\sqrt{(3)^2 + (6)^2 + (-2)^2}} = \frac{3i + 6j - 2k}{\sqrt{9+36+4}} = \frac{3i + 6j - 2k}{\sqrt{49}} = \frac{3i + 6j - 2k}{7} \Rightarrow \hat{r} = \frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k$$

Q#06: If $a = 3i - j - 4k$, $b = 2i + 4j - 3k$ and $c = i + 2j - k$ ***.Find unit vector parallel to 3a−2b+ 4c.***

Solution: Let $\vec{r} = 3a - 2b + 4c$

$$= 3(3i - j - 4k) - 2(-2i + 4j - 3k) + 4(i + 2j - k)$$

$$= 9i - 3j - 12k + 4i - 8j + 6k + 4i + 8j - 4k$$

$$\vec{r} = 17i - 3j - 10k$$

Let $\hat{r}$ ***be unit vector in the direction of vector*** $\vec{r}$ ***. Since***

$$\hat{r} = \frac{\vec{r}}{r} = \frac{17i - 3j - 10k}{\sqrt{(17)^2 + (-3)^2 + (-10)^2}} = \frac{17i - 3j - 10k}{\sqrt{289+9+100}} = \frac{17i - 3j - 10k}{\sqrt{398}}$$

$$\hat{r} = \frac{17}{\sqrt{398}}i - \frac{3}{\sqrt{398}}j - \frac{10}{\sqrt{398}}k$$

Q#07: The position vectors of four points P, Q, R and S are $a, b, 2a + 3b$ and $a - 2b$ ***respectively. Express*** $\vec{PQ}$, $\vec{SQ}$, $\vec{QR}$ and $\vec{PR}$ ***in terms of a and b.***

Solution: Given ***P.v of P = a***

P.v of Q = b

P.v of R = 2a+ 3b

P.v of S = a−2b

Now

$\vec{PQ}$ = ***P.v of Q − P.v of P*** $\Rightarrow \vec{PQ} = b - a$

$\vec{SQ}$ = ***P.v of Q − P.v of S = b – (a−2b) = b−a+ 2b*** $\Rightarrow \vec{SQ} = 3b - a$

$\vec{QR}$ = ***P.v of R − P.v of Q = 2a+3b –b =2a+2b*** $\Rightarrow \vec{QR} = 2(a + b)$

$\vec{PR}$ = ***P.v of R − P.v of P = 2a+3b –a =a+3b*** $\Rightarrow \vec{PR} = a + 3b$

Q#08:Find the value of m and n so that the vector $9i + 7j - 9m\,k$ and $9i - n\,j + 18k$ have same magnitude and direction.

Solution: Let $\vec{a} = 9i + 7j - 9m\,k$ **&** $\vec{b} = 9i - n\,j + 18k$

According to given condition $\vec{a}$ **and** $\vec{b}$ **are parallel vectors.**

Thus $\frac{9}{9} = \frac{7}{-n} = \frac{-9m}{18}$

$\Rightarrow 1 = \frac{-7}{n} = \frac{-m}{2}$

$\Rightarrow 1 = \frac{-7}{n} \Rightarrow n = -7$ **&** $\Rightarrow 1 = \frac{-m}{2} \Rightarrow m = -2$

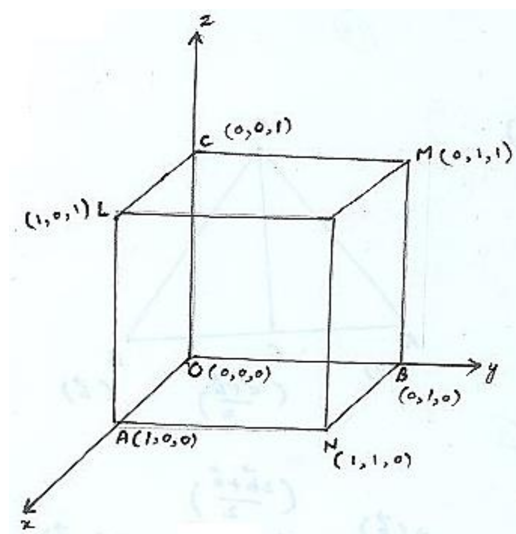
Q#09 : Three edges of a unit cube through the origin O represent the vector i , j , k respectively. Write the diagonal expression for the vectors represented by

(i) The diagonal of the cube, through O.

(ii) The diagonals of the three faces passes through O.

Solution: Let a unit cube whose origin is at point O as shown in figure.

Point of each corner of a cube are represented in the figure as O(0,0,0) ,P(1,1,1) , A(1,0,0), B(0,1,0) ,C (0,0,1), L(1,0,1) , M (0,1,1) and N(1,1,0).



(i) The diagonal of the unit cube is $\overrightarrow{OP}$

Then

$\overrightarrow{OP}$ = ***P.v's of P – P.v's of O = P(1,1,1) – O(0,0,0)*** $= i + j + k$

(ii) The diagonal of three faces of a cube are $\overrightarrow{OL}$ **,** $\overrightarrow{OM}$ **and** $\overrightarrow{ON}$ **.**

Then

$\overrightarrow{OL}$ = ***P.v's of L – P.v's of O = L(1,0,1) – O(0,0,0)*** $= i + 0j + k$

$\overrightarrow{OM}$ = ***P.v's of M – P.v's of O = L(0,1,1) – O(0,0,0)*** $= 0i + j + k$

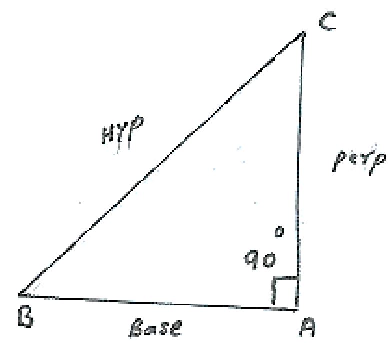
$\overrightarrow{ON}$ = ***P.v's of N – P.v's of O = N(1,1,0) – O(0,0,0)*** $= i + j + 0k$

Q#10:Find the lengths of the sides of a triangle , whose vertices are $A(2,4,-1), B(4,5,1)$ and $C(3,6,-3)$.

and show that the triangle is a right angle triangle.

Solution: Let ΔABC ***whose corner points are*** $A(2,4,-1), B(4,5,1)$ and $C(3,6,-3)$

The length of sides of ΔABC ***are :***



$\overrightarrow{AB}$ = ***P.v's of B – P.v's of A*** $= B(4,5,1) - A(2,4,-1)$

$= (4-2)i + (5-4)j + (1+1)k$

$= 2i + j + 2k$

$|\overrightarrow{AB}| = \sqrt{(2)^2 + (1)^2 + (2)^2} = \sqrt{4+1+4} = \sqrt{9}$

$|\overrightarrow{AB}| = \mathbf{3}$ --------------------------------------***(i)***

$\overrightarrow{BC}$ = ***P.v's of C – P.v's of B*** $= C(3,6,-3) - B(4,5,1)$

$= (3-4)i + (6-5)j + (-3-1)k$

$= -i + j - 4k$

$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (1)^2 + (-4)^2} = \sqrt{1+1+16}$

$|\overrightarrow{BC}| = \sqrt{18}$ ----------------------------------***(ii)***

$\overrightarrow{CA}$ = ***P.v's of A – P.v's of C*** $= A(2,4,-1) - C(3,6,-3)$

$= (2-3)i + (4-6)j + (-1+3)k$

$= -i - 2j + 2k$

$|CA| = \sqrt{(-1)^2 + (-2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9}$

$|\overrightarrow{CA}| = \mathbf{3}$ --------------------------------------***(iii)***

From equ. (i) ,(ii) and (iii)

$|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = (3)^2 + (3)^2 = \mathbf{9+9 = 18} = (\sqrt{18})^2 = |\overrightarrow{BC}|^2$

$|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2$

This show that given triangle is a right angle triangle at point A. because $\angle A = 90^0$ **.**

Q#11:Find a vector whose magnitude is 5 and is in the direction of vector $4i - 3j + k$.

Solution: Let $\vec{A}$ ***be a vector whose magnitude is 5.***

$$\therefore \quad |\vec{A}| = 5$$

& ***let*** $\vec{r} = 4i - 3j + k$

According to given condition, $\vec{A}$ ***be a vector whose magnitude is 5 in the direction of*** $\vec{r}$ ***vector is written as,***

$$\vec{A} = |\vec{A}| \cdot \hat{r}$$

$$= |\vec{A}| \cdot \frac{\vec{r}}{r} \qquad \therefore \hat{r} = \frac{\vec{r}}{r}$$

$$= 5 \cdot \frac{4i-3j+k}{\sqrt{(4)^2+(-3)^2+(1)^2}}$$

$$= 5. \frac{4i-3j+k}{\sqrt{16+9+1}}$$

$$= \frac{20i-15j+5k}{\sqrt{26}}$$

$$\vec{A} = \frac{20}{\sqrt{26}} i - \frac{15}{\sqrt{26}} j + \frac{5}{\sqrt{26}} k$$

Q#12:Find a vector whose magnitude is 2 and is parallel to vector 5i+3j+2k.

Solution: Let $\vec{A}$ ***be a vector whose magnitude is 2***

$$\therefore \quad |\vec{A}| = 2$$

& ***let*** $\vec{r} = 5i + 3j + 2k$

According to given condition, $\vec{A}$ ***be a vector whose magnitude is 2 is parallel to*** $\vec{r}$ ***vector is written as,***

$$\vec{A} = |\vec{A}| \cdot \hat{r}$$

$$= |\vec{A}| \cdot \frac{\vec{r}}{r} \qquad \therefore \hat{r} = \frac{\vec{r}}{r}$$

$$= (2) \cdot \frac{5i+3j+2k}{\sqrt{(5)^2+(3)^2+(2)^2}}$$

$$= (2). \frac{5i+3j+2k}{\sqrt{25+9+4}}$$

$$= \frac{10i+6j+4k}{\sqrt{38}}$$

$$\vec{A} = \frac{10}{\sqrt{38}} i + \frac{6}{\sqrt{38}} j + \frac{4}{\sqrt{38}} k$$

Q#13: Find a vector whose magnitude is that of the vector i−3j+9k and is in the direction of vector 4i−3j+k.

***Solution:* Let** $\vec{A} = i-3j+9k$; $\vec{B} = 4i-3j+k$

Let $\vec{R}$ be the required vector whose magnitude is that of the vector $\vec{A}$ in the direction of $\vec{B}$.

$$\vec{R} = |\vec{A}|.\hat{B}$$

$$= |\vec{A}| . \frac{\vec{B}}{B} \qquad \therefore \hat{B} = \frac{\vec{B}}{B}$$

$$= \sqrt{(1)^2 + (-3)^2 + (9)^2} . \frac{4i-3j+k}{\sqrt{(4)^2 + (-3)^2 + (1)^2}}$$

$$= \sqrt{1+9+81}. \frac{4i-3j+k}{\sqrt{16+9+1}}$$

$$= \sqrt{91}. \frac{4i-3j+k}{\sqrt{26}} = \sqrt{\frac{91}{26}}(4i-3j+k)$$

$$= \sqrt{\frac{7}{2}}(4i-3j+k)$$

$$\vec{R} = 4\sqrt{\frac{7}{2}}\,i - 3\sqrt{\frac{7}{2}}\,j + \sqrt{\frac{7}{2}}\,k$$

Q#14: (i) if vectors 3i + j − k and λi − 4j + 4k are parallel, find the value of λ.

(ii) If vectors 3i + 6j + k and i − mj + $\frac{1}{3}$k are parallel, find the value of m.

(i) if vectors 3i + j − k and λi − 4j + 4k are parallel, find the value of λ.

***Solution*: Let** $\vec{a} = 3i + j - k$ & $\vec{b} = \lambda i - 4j + 4k$

Since $\vec{a}$ and $\vec{b}$ are parallel, therefore their directional components are proportional as

$$\frac{3}{\lambda} = \frac{1}{-4} = \frac{-1}{4} \Rightarrow \frac{3}{\lambda} = \frac{1}{-4} \Rightarrow 3(-4) = \lambda \Rightarrow \boxed{\lambda = -12}$$

(ii) If vectors 3i + 6j + k and i − mj + $\frac{1}{3}$k are parallel, find the value of m.

***Solution:* let** $\vec{a} = 3i + 6j + k$ & $\vec{b} = i - mj + \frac{1}{3}k$

Since $\vec{a}$ and $\vec{b}$ are parallel, therefore their directional components are proportional as

$$\frac{3}{1} = \frac{6}{-m} = \frac{1}{\left(\frac{1}{3}\right)}$$

$$\frac{3}{1} = \frac{6}{-m} = \frac{3}{1} \Rightarrow \frac{3}{1} = \frac{6}{-m} \Rightarrow m = (-6)/3 \Rightarrow \boxed{m = -2}$$

Q#15: Show that the vectors $4i - 6j + 9k$ **and** $-6i + 9j - \frac{27}{2}k$ **are collinear.**

Solution: Let $\vec{a} = 4i - 6j + 9k$ **&** $\vec{b} = -6i + 9j - \frac{27}{2}k$

Multiplying $\vec{b}$ ***with*** $\frac{-2}{3}$

$$\frac{-2}{3}\vec{b} = \frac{-2}{3}\left(-6i + 9j - \frac{27}{2}k\right)$$

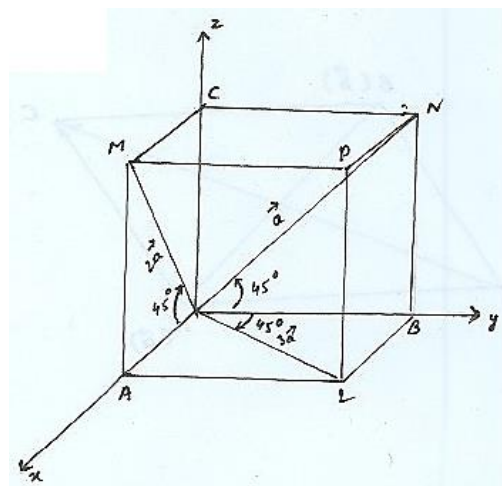
$$\frac{-2}{3}\vec{b} = 4i - 6j + 9k$$

$$\frac{-2}{3}\vec{b} = \vec{a} \qquad \textit{or} \qquad \vec{a} = \frac{-2}{3}\vec{b}$$

This shows that vectors $\vec{a}$ ***and*** $\vec{b}$ ***are collinear.*** $(\vec{a} = \lambda\vec{b})$

Q#16: *Three vectors of magnitude a, 2a, 3a, meet in point and their direction are along the diagonals of adjacent faces of a cube . Determine their resultant and direction cosines.*

Solution: Let i, j ,k be the unit vectors represented by along $\overrightarrow{OA}$, $\overrightarrow{OB}$ ***and*** $\overrightarrow{OC}$ ***and given vectors*** $\vec{a}$, $2\vec{a}$, $3\vec{a}$ ***acting along the diagonal of faces of a cube*** $\overrightarrow{ON}$, $\overrightarrow{OM}$ ***and*** $\overrightarrow{OL}$ ***making an angle of*** 45^0 ***with x,y,z –axis.***



$$\vec{a} = a\cos 45^0\, j + a\sin 45^0\, k = \frac{a}{\sqrt{2}}j + \frac{a}{\sqrt{2}}k \text{ ------------(i)}$$

$$2\vec{a} = 2a\cos 45^0\, i + 2a\sin 45^0\, k = \frac{2a}{\sqrt{2}}i + \frac{2a}{\sqrt{2}}k \text{ ------------(ii)}$$

$$3\vec{a} = 3a\cos 45^0\, i + 3a\sin 45^0\, j = \frac{3a}{\sqrt{2}}i + \frac{3a}{\sqrt{2}}j \text{ ------------(iii)}$$

Let $\vec{r}$ ***be the resultant of*** $\vec{a}$, $2\vec{a}$ ***and*** $3\vec{a}$ ***.then***

$$\vec{r} = \vec{a} + 2\vec{a} + 3\vec{a}$$

$$\vec{r} = \frac{a}{\sqrt{2}}j + \frac{a}{\sqrt{2}}k + \frac{2a}{\sqrt{2}}i + \frac{2a}{\sqrt{2}}k + \frac{3a}{\sqrt{2}}i + \frac{3a}{\sqrt{2}}j$$

$$\therefore \vec{r} = \frac{5a}{\sqrt{2}}i + \frac{4a}{\sqrt{2}}j + \frac{3a}{\sqrt{2}}k$$

$$|\vec{r}| = \sqrt{\left(\frac{5a}{\sqrt{2}}\right)^2 + \left(\frac{4a}{\sqrt{2}}\right)^2 + \left(\frac{3a}{\sqrt{2}}\right)^2} = \sqrt{\frac{25a^2}{2} + \frac{16a^2}{2} + \frac{9a^2}{2}} = \sqrt{\frac{25a^2+16a^2+9a^2}{2}} = \sqrt{\frac{50a^2}{2}} = \sqrt{25a^2}$$

$$|\vec{r}| = 5a$$

Direction cosines of vector $\vec{r}$ ***are***

$$\frac{\left(\frac{5a}{\sqrt{2}}\right)}{5a}, \frac{\left(\frac{4a}{\sqrt{2}}\right)}{5a}, \frac{\left(\frac{3a}{\sqrt{2}}\right)}{5a} \quad \Rightarrow \quad \frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}$$

Q#17: Find the angles which the vector $\vec{a} = 3i - 6j + 2k$ makes with the coordinate axes.

***Solution:** Let vector $\vec{a}$ makes makes an angle α , β and γ with x, y and z-axes.*

Given vector $\vec{a} = 3i - 6j + 2k$

$|\vec{a}| = \sqrt{(3)^2 + (-6)^2 + (2)^2} = \sqrt{9+36+4} = \sqrt{49}$

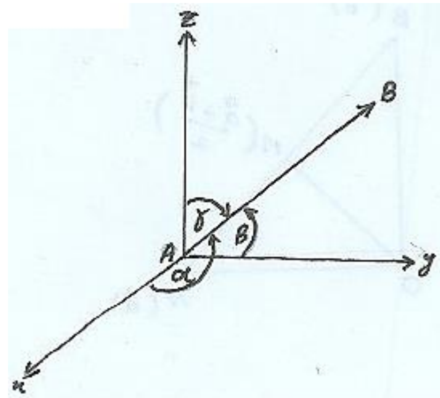
$|\vec{a}| = 7$ *and* $a_x = 3$, $a_y = -6$, $a_z = 2$

By using direction cosines

$\text{Cos}\,\alpha = \frac{a_x}{|\vec{a}|} = \frac{3}{7} \Rightarrow \alpha = \cos^{-1}\left(\frac{3}{7}\right) \Rightarrow \alpha = 71.8^0$

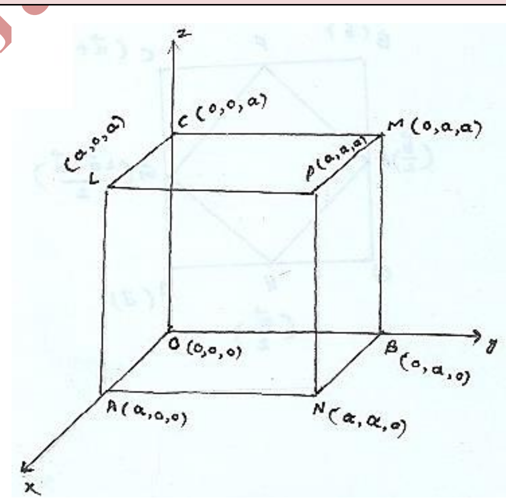
$\text{Cos}\,\beta = \frac{a_y}{|\vec{a}|} = \frac{-6}{7} \Rightarrow \beta = \cos^{-1}\left(\frac{-6}{7}\right) \Rightarrow \beta = 149^0$

$\text{Cos}\,\gamma = \frac{a_y}{|\vec{a}|} = \frac{2}{7} \Rightarrow \gamma = \cos^{-1}\left(\frac{2}{7}\right) \Rightarrow \gamma = 73.3^0$



Q#18: Prove that the sum of three vectors determined by the diagonal of the three faces of a cube passing through the same corner, the vector being directed from the corner ,is twice the vector determined by the diagonal of the cube passing through the same corner.

***Solution:** Let a cube whose length of each side is 'a' . $\overrightarrow{OL}$, $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are the diagonal of the faces of cube and $\overrightarrow{OP}$ be the diagonal of cube passing through point O.*



We have to prove $\overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON} = 2\,\overrightarrow{OP}$

From figure P(a,a,a) , A(a,0,0), B(0,a,0) ,C (0,0,a),

L(a,0,a) , M (0,a,a) and N(a,a,0).

The diagonal of the unit cube is $\overrightarrow{OP}$

Then $\overrightarrow{OP}$ *= P.v of P – P.v of O = P(a ,a ,a) – O(0,0,0) = a i+ a j +a k* -------------------------------------(i)

The diagonal of three faces of a cube are $\overrightarrow{OL}$, $\overrightarrow{OM}$ *and* $\overrightarrow{ON}$.

Then $\overrightarrow{OL}$ *= P.v of L – P.v of O = L(a,0,a) – O(0,0,0) = ai+0j+ak = a i+ a k* -----------------------------(ii)

$\overrightarrow{OM}$*= P.v of M – P.v of O = M(0,a,a) – O(0,0,0) = 0i+a j+ a k = a j+ a k* --------------------------(iii)

$\overrightarrow{ON}$ *=P.v of N – P.v of O= N(a,a,0) – O(0,0,0) = a i+ a j+0k = a i+ a j* ----------------------------(iv)

According to given condition, adding (ii),(iii) and (iv)

$\overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON}$ *= a i + a k + a j +a k+ a i+ a j =2a i +2a j+ 2a k =2(ai + aj +a k)*

$\overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON} = 2\,\overrightarrow{OP}$ *Hence proved.*

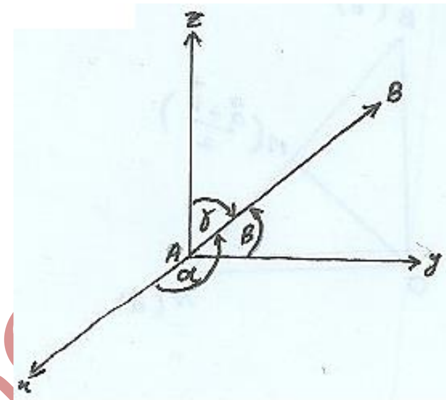
Q#19: (i) Find direction cosines of line joining the points $(3,2,-4)$ and $(1,-1,2)$ **.**

(ii)Prove that the points $-4a + 6b + 10c$, $2a + 4b + 6c$ and $14a - 2c$ **are collinear .**

(i) Find direction cosines of line joining the points $(3,2,-4)$ and $(1,-1,2)$ **.**

Solution: Given points $A(3,2,-4)$ and $B(1,-1,2)$ **.**

Let vector $\vec{a}$ **makes an angle** α, β and γ **with x, y and z-axes.**



$\vec{a} = \overrightarrow{AB}$ **=P.v's of B – P.v's of A** $= B(1,-1,2) - A(3,2,-4)$

$= (1-3)i + (-1-2)j + (2+4)k$

$\vec{a} = -2i - 3j + 6k$

& $|\vec{a}| = \sqrt{(-2)^2 + (-3)^2 + (6)^2} = \sqrt{4+9+36} = \sqrt{49}$

$|\vec{a}| = 7$ **and** $a_x = -2$, $a_y = -3$, $a_z = 6$

Direction cosines:

Cos $\alpha = \frac{a_x}{|\vec{a}|} = \frac{-2}{7}$ **:** **Cos** $\beta = \frac{a_y}{|\vec{a}|} = \frac{-3}{7}$ **:** **Cos** $\gamma = \frac{a_z}{|\vec{a}|} = \frac{6}{7}$

(ii)Prove that the points $-4a + 6b + 10c$, $2a + 4b + 6c$ and $14a - 2c$ **are collinear.**

Solution: Let A$(-4a + 6b + 10c)$, $B(2a + 4b + 6c)$ and $C(14a - 2c)$ **be three points.**

Take A be the initial point of B and C.

Now $\overrightarrow{AB} = \text{P.v's of B} - \text{P.v's of A}$

$= (2a + 4b + 6c) - (-4a + 6b + 10c) = 2a + 4b + 6c + 4a - 6b - 10c$

$= 6a - 2b - 4c$

$\overrightarrow{AC} = \text{P.v's of C} - \text{P.v's of}$

$= (14a - 2c) - (-4a + 6b + 10c) = 14a - 2c + 4a - 6b - 10c$

$= 18a - 6b - 12c$

$\overrightarrow{AC} = 3(6a - 2b - 4c)$

$\overrightarrow{AC} = 3\,\overrightarrow{AB}$

According to above condition , this shows that the given points are collinear .

Q#20: Find the value of x and y. If $x\vec{a} - 5\vec{b} = 3\vec{a} + y\vec{b}$. where $\vec{a}$ and $\vec{b}$ are two collinear vectors.

Solution: Given statement $x\vec{a} - 5\vec{b} = 3\vec{a} + y\vec{b}$

Comparing coefficients of vector $\vec{a}$ and $\vec{b}$ from both sides

$$x = 3 \quad \text{and} \quad y = -5$$

Q#21: Under what condition do the vectors $5x\,i - y\,j + z\,k$ and $xi - 6j + k$ have same magnitude ?

Solution: Let $\vec{a} = 5x\,i - y\,j + z\,k$ **&** $\vec{b} = xi - 6\,j + k$

According to given condition $\vec{a}$ and $\vec{b}$ have same magnitude

$$|\vec{a}| = |\vec{b}|$$

$$\sqrt{(5x)^2 + (-y)^2 + (z)^2} = \sqrt{(x)^2 + (-6)^2 + (1)^2}$$

$$\sqrt{25x^2 + y^2 + z^2} = \sqrt{x^2 + 36 + 1}$$

$$\sqrt{25x^2 + y^2 + z^2} = \sqrt{x^2 + 37}$$

Taking square on both sides

$$25x^2 + y^2 + z^2 = x^2 + 37$$

$$25x^2 + y^2 + z^2 - x^2 - 37 = 0$$

$$24x^2 + y^2 + z^2 - 37 = 0$$ ***This is the required condition.***

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